第３８回 環論および表現論シンポジウム報告集

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愛知工業大学

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２００６年３月
山梨大学

Organizing Committee of The Symposium on Ring Theory and Representation Theory

The symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are Y. Hirano (Okayama Univ.), S. Koshitani (Chiba Univ.), K. Nishida (Shinshu Univ.) and M. Sato (Yamanashi Univ.).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2006 will be held at Hiroshima University in Hiroshima Prefecture for Sep 16-18, and the program will be arranged by Mamoru Kutami (Ymaguchi Univ.).

Concerning several information on ring theory group in Japan containing schedules of meetings and symposiums as well as the addresses of members in the group, you should refer the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/ ring/ (in Japanese)
civil2.cec.yamanashi.ac.jp/ ring/japan/ (in English)

Kenji Nishida
Matsumoto, Japan
December, 2005

CONTENTS

PREFACE

The 38th Symposium on Ring Theory and Representation Theory was held at Aichi Institute of Technology in Aichi Prefecture from September 2 to 4 in 2005. The symposium and the proceedings are financially supported by Toshiyuki Katsura (University of Tokyo) JSPS Grant-in-Aid for Scientific Research (A), No.15204001, Kenji Nishida (Shinsyu University) JSPS Grant-in-Aid for Scientific Research (C), No.17540021, and Masahisa Sato (Yamanashi University) JSPS Grant-in Aid for Scientific Research (C), No.16540019.

This volume consists of the articles presented at the symposium. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the organaizing committee for their helpful suggestions concerning the symposium. Finally we should like to express our gratitude to Professor Sumiyama and his students of Aichi Institute of Technology who contributed in the organization of the symposium.

Takayoshi Wakamatsu
Saitama, Japan
March, 2006

Program of the 38-th Symposium on Ring Theory and Representation Theory(2005)

September 2 (Friday)

[09:45 – 10:30]
Ohnuki, Yosuke (Tokyo Univ. of Agr. and Tech.)
Stable equivalences related with syzygy functors

[10:45 – 11:30]
Nagase, Hiroshi (Nara National College of Tech.)
Hochschild cohomology of stratified algebras

[11:45 – 12:30]
Nishida, Kenji (Shinshu Univ.)
Linkage and duality of modules

[13:45 – 14:45]
Ariki, Susumu (Kyoto Univ.)
Algebras arising from Schur-Weyl type dualities (I)

[15:00 – 15:45]
Yanagawa, Kohji (Osaka Univ.)
Castelnuovo-Mumford regularity for complexes and weakly Koszul modules

[16:00 – 16:45]
Motose, Kaoru (Hirosaki Univ.)
Integral group algebras and cyclotomic polynomials

[17:00 – 18:00]
Rouquier, Raphaël (Université Denis Diderot - Paris 7)
Broué's conjecture: methods and results (I)

September 3 (Saturday)

[09:45 – 10:30]
Iyama, Osamu (Univ. of Hyogo)
Orthogonality of subcategories

[10:45 – 11:30]
Takahashi, Ryo (Meiji Univ.)
A generalization of n-torsionfree modules

[11:45 – 12:30]
Koshitani, Shigeo (Chiba Univ.)
Broué's abelian defect group conjecture

[13:45 – 14:45]
Ariki, Susumu (Kyoto Univ.)
Algebras arising from Schur-Weyl type dualities (II)

[15:00 – 16:00]
Rouquier, Raphaël (Université Denis Diderot - Paris 7)
Broué's conjecture: methods and results (II)

[16:15 – 17:00]
Tachikawa, Hiroyuki (Univ. of Tsukuba)
QF rings and QF associated graded rings

September 4 (Sunday)

[09:45 – 10:45]
Rouquier, Raphaël (Université Denis Diderot - Paris 7)
Broué's conjecture: methods and results (III)

[11:00 – 12:00]
Ariki, Susumu (Kyoto Univ.)
Algebras arising from Schur-Weyl type dualities (III)

ALGEBRAS ARISING FROM SCHUR-WEYL TYPE DUALITIES

SUSUMU ARIKI

Abstract. The aim of this paper is to present algebras which are studied in our field. These algebras deserve more detailed study from various points of view.

1. Basic Example

この節では，Green による Schur 代数から始めて Dipper-James 理論に現れる q-Schur 代数を導入し，Beilinson-Lusztig-MacPherson による q-Schur 代数の幾何的な実現のもとになったアイデアを説明する．

1.1. **Schur 代数.** k を体とし，$A(n) = k[X_{ij}]_{1\le i,j\le n}$ を n^2 変数の多項式環とする．

$$\epsilon : A(n) \to k, \quad \Delta : A(n) \to A(n) \otimes A(n)$$

を以下のように定義すると $(A(n), \epsilon, \Delta)$ は余代数である．

$$\epsilon(X_{ij}) = \delta_{ij}, \quad \Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}.$$

$A(n,r)$ を $A(n)$ の次数 r の斉次多項式全体とすると

$$\Delta(A(n,r)) \subset A(n,r) \otimes A(n,r)$$

であり，$\dim A(n,r) = \binom{n^2+r-1}{r}$ である．

Definition 1. $S(n,r) = \mathrm{Hom}_k(A(n,r), k)$ を Schur 代数とよぶ.

ϵ が $S(n,r)$ の単位元である．また，$A(n,r)$-*mod* $= S(n,r)$-*mod* である．

1.2. **Schur 代数を導入する動機.** k を代数閉体，$G = GL_n$ の関数環を $k[G] = A(n)[\frac{1}{detX}]$ とすると，$G(k) = \mathrm{Hom}_{k\text{-}alg}(k[G], k) = GL_n(k)$ である．

Definition 2. $k[G]$-*comod* を $GL_n(k)$-*mod* とかく．$GL_n(k)$-加群 V が多項式加群とは，余加群射 $\Delta_V : V \to V \otimes k[G]$ の像が $V \otimes A(n)$ に含まれるときをいう．

任意の $GL_n(k)$-加群は $det^{\otimes \mathbb{Z}} \otimes -$ を除いて多項式加群である．

$$V(n,r) = \{v \in V | \Delta_V(v) \in V \otimes A(n,r)\}$$

とおくと，$V = \oplus_{r\ge 0} V(n,r)$ であり，$V(n,r)$ は $A(n,r)$-余加群であるから有限次元 $GL_n(k)$-加群の性質の研究の多くは $S(n,r)$-加群の研究に帰着する．

The paper is in a final form and no version of it will be submitted for publication elsewhere.

1.3. **Schur-Weyl 相互律.** $(\Delta\otimes\mathrm{Id})\circ\Delta=(\mathrm{Id}\otimes\Delta)\circ\Delta$ より $A(n,r)$ は $(S(n,r),S(n,r))$-両側加群である.

$$D(n,r)=k[T_1,\dots,T_n]=\bigoplus_{r\geq 0}D(n,r),\quad T(n,r)=\mathrm{Hom}_{k\text{-}alg}(D(n,r),k)$$

とする. $D(n,r)$ は単項式 $\{T^\mu|\mu=(\mu_1,\dots,\mu_n)\in\mathbb{Z}^n_{\geq 0},|\mu|=r\}$ を基底にもつ. 双対基底 $\{\xi_\mu\}$ は $T(n,r)$ の基底である. $D(n,r)$ は

$$\epsilon(T_i)=1,\quad \Delta(T_i)=T_i\otimes T_i$$

により余代数であるから, $T(n,r)$ は $\binom{n+r-1}{r}$ 次元の代数であり,

$$1=\sum_\mu\xi_\mu,\quad \xi_\mu\xi_\nu=\delta_{\mu\nu}\xi_\mu$$

が成り立つ. $A(n)\to D(n)$ を $X_{ij}\mapsto\delta_{ij}T_i$ で定めると余代数射 $A(n,r)\to D(n,r)$ を誘導し, これは全射であるから, $T(n,r)\subset S(n,r)$ と思うこととする. このとき, $A(n,r)$ は $(T(n,r),S(n,r))$-両側加群であるから, 双対をとれば weight 分解

$$S(n,r)=\bigoplus_\mu S(n,r)_\mu,\quad S(n,r)_\mu=S(n,r)\xi_\mu$$

が得られる. E を $GL_n(k)$ の定義加群とする.

Lemma 3. $r\leq n$ とすると, $\xi_{(1^r)}S(n,r)\xi_{(1^r)}\simeq kS_r$ であり, $(S(n,r),kS_r)$-両側加群同型 $S(n,r)\xi_{(1^r)}\simeq E^{\otimes r}$ が成り立つ.

Theorem 4. (1) $\mathrm{End}_{kS_r}(E^{\otimes r})\simeq S(n,r)$.
(2) $r\leq n$ ならば $\mathrm{End}_{S(n,r)}(E^{\otimes r})\simeq kS_r$.

Definition 5. $\mathrm{Hom}_{S(n,r)}(S(n,r)\xi_{(1^r)},-):S(n,r)\text{-}mod\to kS_r\text{-}mod$ を Schur 関手という.

1.4. **q-analogue.** $k[G]$ を $k_q[G]$ にすると, kS_r は A 型 Hecke 代数に置き換わる.

$$\mathcal{H}_r^A(q)=\bigoplus_{w\in S_r}kT_w,\quad T_wT_i=\begin{cases}T_{ws_i} & (\ell(ws_i)>\ell(w))\\ qT_{ws_i}+(q-1)T_w & (\ell(ws_i)<\ell(w))\end{cases}$$

ただし, $s_i=(i,i+1), T_i=T_{s_i}$ で $\ell(w)$ は w の転倒数である. $T\in\mathrm{End}_k(E^{\otimes 2})$ を

$$Te_i\otimes e_j=\begin{cases}qe_j\otimes e_i & (i\leq j)\\ e_j\otimes e_i+(q-1)e_i\otimes e_j & (i>j)\end{cases}$$

で定めると, $T_i=\mathrm{Id}^{\otimes i-1}\otimes T\otimes\mathrm{Id}^{\otimes r-i-1}$ により $E^{\otimes r}$ は $\mathcal{H}_r^A(q)$-加群になる.

Definition 6. $S_q(n,r)=\mathrm{End}_{\mathcal{H}_r^A(q)}(E^{\otimes r})$ を q-Schur 代数という.

$n\geq r$ ならば, $S_q(n,r)$ は $S_q(r,r)$ に森田同値である.

1.5. **有限体上の一般線形群.** $G = GL_r$, $G(q) = GL_r(\mathbb{F}_q) \supset B(q)$ を Borel 部分群とし，$M = kB(q)\backslash G(q)$ を右 $G(q)$-加群とする．$q \neq 0 \in k$ と仮定すると $\mathcal{H}_r^A(q) \simeq \mathrm{End}_{G(q)}(M)$ であり，M は $(\mathcal{H}_r^A(q), kG(q))$-両側加群である．

$\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}, |\mu| = r$ であるとし，S_μ を μ の行固定化部分群，$x_\mu = \sum_{w \in S_\mu} T_w$ とする．Dipper-James は次を示した．

Theorem 7. $\mathrm{End}_{G(q)}(\oplus_\mu x_\mu M) \simeq S_q(n, r)$ である．

$P_\mu(q)$ を μ に対応する $G(q)$ の放物型部分群とすると $x_\mu M = kP_\mu(q)\backslash G(q)$ であるから，

$$X(q) = \sqcup_\mu G(q)/P_\mu(q) = \{0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{F}_q^r\}$$

とおけば，$S_q(n, r)$ は $X(q) \times X(q)$ 上の畳み込み代数として実現できる．

実際，これは一般的な話である．K, L を有限群 H の部分群とし，$1_K, 1_H$ を単位表現として置換加群を

$$1_K \otimes_{kK} kH = \bigoplus_{h \in K\backslash H} k[Kh], \quad 1_L \otimes_{kL} kH = \bigoplus_{h \in L\backslash H} k[Lh]$$

とかく．$x \in L\backslash H/K$ に対し $\varphi_x \in \mathrm{Hom}_{kH}(1_K \otimes_{kK} kH, 1_L \otimes_{kL} kH)$ を

$$\varphi_x([Kh]) = \sum_{Lh' : h'h^{-1} \in LxK} [Lh']$$

で定めると $\{\varphi_x\}$ は $\mathrm{Hom}_{kH}(1_K \otimes_{kK} kH, 1_L \otimes_{kL} kH)$ の基底である．

次に全単射

$$L\backslash H/K \simeq H\backslash H \times H/L \times K$$

を $LxK \mapsto (L, xK)$, $(xL, yK) \mapsto Lx^{-1}yK$ により定め，LxK に対応する $H/L \times H/K$ の H-軌道を O_x とかく． すると，O_x の特性関数

$$1_x(xL, yK) = \begin{cases} 1 & ((xL, yK) \in O_x) \\ 0 & ((xL, yK) \notin O_x) \end{cases}$$

を用いて

$$\varphi_x([Kh]) = \sum_{Lh'} 1_x(h'^{-1}L, h^{-1}K)[Lh']$$

となる．L, K, J を H の部分群とし，$C^H(H/L \times H/K)$ と $C^H(H/K \times H/J)$ を各々 $H/L \times H/K$ 上および $H/K \times H/J$ 上の H-不変な k-値関数のなすベクトル空間とすると，$f \in C^H(H/L \times H/K), g \in C^H(H/K \times H/J)$ に対して，畳み込み積が

$$f * g(xL, zJ) = \sum_{yK} f(xL, yK) g(yK, zJ)$$

により定義される．

Proposition 8. (1) $\varphi_x \mapsto 1_x$ により次のベクトル空間の同型を得る．

$$\mathrm{Hom}_{kH}(1_K \otimes_{kK} kH, 1_L \otimes_{kL} kH) \simeq C^H(H/L \times H/K)$$

(2) 上の同型のもとで，$\varphi_x \circ \varphi_y$ は $1_x * 1_y$ に対応する．

X を以下のように定め，$\mathbb{F}_q$-値点を $X(q)$ とかく．

$$X = \bigsqcup_{\mu=(\mu_1,\ldots,\mu_n):|\mu|=r} G/P_\mu$$

Corollary 9. $S_q(n,r)$ は $C^{G(q)}(X(q)\times X(q))$ と k-代数として同型である．

$$(0 = V_0 \subset \cdots \subset V_n = \mathbb{F}_q^r,\ \ 0 = W_0 \subset \cdots \subset W_n = \mathbb{F}_q^r) \in X(q)\times X(q)$$

に対し，

$$a_{ij} = \dim \frac{V_i \cap W_j}{V_{i-1}\cap W_j + V_i \cap W_{j-1}}$$

とおくと $A = (a_{ij}) \in Mat_n(\mathbb{Z}_{\geq 0})$ である．

$X(q)\times X(q)$ 中の $G(q)$-軌道は以下のように記述される．

Lemma 10. $X(q)\times X(q) \to \{A = (a_{ij}) \in Mat_n(\mathbb{Z}_{\geq 0}) | \sum_{i,j} a_{ij} = r\}$ の各ファイバーはただひとつの $G(q)$-軌道からなる．

Corollary 9 の畳み込み積による表示をもとに，Beilinson-Lusztig-MacPherson は $\overline{\mathbb{Q}}_l[q,q^{-1}]$ を基礎環にもつ $S_q(n,r)$ の幾何的実現を与えた．ここで q は不定元である．

2. affine 化と退化

この節では，局所体上の代数群の表現論で重要な affine Hecke 代数と, Drinfeld により導入された退化 affine Hecke 代数について説明する．これらの代数ではモジュラー表現があまり研究されていないので，モジュラー表現の研究が進むことが望まれている．

2.1. A 型 Hecke 代数の affine 化.

Definition 11. A 型 Hecke 代数と Laurent 多項式環のテンソル積 $\mathcal{H}_n(q) = \mathcal{H}_n^A(q) \otimes k[X_1^{\pm},\ldots,X_n^{\pm}]$ に

$$T_i X^\lambda = X^{s_i\lambda} T_i + (q-1)\frac{X^\lambda - X^{s_i\lambda}}{1-\frac{X_i}{X_{i+1}}}$$

という交換関係を入れて得られる代数を A 型（拡大）affine Hecke 代数と呼ぶ.

生成元を $X_1^{\pm},\ldots,X_n^{\pm},T_1,\ldots,T_{n-1}$, 基本関係式を

$$(T_i-q)(T_i+1)=0,\quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\quad T_iT_j = T_jT_i\ \ (j\neq i\pm 1).$$

$$X_iX_j = X_jX_i,\quad X_iX_i^{-1} = X_i^{-1}X_i = 1.$$

$$T_iX_iT_i = qX_{i+1},\quad T_iX_j = X_jT_i\ \ (j\neq i,i+1).$$

として定義してもよい．affine Hecke 代数の退化を得るには，$x_ix_j = x_jx_i$ を仮定した上で

$$T_i = s_i + (q-1)t_i + \frac{(q-1)^2}{2}u_i + \cdots,$$

$$X_i = q^{x_i} = 1 + (q-1)x_i + \frac{(q-1)^2}{2}x_i(x_i-1) + \cdots$$

として，基本関係式に代入し $(q-1)$ の２次以上の項を無視する．

Remark 12. 実際，２次の項まで一致させようとすると，まず $T_i^2 = (q-1)T_i + q$ に代入して

$$s_i^2 + (q-1)(2s_it_i) + (q-1)^2(t_i^2 + s_iu_i) + \cdots = 1 + (q-1)(s_i+1) + (q-1)^2t_i + \cdots$$

より $s_i^2 = 1$, $t_i = \frac{s_i+1}{2}$, $u_i = 0$ を得る．すると，

$$\begin{aligned} T_iT_{i+1}T_i = s_is_{i+1}s_i &+ \frac{(q-1)}{2}(3s_is_{i+1}s_i + s_is_{i+1} + s_{i+1}s_i + 1) \\ &+ \frac{(q-1)^2}{4}(3s_is_{i+1}s_i + 2s_is_{i+1} + 2s_{i+1}s_i + 2s_i + s_{i+1} + 2) + \cdots \end{aligned}$$

となるから，$T_iT_{i+1}T_i$ と $T_{i+1}T_iT_{i+1}$ を２次の項まで一致させようとすると $s_i = s_{i+1}$ となって関係式がつぶれすぎてしまう．

さて，$\frac{d}{dq}T_i|_{q=1} = \frac{s_i+1}{2}$, $\frac{d}{dq}X_i|_{q=1} = x_i$ であるから，$T_iX_iT_i = qX_{i+1}$ の両辺を q で微分して $q = 1$ とおくと

$$\frac{s_i+1}{2}s_i + s_ix_is_i + s_i\frac{s_i+1}{2} = 1 + x_{i+1}$$

すなわち，$s_ix_i - x_{i+1}s_i = -1$ を得る．同様に $T_iX_j = X_jT_i$ $(j \neq i, i+1)$ から $s_ix_j = x_js_i$ $(j \neq i, i+1)$ を得る．

そこで，$k[r]$ を１変数多項式環として以下のように退化 affine Hecke 代数を定義する．

Definition 13. 退化 affine Hecke 代数 $\mathcal{H}_n$ とは，生成元が $s_1, \ldots, s_{n-1}, x_1, \ldots, x_n$，基本関係が

$$s_i^2 = 0, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad s_is_j = s_js_i \ \ (j \neq i \pm 1).$$

$$x_ix_j = x_jx_i,$$

$$s_ix_i - x_{i+1}s_i = -r, \quad s_ix_j = x_js_i \ \ (j \neq i, i+1).$$

で定義される $k[r]$-代数である．

対称群の群環と多項式環のテンソル積 $\mathcal{H}_n = kS_n \otimes k[r][x_1, \ldots, x_n]$ に $s_ix_i - x_{i+1}s_i = -r$, $s_ix_j = x_js_i$ $(j \neq i, i+1)$ という交換関係を与えて得られる代数といってもよい．

2.2. **Lusztig の結果．** X を $G = GL_n(\mathbb{C})$ の旗多様体，$\mathfrak{g} = gl_n(\mathbb{C})$,

$$\tilde{X} = \{(x, b) \in \mathfrak{g} \times X | b \text{ は } x\text{-stable}\}$$

とする．$\tilde{X}$ は $(g,t)(x,b) = (t^{-2}Ad(g)x, Ad(g)b)$ により $G \times \mathbb{C}^\times$-多様体である．同様に $\mathfrak{g}$ も $G \times \mathbb{C}^\times$-多様体になり，$p : \tilde{X} \to \mathfrak{g}$ を第１成分への射影とすると p は同変写像である．

$$K = Rp_!\mathbb{C} \in D^{G \times \mathbb{C}^\times}(\mathfrak{g})$$

とおく．次は Lusztig の定理である．

Theorem 14. $\mathrm{End}_{D^{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ は $\mathbb{C}$ 上の退化 affine Hecke 代数 $\mathcal{H}_n$ と同型であり，この同型のもとで，$\mathrm{End}^0_{D^{G\times\mathbb{C}^\times}(\mathfrak{g})}(K)$ は $\mathbb{C}S_n$ と同一視される．

2.3. **荒川・鈴木関手.** $\mathfrak{g} = gl_n(\mathbb{C})$ とする．$\mathcal{O}$ を有限生成 $\mathfrak{g}$-加群であって，ウエイト分解をもち Borel 部分 Lie 環に関して局所有限であるもののなす圏とする．

$$\mathfrak{h} = \mathbb{C}y_1 \oplus \cdots \oplus \mathbb{C}y_n, \quad \mathfrak{h}^* = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$$

とし，$\lambda \in \mathfrak{h}^*$ の定める中心指標を

$$\chi_\lambda : Z(\mathfrak{g}) \simeq \mathbb{C}[y_1, \ldots, y_n]^{S_n} \to \mathbb{C}$$

とする．ここで対称群の $\mathfrak{h}^*$ への作用は dot 作用 $w \circ \lambda = w(\lambda + \rho) - \rho$ である．

$$\mathcal{O} = \oplus_\lambda \mathcal{O}^{[\lambda]}$$

と中心指標に合わせて圏 $\mathcal{O}$ も直和分解される．$M \in \mathcal{O}$ に対し，$M = \oplus_\lambda M^{[\lambda]}$ とかく．

$$\Omega = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \in \mathfrak{g} \otimes \mathfrak{g}$$

と定める．E を $\mathfrak{g}$ の定義表現とする．

Lemma 15. $M \in \mathcal{O}$ に対し，$M \otimes E^{\otimes r}$ は次の作用により退化 affine Hecke 代数 $\mathcal{H}_r$ の表現加群である．

$$\begin{cases} s_i \mapsto \Omega_{i,i+1} \quad (1 \le i < r) \\ x_j \mapsto n - 1 + \sum_{0 \le i < j} \Omega_{ij} \quad (1 \le j \le r) \end{cases}$$

ここで，Ω_{ij} は Ω を i 番めと j 番めのテンソル成分に作用させる作用素であり，M を 0 番めのテンソル成分と数えている．

Theorem 16. μ を支配的整ウエイト，$L(\mu)$ を μ を最高ウエイトにもつ有限次元既約 $\mathfrak{g}$-加群とすると，$L(\mu) \otimes E^{\otimes r}$ は $(U(\mathfrak{g}), \mathcal{H}_r)$-両側加群であって

(a) $\mathrm{End}_{U(\mathfrak{g})}(L(\mu) \otimes E^{\otimes r})$ は $\mathcal{H}_r$ の商代数である．
(b) $\mathrm{End}_{\mathcal{H}_r}(L(\mu) \otimes E^{\otimes r})$ は $U(\mathfrak{g})$ の商代数である．

これは skew 版の Schur-Weyl 相互律であるが，モジュラー版の域には達していないので，精密化が望まれる．

Definition 17. $F_\lambda : \mathcal{O} \to \mathcal{H}_r\text{-}mod$ を次で定め，荒川・鈴木関手と呼ぶ．

$$F_\lambda = \mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), -)$$

ただし，$M(\lambda)$ は Verma 加群である．

順序づけられた multisegment

$$[\mu_1 + n - 1, \lambda_1 + n - 2], \ [\mu_2 + n - 2, \lambda_2 + n - 3], \ldots, [\mu_n, \lambda_n - 1]$$

を考える．ここで λ は分割，$\lambda \supset \mu$ である．μ は行の長さが大きい順に並んでいるとは限らない．この順序づけられた multisegment は $\mathcal{H}_r$ の放物型部分代数 $\mathcal{H}_{\lambda-\mu}$ の 1 次元加群 $\mathbb{C}_{\lambda-\mu}$ を定める．次は Arakawa-Suzuki による．

Theorem 18. $M(\lambda, \mu) = \mathcal{H}_r \otimes_{\mathcal{H}_{\lambda-\mu}} \mathbb{C}_{\lambda-\mu}$ とおくと，$F_\lambda(M(\mu)) \simeq M(\lambda, \mu)$ であり，$F_\lambda(L(\mu))$ は既約か零である．

かれらの結果はいつ $F_\lambda(L(\mu)) \neq 0$ かも教えてくれる．またこのとき $L(\lambda,\mu) = F_\lambda(L(\mu))$ とおけば $L(\lambda,\mu) = \mathrm{Top}(M(\lambda,\mu))$ である．F_λ は skew Schur 関手とよぶべきものであり，やはりモジュラー版の開発が望まれる．

Remark 19. $\mathcal{H}_r$-*mod* $\to Y(gl_n)$-*mod* という関手もあり，Drinfeld 関手と呼ばれる．

Remark 20. $\mathbb{C}[t,t^{-1}]^{\otimes r}$ の $\mathbb{C}[t]$-部分加群の列 $\{L_i\}_{i\in\mathbb{Z}}$ であって

(1) $L_i \otimes_{\mathbb{C}[t]} \mathbb{C}[t,t^{-1}] = \mathbb{C}[t,t^{-1}]^{\otimes r}$,
(2) $\cdots \subset L_i \subset L_{i+1} \subset \cdots$,
(3) $tL_i = L_{i-n}$.

をみたすものを考えれば q-Schur 代数の代わりに affine q-Schur 代数を得る．

3. BC 型への拡張

3.1. **Schur 代数.** $G = GSp_{2n}$ とする．定義加群 E は $e_{1'} = e_{2n}, \ldots, e_{n'} = e_{n+1}$ として
$$E = (\oplus_{i=1}^n ke_i) \oplus (\oplus_{i=1}^n ke_{i'})$$
であり，$(e_i,e_j) = 0 = (e_{i'},e_{j'})$, $(e_i,e_{j'}) = \delta_{ij}$ により交代形式が定義される．G の関数環は $k[G] = k[X_{ij}]_{1\le i,j\le 2n}[\frac{1}{detX}]/I$ であり，$k[G]$-*mod* を $G(k)$-*mod* とかく．ただし，I は次の元で生成される両側イデアルである．
$$\sum_{k=1}^n (X_{ik}X_{jk'} - X_{ik'}X_{jk}),\ \ \sum_{k=1}^n (X_{ki}X_{k'j} - X_{k'i}X_{kj}),\ \ (1 \le i \neq j' \le 2n)$$
$$\sum_{k=1}^n (X_{ik}X_{i'k'} - X_{ik'}X_{i'k}) - \sum_{k=1}^n (X_{kj}X_{k'j'} - X_{k'j}X_{kj'}),\ \ (1 \le i,j \le n)$$
多項式部分 $A(n) = k[X_{ij}]_{1\le i,j\le 2n} + I/I$ は次数環であるから，r 次部分を $A(n,r)$ とかく．ϵ, Δ を A 型のときと同じ式で定義すると $A(n,r)$ は余代数になる．Donkin は次の代数を導入した．

Definition 21. $S(n,r) = \mathrm{Hom}_k(A(n,r),k)$ を symplectic Schur 代数と呼ぶ．

3.2. **Brauer 代数.** symplectic Schur 代数の場合，Schur-Weyl 相互律のパートナーは Brauer 代数である．

Definition 22. $\omega \in k$ とする．Brauer 代数 $B_r(\omega)$ とは生成元 $s_1,\ldots,s_{r-1},e_1,\ldots,e_{r-1}$ と基本関係
$$s_i^2 = 1, \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad s_is_j = s_js_i \ \ (j \neq i \pm 1)$$
$$e_i^2 = \omega e_i, \quad e_ie_{i+1}e_i = e_i, \quad e_{i+1}e_ie_{i+1} = e_{i+1}, \quad e_ie_j = e_je_i \ \ (j \neq i \pm 1)$$
$$e_is_i = e_i = s_ie_i$$
$$s_ie_{i+1}e_i = s_{i+1}e_i, \quad e_{i+1}e_is_{i+1} = e_{i+1}s_i, \quad s_ie_j = e_js_i \ \ (j \neq i \pm 1)$$
で定義される k-代数である．

次の定理は Dipper-Doty-Hu による．

Theorem 23. k を無限体とすると次が成立．

(1) $S(n,r) \simeq \mathrm{End}_{B_r(-2n)}(E^{\otimes r})$.
(2) $r \leq n$ ならば $B_r(-2n) \simeq \mathrm{End}_{S(n,r)}(E^{\otimes r})$.

3.3. **q-analogue.** $k[G]$ を $k_q[G]$ にすると，$B_r(2n)$ は Birman-Murakami-Wenzl 代数に置き換わる．

Definition 24. $q, \lambda \in k^\times$ とする．BMW 代数とは生成元 $T_1, \ldots, T_{r-1}$ と基本関係

$$(T_i - \lambda)(T_i - q)(T_i + q^{-1}) = 0$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \ \ (j \neq i \pm 1)$$

$$E_i T_i^{\pm 1} = T_i^{\pm 1} E_i = \lambda^{\pm 1} E_i$$

$$E_i T_{i\pm 1} E_i = \lambda^{-1} E_i, \quad E_i T_{i\pm 1}^{-1} E_i = \lambda E_i$$

で定義される k-代数である．ただし

$$E_i = 1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}.$$

今考えている C 型の Schur-Weyl 相互律の場合 $\lambda = -q^{-2n-1}$ になる．

3.4. **affine BMW 代数.** BMW 代数の affine 化は Ram により定義された．

Definition 25. $q, \lambda, Q_1, Q_2, \dots \in k^\times$ とする．affine BMW 代数とは生成元が $T_1, \ldots, T_{r-1}, X_1$ で，基本関係が BMW 代数の基本関係に加えて

$$E_1 X_1^r E_1 = Q_r E_1, \quad E_1 X_1 T_1 X_1 = \lambda E_1$$

で定義される k-代数である．

3.5. **退化 BMW 代数.** 退化 affine BMW 代数は，affine BMW 代数が導入される以前にすでに Nazarov により導入されていた．

Definition 26. $\omega, \omega_1, \omega_2, \dots \in k$ とする．affine Wenzl 代数とは，生成元が

$$s_1, \ldots, s_{r-1}, e_1, \ldots, e_{r-1}, x_1, \ldots, x_r$$

で，基本関係が Brauer 代数 $B_r(\omega)$ の基本関係に加えて

$$s_i x_j = x_j s_i \ \ (j \neq i, i+1), \quad e_i x_j = x_j e_i \ \ (j \neq i, i+1)$$

$$s_i x_i - x_{i+1} s_i = e_i - 1 = x_i s_i - s_i x_{i+1}$$

$$e_1 x_1^r e_1 = \omega_r e_1$$

$$e_i(x_i + x_{i+1}) = 0 = (x_i + x_{i+1}) e_i$$

で定義される k-代数である．

Remark 27. $X_{i+1} = T_i X_i T_i$ とおくと，$X_1, \ldots, X_r$ は可換である．そこで，$E_i = 1 - \frac{T_i - T_i^{-1}}{q - q^{-1}}$ より $\frac{d}{dq} T_i|_{q=1} = 1 - e_i$ であることに注目し，$X_i = q^{2x_i}$ とおいて $X_{i+1} = T_i X_i T_i$ を q で微分すれば $s_i x_i - x_{i+1} s_i = e_i - 1$ が得られる．

Remark 28. $\mathfrak{g} = sp_{2n}$ とすると，$L(\mu) \otimes E^{\otimes r}$ には affine Wenzl 代数が作用し，$\mathfrak{g}$ の作用と可換である．つまりここにも Schur-Weyl 相互律がある．

3.6. **affine Sergeev 代数.**

Definition 29. Clifford 超代数 C_n とは次数が奇の生成元 $c_1, \dots, c_n$ と基本関係

$$c_i^2 = 1, \quad c_i c_j = -c_j c_i \ \ (i \neq j)$$

で定義された k-超代数である.

対称群の捩れ群環 T_n を生成元が $t_1, \dots, t_{n-1}$ で基本関係が

$$t_i^2 = 1, \quad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \quad t_i t_j = -t_j t_i \ \ (j \neq i, i \pm 1)$$

で定義される k-代数として定義する.

Remark 30. $H^2(S_n, \mathbb{C}^\times) \simeq \mathbb{Z}/2\mathbb{Z}$ $(n \geq 5)$ である.

Definition 31. Sergeev 超代数とは，超テンソル積 $T_n \otimes C_n$ に交換関係

$$t_i c_j = -c_j t_i$$

を与えて得られる超代数である.

Lemma 32. Sergeev 超代数は kS_n と C_n の半直積に同型である.

Definition 33. affine Sergeev 超代数とは，偶生成元 $s_1, \dots, s_{r-1}, x_1, \dots, x_r$ と奇生成元 $c_1, \dots, c_r$ で生成され，基本関係が以下のように与えられるものをいう.

(1) $s_1, \dots, s_{r-1}$ は対称群の基本関係式をみたす.
(2) $x_1, \dots, x_r$ は可換.
(3) $c_1, \dots, c_r$ は Clifford 関係式をみたす.
(4) $s_i c_j = c_j s_i$ $(j \neq i, i+1)$, $s_i c_i = c_{i+1} s_i$, $s_i c_{i+1} = c_i s_i$.
(5) $s_i x_j = x_j s_i$ $(j \neq i, i+1)$.
(6) $s_i x_i - x_{i+1} s_i = -c_i c_{i+1} - 1$.

定義式をみれば affine Wenzl 代数との類似性は明らかであろう．前節までに紹介してきた A 型 affine Hecke 代数，A 型退化 affine Hecke 代数，affine Wenzl 代数は，すべて有限次元モジュラー既約表現の分類が $A^{(1)}$ 型の柏原クリスタルで記述されることがわかっているが，Brundan-Kleshchev の結果によれば，affine Sergeev 代数のモジュラー既約表現の分類は $A^{(2)}$ 型の柏原クリスタルで記述される.

3.7. **q-analogue.**

Sergeev 超代数の q-analogue も存在する.

Definition 34. Hecke-Clifford 代数とは，生成元が $T_1, \dots, T_{r-1}, C_1, \dots, C_r$ で基本関係が以下のように与えられるものをいう.

(1) $T_1, \dots, T_{r-1}$ は $\mathcal{H}_r^A(q)$ の基本関係式をみたす.
(2) $C_1, \dots, C_r$ は Clifford 関係式をみたす.
(3) $T_i C_j = C_j T_i$ $(j \neq i, i+1)$.
(4) $T_i C_{i+1} = C_i T_i - (q - q^{-1})(C_i - C_{i+1})$.

さらに，affine Sergeev 超代数の q-analogue も存在して affine Sergeev 超代数はその退化になっているのであるが，ここでは定義は省略する.

Remark 35. Nazarov によれば，affine Sergeev 超代数の超加群のなす圏から Queer Lie 超代数の Yangian の超加群のなす圏への Drinfeld 関手が存在する.

4. 結び

以上紹介してきたことからわかるように，不思議な代数がたくさん存在していて，柏原クリスタルと関係したり，幾何のにおいがしたりして，もう少し理解するといいことがありそうに思われます．affine Hecke 代数については巡回商という考え方で有限次元代数の理論に帰着させていろいろ結果を得ました．最近では，affine Wenzl 代数に同じ手法を使って，すべての有限次元既約表現を構成することに成功しましたので最後にこの結果だけ報告して終わりとします．これは Mathas と Rui との共同研究です．詳しくは論文をご覧下さい．

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ORTHOGONALITY OF SUBCATEGORIES

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Abstract. Let G be a finite small subgroup of $\mathrm{SL}_d(k)$ and $S := k[[x_1, ..., x_d]]$. We will discuss (i)–(iv) below.
(i) maximal $(d-2)$-orthogonal subcategories of $\mathrm{CM}\, S^G$,
(ii) non-commutative crepant resolutions of S^G,
(iii) tilting $S * G$-modules,
(iv) Fomin-Zelevinsky mutation.

0. 導入

以下 k を標数 0 の体とし, G を $\mathrm{SL}_d(k)$ の有限部分群とする. S で冪級数環 $k[[x_1, \cdots, x_d]]$ を表し, S^G で G の不変式環を表す. 以下 S^G が孤立特異点であると仮定する. また $\mathrm{CM}\, S^G$ で maximal Cohen-Macaulay S^G-加群（以下, 単に $\mathrm{CM}S^G$-加群と呼ぶ）の成す圏を表す. 有限生成 S^G-加群の圏 $\mathrm{mod}\, S^G$（特に $\mathrm{CM}\, S^G$）においては, 直既約分解に関する Krull-Schmidt 型の定理が成立する. 本文の出発点は Auslander による, 2 次元における次の定理である [A3][Y1].

0.1 定理 $d = 2$ とする.

(1) S^G は**有限表現型**, 即ち直既約 CM S^G-加群は同型を除いて有限個しか存在しない.

(2) S^G の Auslander-Reiten quiver は G の McKay quiver に一致する. ここで各々の quiver の定義は略すが（1.11 参照）以下の要素から成るものである.

	S^G の Auslander-Reiten quiver	G の McKay quiver
頂点	直既約 CM S^G-加群	既約 G-加群
矢印	概分裂完全列で決める	$(k^d) \otimes -$ で決める

しかし一方で $d \geq 3$, $G \neq 1$ の時, S^G は決して有限表現型にならない事が証明される [AR2]. より強くその表現は, 表現型理論 [CB] における有限-tame-wild の 3 分割中, 最も難しい wild と呼ばれるクラスに属する事さえ分かる.

本文の動機は, 表現型理論とは異なる視点から $\mathrm{CM}\, S^G$ を理解する方法を模索する事にある. そのために S^G とともに本文における主役である**捩れ群環** $S * G$ を導入する. これは G を基底とする自由 S-加群に, 積を

$$(sg)(s'g') = (sg(s'))(gg') \qquad (s, s' \in S,\ g, g' \in G)$$

と定めたものである. McKay quiver は $S * G$ の構造を与える quiver に他ならない.

表題にある Auslander-Reiten 理論は, 有限次多元環や可換 Cohen-Macaulay 環, 或いはそれらを統合した整環の表現論における基本理論である [ARS][Y1]. その典型が 2 次元における定理 0.1 であるが, d 次元の類似を考察すると, 以下の 1 章で紹介するように Auslander-Reiten 理論の高次元版とでも呼ぶべき現象が観察される事が分かる [I1,2]. 一方で代数幾

The detailed version of this paper will be submitted for publication elsewhere.

何学において S^G の特異点解消の導来圏の研究は, McKay 対応として近年非常に盛んである [KV][BKR]. 我々の考察する捩れ群環 $S*G$ 上の加群は, McKay 対応に現れる k^d（の原点での完備化）上の G-同変連接層と見なされる．また 2 次元 McKay 対応において基本的な Artin-Verdier 理論 [AV] は, 直既約 CMS^G-加群と S^G の特異点解消の例外集合の間の対応を与えるが, その一種の高次元版が Van den Bergh [V1,2] により非可換クレパント解消 (2.3) として導入された．これらを含めた観点から, 本文では圏 CM S^G や $D^b(\mathrm{mod}\, S*G)$ を 0.1 の高次元版として調べる．その際に基本となる概念は, Kontsevich による homological ミラー対称性予想の研究に端を発する, 三角圏に対する Calabi-Yau 条件である.

3 章は吉野雄二氏との共同研究 [IY][I3][Y2], 2・4 章は I.Reiten との共同研究 [IR] です. また McKay 対応に関して石井亮氏はじめ多くの方々に御教示を頂いた事を感謝します.

以下特に断らない限り Λ, Γ 等は**整環**（**=CM 多元環**）, 即ち d 次元完備正則局所環 R 上の多元環であり, R-加群として有限生成射影的なものを表す. S^G や $S*G$ を典型例として想定している．整環 Λ 上の **CM 加群**とは, Λ-加群のうち R-加群として有限生成射影的なものの事である．また整環 Λ が**孤立特異点**であるとは, $\mathrm{gl.dim}\, \Lambda \otimes_R R_{\mathfrak{p}} = \mathrm{ht}\, \mathfrak{p}$ が R の全ての極大でない素イデアル $\mathfrak{p}$ に対して成立する事であり, 整環 Λ が**対称**であるとは, (Λ, Λ)-加群として $\mathrm{Hom}_R(\Lambda, R) \simeq \Lambda$ である事を意味する．例えば可換 Gorenstein 環は対称整環である.

1. 高次元 Auslander-Reiten 理論の試み

一般に加法圏 $\mathcal{C}$ の直既約対象の同型類の全体を $\mathrm{ind}\, \mathcal{C}$ で表す．また $X \in \mathcal{C}$ に対して

$$\mathrm{add}\, X := \{Y \in \mathcal{C} \mid Y \text{ は } X^n\ (n > 0) \text{ の直和因子}\}$$

とおく．X が $\mathcal{C}$ の**加法生成元**であるとは, $\mathcal{C} = \mathrm{add}\, X$ となる事である.

まずは S^G と $S*G$ の関係を少し詳しく観察してみる.

1.1 (1) $\mathrm{gl.dim}\, S*G = d = \mathrm{depth}\, S*G$ が成立する．即ち $S*G$ は非可環な正則環と呼ぶに相応しい性質を有する.

(2) $\mathrm{End}_{S^G}(S) = S*G$ が成立し, さらに $d = 2$ ならば, S は CM S^G の加法生成元である.

即ち CM S^G の加法生成元 S の準同型環として $S*G$ が現れる．このような有限表現型である S^G と大域次元が 2 である $S*G$ の関係を観察すると, 有限次多元環における類似である, やはり Auslander による次の定理 [A1][ARS] が思いつく.

1.2 定理 有限表現型有限次多元環 Λ の森田同値類と, 大域次元が 2 以下で dominant 次元が 2 以上の有限次多元環 Γ の森田同値類の間に一対一対応が存在する．それは Λ に対して $\mathrm{mod}\, \Lambda$ の加法生成元 M をとり $\Gamma := \mathrm{End}_\Lambda(M)$ と置く事により与えられる.

ここでは dominant 次元の定義は与えないが, 自己入射分解によって定義されるものであり, 後藤-西田 [GN] により depth の一種の類似とみなす事が可能である事のみ注意しておく.

1.3 Auslander-Reiten 理論のアイデアを一言で述べるとすれば, 加群圏 $\mathrm{mod}\, \Lambda$ 上の関手圏における単純関手の射影分解の考察, となろう．例えば $\mathrm{mod}\, \Lambda$ における概分裂完全列の存在は,（特定の例外を除いて）単純関手が自己双対性を有する長さ 2 の射影分解を持つという事に他ならない．ここで 2 という数字が現れたのは, $\mathrm{mod}\, \Lambda$ がアーベル圏であるためその関手圏の大域次元が 2 となるからである．この意味で Auslander-Reiten 理論は 2 次元的な理論であるといえる.

最も典型的なのはΛが有限表現型の場合であり, その場合$\mathrm{mod}\,\Lambda$上の関手圏は, 上記のΓ上の加群圏$\mathrm{mod}\,\Gamma$と一致し, $\mathrm{mod}\,\Lambda$における概分裂完全列の存在は, 単純Γ-加群が自己双対性を有する長さ2の射影分解を持つ事に対応する [I1]. その意味で1.2はAuslander-Reiten理論の起点であり, 実際1.2を与えたAuslanderの講義録 [A1] の後, Auslander-Reitenによる一連の論文 [A2][AR1] により, 所謂Auslander-Reiten理論の骨格が出来上がった.

さて1.1も1.2も「表現論的な圏の加法生成元の準同型環を取る事により, ホモロジカルに良い性質を持った環が現れる」という事を意味している. それでは逆に「ある種のホモロジカルに良い環を与えた時に, それを準同型環として実現するような, 表現論的な圏が存在する」のでは無いだろうか　勿論それは条件の良さに依存するであろうが, ではどの様な条件であれば存在するのか調べてみたくなる. 1.1,1.2では大域次元が2の環を扱っていたが, それではある種の大域次元nの環に対応する圏においては, 何か「n次元Auslander-Reiten理論」とでも呼ぶべき現象が存在するのでは無いだろうか

そのために次の概念を導入する.

1.4 定義　以下本文では$\mathrm{CM}\,\Lambda$の部分圏$\mathcal{C}$としては, 充満かつ同型・直和・直和因子で閉じたもののみ考える事にする. ゆえに$\mathcal{C}$は$\mathrm{ind}\,\mathcal{C}$によって一意的に決定される.

(1) $\mathrm{CM}\,\Lambda$の部分圏$\mathcal{C}$が**関手的有限**であるとは, 任意の$X\in\mathrm{CM}\,\Lambda$に対して射$f:Y\to X$及び$g:X\to Z$で, $Y,Z\in\mathcal{C}$かつ

$$\mathrm{Hom}_\Lambda(\ ,Y)\xrightarrow{\cdot f}\mathrm{Hom}_\Lambda(\ ,X)\to 0,\quad \mathrm{Hom}_\Lambda(Z,\ )\xrightarrow{g\cdot}\mathrm{Hom}_\Lambda(X,\ )\to 0$$

が$\mathcal{C}$上完全となるものが存在する事である. 例えば$\#\,\mathrm{ind}\,\mathcal{C}<\infty$ならば, $\mathcal{C}$は関手的有限である事は容易に分かる.

(2) 関手的有限な部分圏$\mathcal{C}$が

$$\begin{aligned}\mathcal{C} &= \{X\in\mathrm{CM}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(\mathcal{C},X)=0 \text{ for any } i\ (0<i\le n)\}\\ &= \{X\in\mathrm{CM}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(X,\mathcal{C})=0 \text{ for any } i\ (0<i\le n)\}\end{aligned}$$

を満たす時, $\mathcal{C}$を**極大n-直交部分圏**と呼ぶ事にする. $\mathcal{C}$上では$\mathrm{Ext}^i_\Lambda(\ ,\ )\ (0<i\le n)$が全て消え, またそのような性質を持つ部分圏の中で極大である.

この時, 有限次多元環において, 次の定理1.5が成立するが, $\mathrm{CM}\,\Lambda$自身は$\mathrm{CM}\,\Lambda$の唯一つの極大0-直交部分圏である事に注意すると, 1.5は1.2の一種の高次元化になっている事が分かる.

1.5 定理 有限次多元環の加群圏の極大$(n-1)$-直交部分圏$\mathcal{C}$で$\#\,\mathrm{ind}\,\mathcal{C}<\infty$となるものの同値類と, 大域次元が$(n+1)$以下でdominant次元が$(n+1)$以上の有限次多元環$\Gamma$の森田同値類の間に一対一対応が存在する. それは$\mathcal{C}$の加法生成元$M$をとり$\Gamma:=\mathrm{End}_\Lambda(M)$と置く事により与えられる.

そこで1.3に述べた様に, 極大$(n-1)$-直交部分圏では何らかの意味においてAuslander-Reiten理論の$(n+1)$-次元版が存在するのでは無いかと期待したくなるが, 以下それを考察する.

1.6 定義　$\mathrm{CM}\,\Lambda$の部分圏$\mathcal{C}$をfactor throughする射全体から成る$\mathrm{CM}\,\Lambda$のイデアルを$[\mathcal{C}]$と表す. **安定圏**$\underline{\mathrm{CM}}\Lambda$と**余安定圏**$\overline{\mathrm{CM}}\Lambda$を

$$\underline{\mathrm{CM}}\Lambda:=(\mathrm{CM}\,\Lambda)/[\mathrm{add}\,\Lambda],\quad \overline{\mathrm{CM}}\Lambda:=(\mathrm{CM}\,\Lambda)/[\mathrm{add}\,\mathrm{Hom}_R(\Lambda,R)]$$

と定める. $\mathrm{CM}\,\Lambda$の部分圏$\mathcal{C}$に対し, 対応する$\underline{\mathrm{CM}}\Lambda$及び$\overline{\mathrm{CM}}\Lambda$の部分圏をそれぞれ$\underline{\mathcal{C}}$及び$\overline{\mathcal{C}}$と表す.

1.7 定理 $\mathrm{CM}\,\Lambda$ の極大 $(n-1)$-直交部分圏 $\mathcal{C}$ に対し, 圏同値 *n-Auslander-Reiten translation*

$$\tau_n : \underline{\mathcal{C}} \to \overline{\mathcal{C}}$$

及び関手的同型 *n-Auslander-Reiten* **双対**

$$\underline{\mathrm{Hom}}_\Lambda(X,Y) \simeq \mathrm{Ext}^n_\Lambda(Y,\tau_n X)^*, \quad (X \in \mathcal{C},\ Y \in \mathrm{CM}\,\Lambda)$$

が存在する.

Auslander-Reiten 双対は Serre 双対の類似と捉える事が出来る [RV] が, 以下では 3.3 で用いる. Auslander-Reiten 双対より, Auslander-Reiten 理論において基本的な概分裂完全列の存在定理の, 以下の様な類似が従う. 但し J は圏 $\mathrm{CM}\,\Lambda$ の Jacobson 根基を表す.

1.8 定理 $\mathrm{CM}\,\Lambda$ の極大 $(n-1)$-直交部分圏 $\mathcal{C}$ は n-**概分裂完全列**を持つ. 即ち任意の非射影的な $X \in \mathrm{ind}\,\mathcal{C}$ に対して完全列

$$0 \to \tau_n X \overset{f_n}{\to} C_{n-1} \overset{f_{n-1}}{\to} C_{n-2} \overset{f_{n-2}}{\to} \cdots \overset{f_2}{\to} C_1 \overset{f_1}{\to} C_0 \overset{f_0}{\to} X \to 0 \qquad (C_i \in \mathcal{C},\ f_i \in J)$$

で, 以下が $\mathcal{C}$ 上完全となるものが同型を除いて一意に存在する.

$$0 \to \mathrm{Hom}_\Lambda(\ ,\tau_n X) \overset{\cdot f_n}{\to} \mathrm{Hom}_\Lambda(\ ,C_{n-1}) \overset{\cdot f_{n-1}}{\to} \cdots \overset{\cdot f_1}{\to} \mathrm{Hom}_\Lambda(\ ,C_0) \overset{\cdot f_0}{\to} J(\ ,X) \to 0$$

$$0 \to \mathrm{Hom}_\Lambda(X,\ ) \overset{f_0\cdot}{\to} \mathrm{Hom}_\Lambda(C_0,\ ) \overset{f_1\cdot}{\to} \cdots \overset{f_{n-1}\cdot}{\to} \mathrm{Hom}_\Lambda(C_{n-1},\ ) \overset{f_n\cdot}{\to} J(\tau_n X,\ ) \to 0$$

1.9 n-概分裂完全列の存在は, $\mathcal{C}$ 上の関手圏において, 非射影的な直既約加群に対応する単純関手が, 長さ n の自己双対性を有する射影分解を持つ事に他ならない.

さて $X \in \mathcal{C}$ の *sink* とは, $\mathcal{C}$ の射 $f \in J(Y,X)$ で, $\mathrm{Hom}_\Lambda(\ ,Y) \overset{\cdot f}{\to} J(\ ,X) \to 0$ が完全となり, かつ $Z \to 0$ $(Z \neq 0)$ の形の直和因子を持たないものの事である. それは存在すれば同型を除いて一意である. 1.8 により非射影的な $X \in \mathrm{ind}\,\mathcal{C}$ は sink f_0 を持つ. それでは射影的な X に対してはどうであろうか　実はその場合も sink は存在するのであるが, 一般には 1.8 のような自己双対性を持つ列にまで延長しない.

しかし Auslander-Reiten 理論においては, 2 次元の場合に限って**基本列**と呼ばれる特別な列が存在した事を思い出そう [A3][Y1]. それは全ての単純関手が, 長さ 2 の自己双対性を有する射影分解を持つ事を意味する. その事の極大 $(n-1)$-直交部分圏における類似として次の 1.10 が成立する. $n=1$, $d=2$ の場合が通常の Auslander-Reiten 理論である. ここで ν は**中山関手** $\mathrm{Hom}_R(\mathrm{Hom}_\Lambda(\ ,\Lambda),R)$ を表すが, それは射影 CM Λ-加群の圏と入射 CM Λ-加群の圏の間の同値を与える.

1.10 定理 $d=n+1$ ならば, $\mathrm{CM}\,\Lambda$ の極大 $(n-1)$-直交部分圏 $\mathcal{C}$ は n-**基本列**を持つ. 即ち任意の射影加群 $X \in \mathrm{ind}\,\mathcal{C}$ に対して完全列

$$0 \to \nu X \overset{f_n}{\to} C_{n-1} \overset{f_{n-1}}{\to} C_{n-2} \overset{f_{n-2}}{\to} \cdots \overset{f_2}{\to} C_1 \overset{f_1}{\to} C_0 \overset{f_0}{\to} X \qquad (C_i \in \mathcal{C},\ f_i \in J)$$

で, 以下が $\mathcal{C}$ 上完全となるものが同型を除いて一意に存在する.

$$0 \to \mathrm{Hom}_\Lambda(\ ,\nu X) \overset{\cdot f_n}{\to} \mathrm{Hom}_\Lambda(\ ,C_{n-1}) \overset{\cdot f_{n-1}}{\to} \cdots \overset{\cdot f_1}{\to} \mathrm{Hom}_\Lambda(\ ,C_0) \overset{\cdot f_0}{\to} J(\ ,X) \to 0$$

$$0 \to \mathrm{Hom}_\Lambda(X,\ ) \overset{f_0\cdot}{\to} \mathrm{Hom}_\Lambda(C_0,\ ) \overset{f_1\cdot}{\to} \cdots \overset{f_{n-1}\cdot}{\to} \mathrm{Hom}_\Lambda(C_{n-1},\ ) \overset{f_n\cdot}{\to} J(\nu X,\ ) \to 0$$

また ν は $\mathcal{C}$ 上の自己圏同値であり, それは τ_n の持ち上げを与える.

1.11 定義 $\mathrm{CM}\,\Lambda$ の極大 $(n-1)$-直交部分圏 $\mathcal{C}$ の *Auslander-Reiten quiver* $\mathfrak{A}(\mathcal{C})$ を定める．簡単のため R の剰余体 k が代数的閉体であるとする．$\mathfrak{A}(\mathcal{C})$ の頂点集合は $\mathrm{ind}\,\mathcal{C}$ とする．$X, Y \in \mathrm{ind}\,\mathcal{C}$ に対して，Y の sink $f : Z \to Y$ をとり，Z の直既約分解に現れる X の個数を d_{XY} とする時，d_{XY} 本の矢印を X から Y に引く．

この時次が成立する．特に 0.1 は $d = 2$ の場合とみなす事が出来る．

1.12 定理 k を標数 0 の体，G を $\mathrm{GL}_d(k)$ の有限部分群，$S := k[[x_1, \cdots, x_d]]$ とし，不変式環 S^G が孤立特異点であると仮定する．すると $\mathcal{C} := \mathrm{add}\,S$ は $\mathrm{CM}\,S^G$ の極大 $(d-2)$-直交部分圏であり，その Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ は G の McKay quiver と一致する．

2．傾斜複体と非可換クレパント解消

1 章では，Auslander-Reiten 理論の高次元化という視点から極大直交部分圏を導入したが，この章では表面上異なる視点からそれを考察する．本章と 4 章では導来圏まで考察する範囲を拡張し，その中の扱いやすい部分として CM 加群の圏がある，という見方をする．

さて代数幾何学においては，代数多様体の導来圏の間の三角同値を作る際の手法として，Fourier-向井変換が広く用いられており，最近では Kapranov-Vasserot [KV], Bridgeland-King-Reid [BKR] らによる McKay 対応の構成に用いられた．一方で多元環論においては，

傾斜複体が多元環の導来圏の三角同値を構成する際の基本的な手法であるが，それは Fourier-向井変換のアフィン版と捉えるのが妥当と思われる．それは歴史的には Bernstein-Gelfand-Ponomarev による Gabriel の定理の証明に現れた鏡映関手に始まる [BGP]．鏡映関手は通常のルート系における鏡映の圏論的実現であり，Auslander-Platzeck-Reiten による一般化・抽象化 [APR] を経て，Brenner-Butler 及び宮下により傾斜加群の概念として加群論的に定式化された [BB][M]．一方，傾斜加群の導来圏的研究は Happel 及び Cline-Parshall-Scott に始まり [H][CPS]，間もなく Rickard [R] により，古典的森田理論における射影生成加群の概念の導来圏版である傾斜複体の概念が，2.6 に示す導来森田定理とともに与えられた．

2.1 定義 $T \in D^b(\mathrm{mod}\,\Lambda)$ が**傾斜複体**であるとは，以下の 3 条件が成立する事．

(i) $\mathrm{pd}\,T < \infty$.

(ii) $\mathrm{Hom}(T, T[i]) = 0$ $(i \neq 0)$.

(iii) T は，射影次元有限の対象全体からなる部分圏 $D^b(\mathrm{mod}\,\Lambda)_{fpd}$ を生成する．

加群である様な傾斜複体を**傾斜加群**と呼ぶ．以下簡単のために，導来圏が三角同値である事を，**導来同値**と呼ぶ．

2.2 定理 環 Λ と Γ に対し，以下の条件は同値である．

(1) Λ と Γ は導来同値である．

(2) Λ の傾斜複体 T が存在して，$\mathrm{End}(T)$ は Γ に同型．

さてここで Bondal-Orlov [BO] により予想された，3 次元端末特異点のクレパント解消の間の導来同値に関する，Bridgeland の定理 [B1] を思い出そう．Van den Bergh は [V1,2] において Bridgeland の定理の別証明を与えたが，そこで用いられている概念が次である．

2.3 定義 Λ の**非可換クレパント解消** (*non-commutative crepant resolution*) とは，reflexive Λ-加群 M で $\Gamma := \mathrm{End}_\Lambda(M)$ が $\mathrm{gl.dim}\,\Gamma = d = \mathrm{depth}\,\Gamma$ を満たすものの事である．ただし，本来の定義では Γ が Λ の非可換クレパント解消なのであるが，本文では便宜上 Γ を与える M の方を，そう呼ぶ事にする．

これは代数幾何学におけるクレパント解消（強調して可換クレパント解消と呼ぶ事にする）とは随分異なる様に見えるが, Van den Bergh は論文 [V1,2] において, 実際に可換クレパント解消と非可換クレパント解消の間のある種の対応を与えている. 可換から非可換を作る方法は 2 次元の Artin-Verdier 理論 [AV] の高次元化であり, 非可換から可換を作る方法には, McKay 対応における G-Hilbert スキーム [IN][INj] や G-constellation のモジュライの一般化である, Γ-加群のモジュライを用いている.

2.4 例 $\mathrm{SL}_d(k)$ の有限部分群 G に対して, $\mathrm{End}_{S^G}(S) = S * G$ 及び $\mathrm{gl.dim}\, S * G = d = \mathrm{depth}\, S * G$ が成立していた (1.1) ので, S は S^G の非可換クレパント解消である.

一方で 1.12 より $\mathrm{add}\, S$ は $\mathrm{CM}\, S^G$ の極大 $(d-2)$-直交部分圏でもあった. この事は偶然ではなく 2.5 により説明される. 非可換クレパント解消の方が極大 $(d-2)$-直交部分圏よりも, Cohen-Macaulay でなく reflexive としている分だけ一般なのである.

2.5 定理 $M \in \mathrm{CM}\, \Lambda$ に対し, M が Λ の非可換クレパント解消でかつ $\Lambda \oplus \mathrm{Hom}_R(\Lambda, R) \in \mathrm{add}\, M$ である事と, $\mathrm{add}\, M$ が $\mathrm{CM}\, \Lambda$ の極大 $(d-2)$-直交部分圏である事は同値である.

2.6 Van den Bergh は [V1,2] において, Bondal-Orlov [BO] による予想を一般化した次を予想し, 実際に Bridgeland らの手法 [B1][BKR] を拡張する事により, 3 次元端末特異点を含んだ場合に対して証明を与えた.

問題 Λ の全ての（可換・非可換）クレパント解消は導来圏が三角同値であるか？即ち Λ の全ての可換クレパント解消を X_i $(i \in I)$ とし, 全ての非可換クレパント解消を M_j $(j \in J)$ とした時に, 導来圏 $D^b(\mathrm{Coh}\, X_i)$ $(i \in I)$, $D^b(\mathrm{mod}\, \mathrm{End}_\Lambda(M_j))$ $(j \in J)$ は全てが三角同値ではないか？

この問題のうち非可換クレパント解消に関する部分に対して, 最近得た結果を報告する [IR]. 特に M が Cohen-Macaulay である場合等は, 極めて容易に証明される事柄である [I2].

2.7 定理 Λ を 3 次元孤立特異点とし, M と N を Λ の非可換クレパント解消とする. この時, $\mathrm{Hom}_\Lambda(M, N)$ は射影次元 1 以下の傾斜 $(\mathrm{End}_\Lambda(M), \mathrm{End}_\Lambda(N))$-加群である. 特に全ての非可換クレパント解消は導来同値.

2.8 Λ が対称整環である場合は, より強く次が言える.

定理 対称整環 Λ を 3 次元孤立特異点とし, M を Λ の非可換クレパント解消とする. この時, Λ の非可換クレパント解消全体と, $\mathrm{End}_\Lambda(M)$ の射影次元 1 以下の傾斜加群全体の間に一対一対応が存在する. それは $N \mapsto \mathrm{Hom}_\Lambda(M, N)$ により与えられる.

$M = \Lambda$ とする事により, 次が分かる.

2.9 系 対称整環 Λ が $\mathrm{gl.dim}\, \Lambda = 3$ を満たせば, Λ の非可換クレパント解消と Λ の射影次元 1 以下の傾斜加群とは一致する.

また, 傾斜加群の一般論等より次が従う.

2.10 系 G を $\mathrm{SL}_3(k)$ の有限部分群とし, $k^3 \backslash \{0\}$ に自由に作用するとする. G の共役類の個数を g とする.

(1) S^G の任意の非可換クレパント解消の非同型な直既約直和因子の個数は g に等しい.

(2) 任意の reflexive S^G-加群で rigid (i.e. $\mathrm{Ext}^1_{S^G}(M, M) = 0$) であるものは, ある非可換クレパント解消の直和因子である. 特に M の非同型な直既約直和因子の個数は g 以下.

特に, S^G の非可換クレパント解消の考察は, rigid reflexive S^G-加群の考察と同等である.

3. mutation

Gorodentsev-Rudakov [GR] による $\mathbb{P}^2$ 上の例外ベクトル束の分類では, 鏡映関手に類似した *mutation* と呼ばれる圏論的手法が用いられている. それは Bondal-Kapranov [BK] による Serre 双対を持つ三角圏への応用を経て, Seidel-Thomas [ST] により特別な Fourier-向井変換である *twist* **関手**の構成へと繋がった. mutation は braid 群の生成系の圏論的実現であり, また興味深い事に Auslander-Reiten による古典的近似理論の一端と捉える事も可能である. 一方でごく最近 Fomin-Zelevinsky [FZ1,2] により, root 系と深く関係した cluster algebra が導入されたが, そこには mutation (3.5) と呼ばれる組み合わせ的操作が現れる. その圏論的実現が Buan-Marsh-Reineke-Reiten-Todorov [BMRRT], Geiss-Leclerc-Schröer [GLS] 等で与えられている. 本章で扱う mutation もその一種である.

この章では Λ が Gorenstein（左 Λ-加群として $\mathrm{Hom}_R(\Lambda, R) \simeq \Lambda$）であると仮定し, $\mathrm{CM}\,\Lambda$ の極大直交部分圏に対して mutation を定める事を試みる. Λ が Gorenstein である場合には, 1.6 で定めた安定圏 $\underline{\mathrm{CM}}\Lambda$ が, 三角圏の構造を持つ事が容易に分かる. mutation が上手く定まるために, 三角圏 $\underline{\mathrm{CM}}\Lambda$ が次に示す Calabi-Yau 条件を満たす事を要請する.

3.1 Bondal-Kapranov [BK] による, d 次元非特異射影多様体 X における Serre 双対

$$\mathrm{Hom}(F,G) \simeq \mathrm{Hom}(G, \omega_X \overset{\mathbf{L}}{\otimes} F[d])^* \quad (F,G \in D^b(\mathrm{Coh}\,X))$$

を思い出そう. X が Calabi-Yau 多様体の時に右辺は $\mathrm{Hom}(G, F[d])^*$ となるが, これを一般の三角圏に適用して Kontsevich は次の定義 (1) を与えた.

定義 (1) 三角圏 $\mathcal{T}$ が *n-Calabi-Yau* $(n \geq 0)$ であるとは, 関手的同型

$$\mathrm{Hom}_{\mathcal{T}}(F,G) \simeq \mathrm{Hom}_{\mathcal{T}}(G, F[n])^* \quad (F,G \in \mathcal{T})$$

が存在する事.

(2) 特に, 多元環 Γ 上の長さ有限の加群の導来圏 $D^b(\mathrm{flmod}\,\Gamma)$ が n-Calabi-Yau である時, Γ を *n-Calabi-Yau* **多元環**と呼ぶ. この時 $\mathrm{gl.dim}\,\Gamma = n$ が成立する. また n-Calabi-Yau 多元環全体は, 導来同値で閉じている.

3.2 例 G を $\mathrm{SL}_d(k)$ の有限部分群とする.

(1) 安定圏 $\underline{\mathrm{CM}}S^G$ は $(d-1)$-Calabi-Yau である.

(2) $S * G$ は d-Calabi-Yau 多元環である. より一般に d 次元対称整環 Γ で $\mathrm{gl.dim}\,\Gamma = d$ となるものは, d-Calabi-Yau 多元環である.

3.3 定義 以下安定圏 $\underline{\mathrm{CM}}\Lambda$ が n-Calabi-Yau であると仮定する. この時, $\mathrm{CM}\,\Lambda$ の極大 $(n-1)$-直交部分圏 $\mathcal{C}$ と射影的でない $X \in \mathrm{ind}\,\mathcal{C}$ から, 別の極大 $(n-1)$-直交部分圏を以下のようにして構成する. まず 1.8 で述べた n-概分裂完全列

$$0 \to \tau_n X \overset{f_n}{\to} C_{n-1} \overset{f_{n-1}}{\to} C_{n-2} \overset{f_{n-2}}{\to} \cdots \overset{f_2}{\to} C_1 \overset{f_1}{\to} C_0 \overset{f_0}{\to} X \to 0$$

をとる. ここで 1.7 の n-Auslander-Reiten 双対と 3.1 の Serre 双対を比較すると $\tau_n X = X$ が分かる. また $X_i := \mathrm{Im}\, f_i$ $(i \in \mathbb{Z}/n\mathbb{Z})$ は互いに非同型な直既約加群である事が容易に分かる. $\mathrm{CM}\,\Lambda$ の部分圏を

$$\mathrm{ind}\,\mu_X^i(\mathcal{C}) = (\mathrm{ind}\,\mathcal{C} \backslash \{X\}) \cup \{X_i\} \quad (0 \leq i < n)$$

と定める. この操作 μ_X^i の事を *mutation* と呼ぶ.

一方 X が**ループ**を持たないとは, $X \notin \operatorname{add}\bigoplus_{i=1}^{n-1} C_i$ が成立する事とする. 全ての $X \in \operatorname{ind}\mathcal{C}$ がループを持たない時, $\mathcal{C}$ もループを持たないと言う. この時 $\mathcal{C}$ の Auslander-Reiten quiver (1.11) はループを持たず, $n=2$ の時は逆も成立する.

3.4 定理 X がループを持たないならば, 以下が成立する.
(1) $\mu_X^i(\mathcal{C})$ $(i \in \mathbb{Z}/n\mathbb{Z})$ は $\operatorname{CM}\Lambda$ の極大 $(n-1)$-直交部分圏である.
(2) $\operatorname{ind}\mathcal{C}\backslash\{X\}$ を含む $\operatorname{CM}\Lambda$ の極大 $(n-1)$-直交部分圏は, (1) の n 個に限る.
(3) $\mu_X^i(\mathcal{C}) = \mu_{X_{i-1}}^1 \circ \cdots \circ \mu_{X_1}^1 \circ \mu_X^1(\mathcal{C})$.

(3) より $(\mu^1)^n = \mu^n = $id が成立する. 特に $n=2$ の時, μ^1 は鏡映の類似と見なされ, また, $\mathcal{C}$ の Auslander-Reiten quiver $\mathfrak{A}(\mathcal{C})$ から, $\mu_X^1(\mathcal{C})$ の Auslander-Reiten quiver $\mathfrak{A}(\mu_X^1(\mathcal{C}))$ を求める事が可能である. それを次に説明する.

3.5 定義 (1) 整数係数歪対称行列 $A = (a_{ij})_{1\le i,j\le l}$ に対して, 次の様にして quiver を対応させる. 頂点の集合を $\{1,2,\cdots,l\}$ とする. $a_{ij} > 0$ ならば i から j へ a_{ij} 本の矢印を描く. $a_{ij} < 0$ ならば j から i へ $-a_{ij} (= a_{ji})$ 本の矢印を描く.
この対応により整数係数歪対称行列と, ループと長さ 2 のサイクルを持たない quiver が一対一に対応する.
(2) 整数係数の歪対称行列 $A = (a_{ij})_{1\le i,j\le l}$ と $k = 1,2,\cdots,l$ から, 新しい行列 $\mu_k(A) = (b_{ij})_{1\le i,j\le l}$ を次のように定める.

$$b_{ij} := \begin{cases} -a_{ij} & (i=k \text{ または } j=k) \\ a_{ij} + \frac{1}{2}(a_{ik}|a_{kj}| + |a_{ik}|a_{kj}) & (\text{else}) \end{cases}$$

この時, $\mu_k(A)$ も歪対称行列で $\mu_k \circ \mu_k(A) = A$ を満たす事が分かる. この操作 μ_k の事を *Fomin-Zelevinsky mutation* と呼ぶ [FZ1,2].

3.6 定理 $\underline{\operatorname{CM}}\Lambda$ が 2-Calabi-Yau であると仮定する. $\operatorname{CM}\Lambda$ の極大 1-直交部分圏 $\mathcal{C}$ に対し, $\operatorname{ind}\mathcal{C} = \{1,2,\cdots,l\}$ とし $X \in \operatorname{ind}\mathcal{C}$ と自然数 k が対応するとする. もし $\mathcal{C}$ と $\mu_X^1(\mathcal{C})$ がともにループと長さ 2 のサイクルを持たないならば, $\mathfrak{A}(\mu_X^1(\mathcal{C})) = \mu_k(\mathfrak{A}(\mathcal{C}))$ が成立する.

3.7 例 $G = \langle\sigma\rangle \subset \mathrm{SL}_3(k)$ $(\sigma = \operatorname{diag}(\omega,\omega,\omega), \omega^3 = 1)$ とする. この時, $S_i := \{x \in S \mid \sigma(x) = \omega^i X\}$ $(i \in \mathbb{Z}/3\mathbb{Z})$ と置くと, $S^G = S_0$ で, S^G-加群として $S \simeq S_0 \oplus S_1 \oplus S_2$ と分解する. 1.12 より $\operatorname{add} S$ は $\operatorname{CM} S^G$ の極大 1-直交部分圏であるが, その mutation を計算すると下図左のようになる. ただし, 部分圏 $\mathcal{C}$ を $\operatorname{ind}\mathcal{C}$ によって表しており, Ω と Ω^- は S^G-加群としての syzygy と cosyzygy である. mutation の定義より, 常に $\#\operatorname{ind}\mathcal{C} = 3$ と $S_0 \in \operatorname{ind}\mathcal{C}$ が成立する事に注意せよ.
また, 各極大 1-直交部分圏の Auslander-Reiten quiver を描いたものが下図右である. 例えば $\mathfrak{A}(\operatorname{add} S)$ は 1.12 より G の McKay quiver により与えられ, それは歪対称行列 $\begin{pmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{pmatrix}$ に対応する. この行列に μ_1 を施すと $\begin{pmatrix} 0 & -3 & 3 \\ 3 & 0 & -6 \\ -3 & 6 & 0 \end{pmatrix}$ となり, これが $\mathfrak{A}(\operatorname{add}\Omega S_1 \oplus S_2 \oplus S_0)$

を与える. この様にして全ての Auslander-Reiten quiver が計算される.

$$
\begin{array}{c}
\vdots\\
\{\Omega^{-2}S_1,\Omega^{-2}S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega^{-2}S_1}}\ \ \uparrow^{\mu^1_{\Omega^{-}S_1}}\\
\{\Omega^{-}S_1,\Omega^{-2}S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega^{-2}S_2}}\ \ \uparrow^{\mu^1_{\Omega^{-}S_2}}\\
\{\Omega^{-}S_1,\Omega^{-}S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega^{-}S_1}}\ \ \uparrow^{\mu^1_{S_1}}\\
\{S_1,\Omega^{-}S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega^{-}S_2}}\ \ \uparrow^{\mu^1_{S_2}}\\
\{S_1,S_2,S_0\}\\
\downarrow^{\mu^1_{S_1}}\ \ \uparrow^{\mu^1_{\Omega S_1}}\\
\{\Omega S_1,S_2,S_0\}\\
\downarrow^{\mu^1_{S_2}}\ \ \uparrow^{\mu^1_{\Omega S_2}}\\
\{\Omega S_1,\Omega S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega S_1}}\ \ \uparrow^{\mu^1_{\Omega^2 S_1}}\\
\{\Omega^2 S_1,\Omega S_2,S_0\}\\
\downarrow^{\mu^1_{\Omega S_2}}\ \ \uparrow^{\mu^1_{\Omega^2 S_2}}\\
\{\Omega^2 S_1,\Omega^2 S_2,S_0\}\\
\vdots
\end{array}
$$

$$
\begin{array}{ccc}
\Omega^{-2}S_2 & & 3\\
\downarrow^{102} & & \\
S_0 & \xrightarrow{39} & \Omega^{-2}S_1
\end{array}
\qquad
\begin{array}{ccc}
\Omega^{-2}S_2 & & 3\\
\uparrow^{15} & & \\
S_0 & \xleftarrow{39} & \Omega^{-}S_1
\end{array}
$$

$$
\begin{array}{ccc}
\Omega^{-}S_2 & & 3\\
\downarrow^{15} & & \\
S_0 & \xrightarrow{6} & \Omega^{-}S_1
\end{array}
\qquad
\begin{array}{ccc}
\Omega^{-}S_2 & & 3\\
\uparrow^{3} & & \\
S_0 & \xleftarrow{6} & S_1
\end{array}
$$

$$
\begin{array}{ccc}
S_2 & & 3\\
\downarrow^{3} & & \\
S_0 & \xrightarrow{3} & S_1
\end{array}
\qquad
\begin{array}{ccc}
S_2 & & 3\\
\uparrow^{6} & & \\
S_0 & \xleftarrow{3} & \Omega S_1
\end{array}
$$

$$
\begin{array}{ccc}
\Omega S_2 & & 3\\
\downarrow^{6} & & \\
S_0 & \xrightarrow{15} & \Omega S_1
\end{array}
\qquad
\begin{array}{ccc}
\Omega S_2 & & 3\\
\uparrow^{39} & & \\
S_0 & \xleftarrow{15} & \Omega^2 S_1
\end{array}
$$

$$
\begin{array}{ccc}
\Omega^2 S_2 & & 3\\
\downarrow^{39} & & \\
S_0 & \xrightarrow{102} & \Omega^2 S_1
\end{array}
$$

3.8 問題 興味深い問題が 2 つある.

(1) $\mathrm{CM}\,\Lambda$ の全ての極大 $(n-1)$-直交部分圏はループを持たないか?

(2)(transitivity) $\mathrm{CM}\,\Lambda$ の全ての極大 $(n-1)$-直交部分圏は, $\mathrm{add}\,S$ から始めて mutation の繰り返しによって得られるか?

これに関しては, 次の場合のみ分かっている.

3.9 定理 G を 3.7 にあるものならば, $\mathrm{CM}\,S^G$ の極大 1-直交部分圏は, 3.7 で挙げたものが全てである. 特に, 3.8(1)(2) はともに正しい.

4. d-Calabi-Yau 多元環上の傾斜複体

G を $\mathrm{SL}_d(k)$ の有限部分群とする時, 包含関係

$$
\begin{array}{rl}
& \{\mathrm{CM}\,S^G \text{ の極大 } (d-2)\text{-直交部分圏}\} \overset{2.5}{\subseteq} \{S^G \text{ の非可換クレパント解消}\}\\
\overset{2.8}{\simeq} & \{S*G \text{ の射影次元 1 以下の傾斜加群}\} \subseteq \{S*G \text{ の傾斜複体}\}
\end{array}
$$

より, $S*G$ 上の傾斜複体を決定する事が一つの目標であるが, この章では現段階で分かっている事を簡潔に述べる.

4.1 T を (Γ, Γ')-加群の両側傾斜複体, T' を (Γ', Γ'')-加群の両側傾斜複体とする．この時, $T \overset{\mathbf{L}}{\otimes}_{\Gamma'} T'$ は (Γ, Γ'')-加群の両側傾斜複体であり, また $T^{-1} := \mathbf{R}\mathrm{Hom}_R(T, R)$ は (Γ', Γ)-加群の両側傾斜複体で, $T \overset{\mathbf{L}}{\otimes}_{\Gamma'} T^{-1} \simeq \Gamma$ と $T^{-1} \overset{\mathbf{L}}{\otimes}_{\Gamma} T \simeq \Gamma'$ を満たす.

これより, Γ と導来同値な多元環の森田同値類を対象とし, $\mathrm{Hom}(\Gamma', \Gamma'')$ は (Γ', Γ'')-加群の両側傾斜複体の同型類, 射の合成は $\overset{\mathbf{L}}{\otimes}$ とする圏を考える事ができる．この圏では全ての射は可逆であり, 特に群 $\mathrm{End}(\Gamma)$ は Γ の**導来 Picard 群**と呼ばれる [Ye]. 導来 Picard 群は, 導来圏 $D^b(\mathrm{mod}\,\Gamma)$ の自己同型群の部分群を成し, 近年盛んに研究されている [LM][MY][RZ].

$\Gamma = S * G$ の場合にこの圏の構造を決める事が一つの目標となるが, 以下, 圏の生成元の候補となる傾斜複体の構成に付いて述べる．鍵となる事実は, $S * G$ が d-Calabi-Yau 多元環である事である.

4.2 定義 Γ を d-Calabi-Yau 多元環 (3.1) とし, 森田同値なものに取り替える事により basic, 即ち Γ の Jacobson 根基による商が斜体の直和であると仮定する.

この時, 左 Γ-加群 P で直既約射影的であるものを任意に選ぶと, P は直既約なので唯一つの極大部分加群 P' を持つ．$\Gamma \simeq P \oplus Q$ と直和分解して, 左 Γ-加群 $\mu_P^i(\Gamma)$ を

$$\mu_P^i(\Gamma) := \Omega^{i+1}(P/P') \oplus Q \quad (0 \leq i < d)$$

と定める．この操作 μ_P^i を, 3.3 と同様に *mutation* と呼ぶ.

一方, $\mathrm{gl.dim}\,\Gamma = d$ なので P/P' は極小射影分解

$$0 \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to P/P' \to 0$$

を持つが, Γ が d-Calabi-Yau である事より, $P_d \simeq P_0 = P$ である事が容易に分かる．P が**ループ**を持たないとは, $P \notin \mathrm{add} \bigoplus_{i=1}^{d-1} P_i$ が成立する事とする．全ての直既約射影加群がループを持たない時, Γ もループを持たないという.

4.3 命題 P がループを持たなければ, $\mu_P^i(\Gamma)$ $(0 \leq i < d)$ は射影次元 $d - i - 1$ の傾斜 Γ-加群である．特に $\mathrm{End}_\Gamma(\mu_P^i(\Gamma))$ も d-Calabi-Yau 多元環である.

4.4 注意 この mutation は, 以下の様にして 3 章で定めたものの一般化とみなされる.

Λ を d 次元対称整環とする．$\mathcal{C}$ を $\mathrm{CM}\,\Lambda$ の極大直交 $(d-2)$-部分圏とし, M を $\mathcal{C}$ の加法生成元とする．この時, $\Gamma := \mathrm{End}_\Lambda(M)$ は d-Calabi-Yau 多元環である事がわかる．任意の非射影的な $X \in \mathrm{ind}\,\mathcal{C}$ に対し, $P := \mathrm{Hom}_\Lambda(M, X)$ は直既約射影 Γ-加群であり, 任意の i $(0 < i < d)$ に対して, $\mu_X^i(\mathcal{C})$ のある加法生成元 N は $\mu_P^i(\Gamma) = \mathrm{End}_\Lambda(N)$ を満たす.

また, Γ（正確には $\mathrm{add}\,\Gamma$）の Auslander-Reiten quiver も 1.11 と全く同様に定義されるが, $d = 3$ の場合, Γ と $\mu_P^1(\Gamma)$ の Auslander-Reiten quiver の変化に関して, 3.6 と同様の事柄が成立する.

4.5 $i = 0$ とした $\mu_P^0(\Gamma)$ は Γ の極大両側イデアルであり, P を変える事により全ての Γ の極大両側イデアルが現れる．これらに関して以下の興味深い事実が成立するが, これは McKay 対応 [KV] を介した, Seidel-Thomas の twist 関手 [ST] の持つ性質の言い換えでもある.

命題 G を $\mathrm{SL}_2(k)$ の有限部分群とし, $\Gamma := S * G$ の極大両側イデアルを $\mathfrak{m}_0, \cdots, \mathfrak{m}_n$ とすると, これらは G の既約表現と一対一に対応し, 以下の braid 関係式が成立する.

(i) $\mathfrak{m}_i$ と $\mathfrak{m}_j$ が G の McKay quiver で隣接していない場合, $\mathfrak{m}_i \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_j \simeq \mathfrak{m}_j \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_i$.

(ii) 隣接している場合, $G \not\simeq \mathbb{Z}/2\mathbb{Z}$ ならば, $\mathfrak{m}_i \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_j \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_i \simeq \mathfrak{m}_j \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_i \overset{\mathbf{L}}{\otimes}_\Gamma \mathfrak{m}_j$.

石井-上原 [IU] において, A_n 型の 2 次元単純特異点の最小解消の導来圏の, Fourier-向井変換の成す自己同型群の生成系が決定されている. McKay 対応を介する事により, A_n 型の $S*G$ の導来 Picard 群の生成系として, $\mathfrak{m}_0, \cdots, \mathfrak{m}_n$ に $\mathrm{Aut}(S*G)$ や shift を施した全体が取れると思われるが, これを直接加群論的に示す事は興味深い.

4.6 一方, 射影次元 1 以下の傾斜加群をのみを考えると, 次のような結果を得る. ここで Γ-加群 M と N が加法同値であるとは, $\mathrm{add}\, M = \mathrm{add}\, N$ となる事とする.

定理 G を $\mathrm{SL}_2(k)$ の有限部分群とすると, 射影次元 1 以下の傾斜 $S*G$-加群の加法同値類と, G に対応するアフィン Weyl 群の間に一対一対応が存在する.

特に射影次元 1 以下の傾斜 $S*G$-加群は, 加法同値を除いて, 極大両側イデアルを適当に掛け合わせる事により得られる事が分かる. ここで「掛け合わせる」とはイデアルとしての積を取る事であり, 無駄の無い掛け合わせ方の場合は $\otimes_{S*G}$ や $\overset{\mathbf{L}}{\otimes}_{S*G}$ とも一致する.

4.7 以下 $d = 3$ の場合を考察する. 3-Calabi-Yau 多元環 $\Gamma_0 := \Gamma$ に対し, 以下のように 4.3 を繰り返す事により, 射影次元 1 の傾斜 Γ-加群を構成する事ができる.
Γ_n まで構成された時, 左 Γ_n-加群 Γ_n の直既約直和因子 P_n を取り,

$$T_n := \mu^1_{P_n}(\Gamma_n), \quad \Gamma_{n+1} := \mathrm{End}_{\Gamma_n}(T_n)$$

と置く. これを繰り返して 3-Calabi-Yau 多元環 $\Gamma_0, \Gamma_1, \Gamma_2, \cdots$ と傾斜 (Γ_n, Γ_{n+1})-加群 T_n を得る. この時,

$$(T_0 \otimes_{\Gamma_1} T_1 \otimes_{\Gamma_2} \cdots \otimes_{\Gamma_n} T_n)^{**}$$

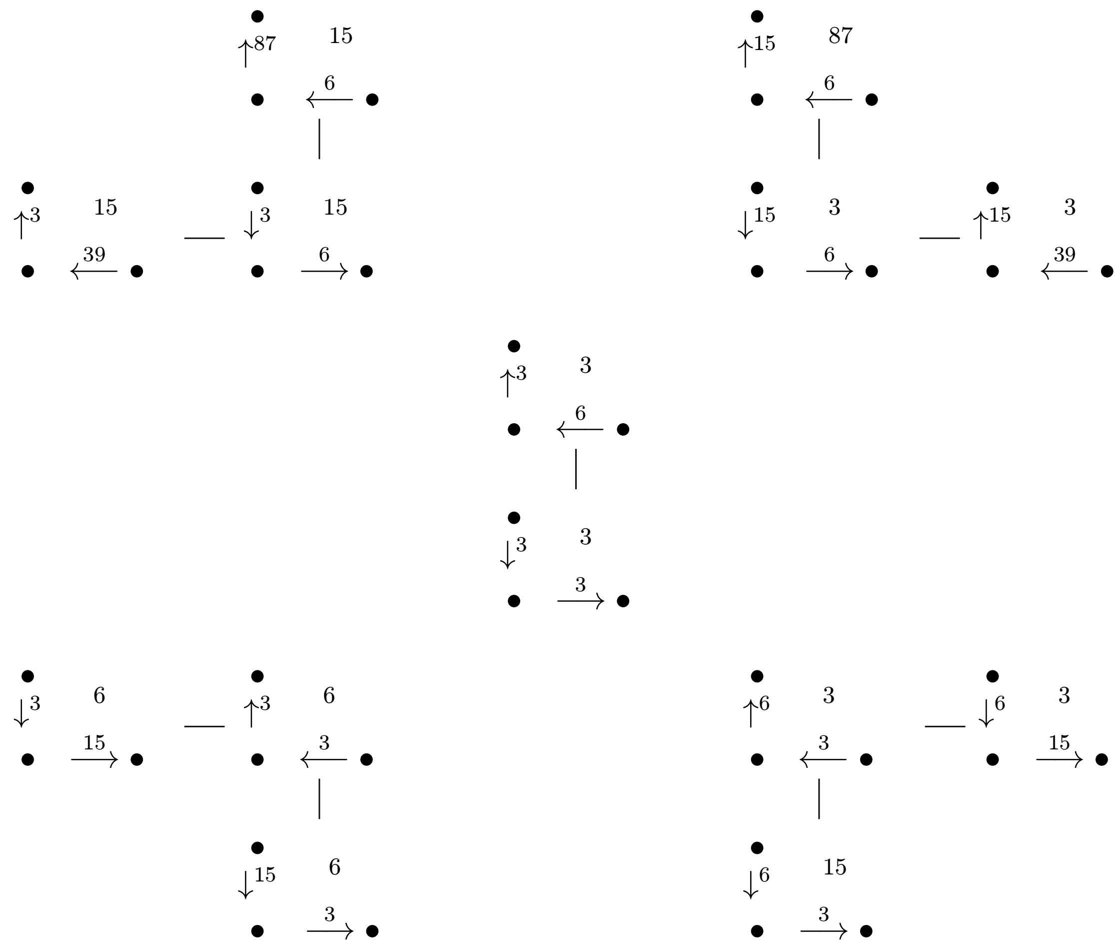
は射影次元 1 以下の傾斜 (Γ, Γ_{n+1})-加群である事が示される. また

$$T_0 \overset{\mathbf{L}}{\otimes}_{\Gamma_1} T_1 \overset{\mathbf{L}}{\otimes}_{\Gamma_2} \cdots \overset{\mathbf{L}}{\otimes}_{\Gamma_n} T_n$$

は (Γ, Γ_{n+1})-加群の両側傾斜複体である.
これらで射影次元 1 以下の傾斜 Γ-加群及び Γ の傾斜複体が, 加法同値を除いて全て尽くされるか否かは, 興味深い問題と思われる.

4.8 例 $G = \langle \sigma \rangle \subset \mathrm{SL}_3(k)$ ($\sigma = \mathrm{diag}(\omega, \omega, \omega), \omega^3 = 1$) とする. この時, $\Gamma := S*G$ に対して 4.7 を適用して得られる 3-Calabi-Yau 多元環の Auslander-Reiten quiver を, 4.4 により描いたものの一部が次図である. これは 3 分木 (Markov tree) であり, 2 つの quiver を結ぶ辺には 1 つの傾斜加群が対応している. 矢印の本数に着目する事により, Markov 等式

$x^3+y^3+z^3=xyz$ の全ての自然数解が現れている．3.7 の図は, 下図の中の一系列である．



この様な構造は [GR][B2] にも現れているもので, より一般の G に対してどの様な構造が現れるのかは非常に興味深い．Bridgeland [B3] により定義された三角圏の安定性条件の空間の構造と関係していると思われる．

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BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE

SHIGEO KOSHITANI

First of all, the result which will be presented here right now is actually a joint work with **Naoko Kunugi** and **Katsushi Waki** [9], which should show up as an ordinary paper in near future, hopefully. In representation theory of finite groups, especially in modular representation theory of finite groups there are quite a few problems and conjectures which are pretty much interesting and important. I believe that most of them have origins which are (were) due to Richard Brauer (1901–77) who should have been a unique pioneer of modular representation theory of finite groups. However, I guess most people might agree that the following conjectures should be the ones if we have to choose three of them from the problems and conjectures. Namely, *Alperin's weight conjecture* (1986), *Dade's conjecture* (1990) and *Broué's abelian defect group conjecture* (Broué's ADGC) (1988). In this short note we shall focus on Broué's ADGC, particularly.

Michel Broué around late eighties announced the following conjecture, which is nowadays well-known and is called Broué's ADGC (abelian defect group conjecture). That is,

Broué's abelian defect group conjecture (ADGC) (see [2], [3], [4], [5])**.** Let p be a prime and let G be a finite group. Let a triple $(\mathcal{O}, \mathcal{K}, k)$ be a p-modular system, namely, $\mathcal{O}$ is a complete discrete valuation ring of rank one, $\mathcal{K}$ is the quotient field of $\mathcal{O}$ with characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ with characteristic p. We assume moreover that the p-modular system $(\mathcal{O}, \mathcal{K}, k)$ is big enough for all subgroups of G, namely, $\mathcal{K}$ and k are both splitting fields for all subgroups of G. Now, let A be a block algebra of the group algebra $\mathcal{O}G$ with defect group P, and set $H = N_G(P)$, the normalizer of P in G. Let B be a block algebra of $\mathcal{O}H$ which is the Brauer correspondent of A, and hence B has the same defect group P. If the defect group P of A is abelian (commutative), then the algebras A and B should be derived (Rickard) equivalent, namely,

$$\mathrm{D}^b(\text{mod-}A) \simeq \mathrm{D}^b(\text{mod-}B) \quad \text{(equivalent)}$$

as triangulated categories, where mod-A is the category of finitely generated right A-modules, and $\mathrm{D}^b(\mathfrak{A})$ is the bounded derived category of an abelian category $\mathfrak{A}$.

The detailed version of this paper will be submitted for publication elsewhere.

As well-known there is a wonderful and beautiful result due to Jeremy Rickard ([12], [13]), which characterizes such a derived equivalence completely and that is a generalization of a Morita equivalence from modules to complexes of modules. In fact, in the conjecture above, a stronger conclusion is expected. That is, *derived (Rickard) equivalent* could be replaced by *splendidly (Rickard) equivalent*, which is due to Jeremy Rickard [13]. As far as we know, so far there have never existed any counter-example even to the stronger version of Broué's ADGC (or we might want to call it Rickard's version of Broué's ADGC).

There are several results, where Broué's ADGC has been checkd in particular cases. We do not want to mention them completely in detail, however, we want to say a few words on a specific case. Namely, for the case where our abelian defect group P is just $C_3 \times C_3$, the elementary abelian group of order 9. The auther with Naoko Kunugi finally have proved that Broué's ADGC is true for the case where our block algebra A is the principal block algebra and our defect group of A is $P = C_3 \times C_3$ (and it turns out that P is a Sylow 3-subgroup of G), see [6]. However, we should confess that we used a lot of iniciated wonderful and important works done by L. Puig, T. Okuyama, H. Miyachi, ... and also the classification of finite simple groups (which we do not like, to be honest, as a matter of fact, but we had no the other choice, life is tough ...). And then, we were successful to check that Broué's ADGC is true also for non-principal block algebras A with the same defect group $P = C_3 \times C_3$ for specific sporadic simple groups G, see [7] and [8].

Anyhow, our work presented here is a sort of continuation of this project. Our main result is the following:

Theorem (Koshitani-Kunugi-Waki, 2005 [9])**.** *Keep the notation as in the conjecture Broué's ADGC above. Let $p = 3$, and let G be the Janko simple group J_4. Let A be a unique non-principal block algebra of $\mathcal{O}G$ with defect group $P = C_3 \times C_3$. Then, there exists a splendid Rickard equivalence between A and its Brauer correspondent block algebra B in $\mathcal{O}N_G(P)$. This means that Broué's ADGC holds for J_4 at least for the particilar block A, and it turns out that Broué's ADGC holds for all primes p and for all p-block algebras of J_4.*

Remark. To prove our main result, results of Okuyama in [10] and [11] play imporant rôles. In order to know that Broué's ADGC holds for all primes p and for all p-blocks of J_4 after we check it for our particular one single non-principal 3-block of J_4, we need a lemma which is stated in [1].

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INTEGRAL GROUP ALGEBRAS AND CYCLOTOMIC POLYNOMIALS [1]

KAORU MOTOSE

Recently, using cyclotomic polynomials, Z. Marciniak and S. K. Sehgal [3] obtained excellent results about units in integral group rings of cyclic groups. In this paper, we shall give some improvements and alternative proofs of their results.

Let $\mathbb{Z}G$ be the group algebra of a finite abelian group G over the ring $\mathbb{Z}$ of rational integers. It is well known that the units of *finite* order in $\mathbb{Z}G$ have the form $\pm g$ for some $g \in G$ (see [1], p. 262). We study the form of units of *infinite* order in $\mathbb{Z}G$ where $G = \langle\sigma\rangle$.

Let $\Phi_m(x)$ be cyclotomic polynomial of order m defined inductively by

$$X^m - 1 = \prod_{d|m} \Phi_d(x).$$

Z. Marciniak and S.K. Sehgal [3] construct many units of infinite order using cyclotomic polynomials. These units cover the alternative units, the Hoechsmann units [3] and Yamauchi's results [4].

In this paper, we study the Euclidean algorithm for cyclotomic polynomials in $\mathbb{Z}[x]$, and we have easy applications to some their results in [3]. The following are well known units. Units in 1, 2 are covered by cyclotomic polynomials.

1. The alternating units:

$$\Phi_{2k}(\sigma) = 1 - \sigma + \sigma^2 - \cdots + (-1)^k\sigma^k$$

   where k is odd and $(2k, |G|) = 1$.

2. The Hoechsmann units (the constructible units) (see also K. Yamauchi [4]).

$$\frac{\sigma^{k\ell} - 1}{\sigma^k - 1} \cdot \frac{\sigma - 1}{\sigma^\ell - 1} = \frac{1 + \sigma^k + \sigma^{2k} + \cdots + \sigma^{(\ell-1)k}}{1 + \sigma + \sigma^2 + \cdots + \sigma^{\ell-1}}$$

   where $k,\ \ell \geq 2,\ (k\ell, |G|) = 1$ and $(k, \ell) = 1$.

3. Bass cyclic units,

$$(1 + \sigma + \cdots + \sigma^{k-1})^m - \ell(1 + \sigma + \cdots + \sigma^{|G|-1})$$

   where $k > 1$ and $k^m = 1 + \ell|G|$.

Since the group algebra $\mathbb{Z}G$ are isomorphic to $\mathbb{Z}[x]/(x^n - 1)\mathbb{Z}[x]$, our study on units in $\mathbb{Z}G$ is equivalent to find polynomials $f(x) \in \mathbb{Z}[x]$ satisfying

$$f(x)u(x) + (x^n - 1)v(x) = 1, \text{ where } u(x), v(x) \in \mathbb{Z}[x].$$

For relatively prime polynomials $f(x)$ and $g(x)$ over a field K, it is easy to compute polynomials $u(x), v(x) \in K[x]$ by Euclidean algorithm such that

$$f(x)u(x) + g(x)v(x) = 1.$$

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However, over $\mathbb{Z}[x]$, situation is different from this. Of course we can compute $u(x), v(x) \in \mathbb{Q}[x]$ by Euclidean algorithm for relatively prime polynomials $f(x), g(x) \in \mathbb{Z}[x]$. Thus we have

$$f(x)u_0(x) + g(x)v_0(x) = a$$

where $u_0(x), v_0(x) \in \mathbb{Z}[x]$ and $0 \neq a \in \mathbb{Z}$.

For example, we obtain for cyclotomic polynomials $\Phi_3(x) = x^2 + x + 1, \Phi_6(x) = x^2 - x + 1$,

$$\Phi_3(x)(1-x) + \Phi_6(x)(x+1) = 1 - x^3 + 1 + x^3 = 2$$

and we can easily show there is no polynomials $u(x), v(x) \in \mathbb{Z}[x]$ such that

$$\Phi_3(x)u(x) + \Phi_6(x)v(x) = 1.$$

In fact $1 = \Phi_6(\omega)v(\omega) = -2\omega v(\omega) = -2\bar{\omega}v(\bar{\omega})$ for two roots $\omega, \bar{\omega}$ of $\Phi_3(x)$. We have a contradiction such that $1 = 4 \cdot v(\omega)v(\bar{\omega})$ and $v(\omega)v(\bar{\omega})$ is an integer.

Thus it is natural to consider the next problem.

For given polynomials $f(x),\ g(x) \in \mathbb{Z}[x]$, does there exist polynomials $u(x),\ v(x) \in \mathbb{Z}[x]$ such that

$$f(x)u(x) + g(x)v(x) = 1 \ ?$$

It is easy for $f(x) = x$ and $g(x) = x^n - 1$. But in general, it seems to be difficult for me because the ring $\mathbb{Z}[x]$ is not Euclidean though it is a unique factorization ring. In this paper, we shall answer to this problem in case $f(x)$ and $g(x)$ are cyclotomic polynomials for units in $\mathbb{Z}G$.

If $m \neq n$, then we have $\Phi_m(x)u(x) + \Phi_n(x)v(x) = 1$ in $\mathbb{Q}[x]$ since $\Phi_m(x), \Phi_n(x)$ are distinct irreducible polynomials in $\mathbb{Q}[x]$. Over $\mathbb{Z}[x]$, we can see the next theorem.

Theorem 1. *Assume $n > m \geq 1$. Then we have*

(1) *If m is not a devisor of n, then there exist $u(x), v(x) \in \mathbb{Z}[x]$ such that*

$$\Phi_m(x)u(x) + \Phi_n(x)v(x) = 1.$$

(2) *If m is a divisor of n, then we set $n = mk$ and k_0 is the product of all distinct prime divisors k. There exist $u(x), v(x) \in \mathbb{Z}[x]$ such that*

$$\Phi_m(x)u(x) + \Phi_n(x)v(x) = \Phi_{k_0}(1).$$

Proof. (1) If we set $n = mq + r,\ 0 \leq r < m$, then we have easily

$$x^n - 1 = (x^m - 1) \cdot (\frac{x^{mq} - 1}{x^m - 1} \cdot x^r) + x^r - 1.$$

Hence, we can use Euclidean algorithm in $\mathbb{Z}[x]$ and so

$$(x^n - 1)s(x) + (x^m - 1)t(x) = x^d - 1, \quad \text{for some } s(x), t(x) \in \mathbb{Z}[x]$$

where $d = (n, m)$. Thus we have

$$\frac{x^n - 1}{x^d - 1}s(x) + \frac{x^m - 1}{x^d - 1}t(x) = 1.$$

Therefore, we obtain the next equation excluding cases $m|n$

$$\Phi_n(x)u(x) + \Phi_m(x)v(x) = 1 \ \text{ for some } u(x), v(x) \in \mathbb{Z}[x].$$

(2) Since $x-1$ divides $\Phi_{k_0}(x)-\Phi_{k_0}(1)$ in $\mathbb{Z}[x]$, we have $x^{hm}-1$ and so $\Phi_m(x)$ divides $\Phi_{k_0}(x^{hm})-\Phi_{k_0}(1)$ where $h=\frac{k}{k_0}$. Let n_0 be the product of all distinct prime divisors n. We set $n_0=\ell k_0$ and

$$u(x)=\frac{\Phi_{k_0}(1)-\Phi_{k_0}(x^{hm})}{\Phi_m(x)} \text{ and } v(x)=\prod_{d|\ell,\ d<\ell}\Phi_{k_0 d}(x^{\frac{n}{n_0}}).$$

Then $u(x)$ and $v(x)\in\mathbb{Z}[x]$. Noting $\frac{n}{n_0}\ell=\frac{k}{k_0}m=hm$ and $(\ell,k_0)=1$, we have

$$\begin{aligned}\Phi_m(x)u(x)+\Phi_n(x)v(x) &= \Phi_m(x)u(x)+\Phi_{n_0}(x^{\frac{n}{n_0}})\prod_{d|\ell,\ d<\ell}\Phi_{k_0 d}(x^{\frac{n}{n_0}})\\ &= \Phi_{k_0}(1)-\Phi_{k_0}(x^{hm})+\Phi_{k_0}((x^{\frac{n}{n_0}})^{\ell})\\ &= \Phi_{k_0}(1).\end{aligned}$$

Let m be a natural number and let q be a power of a prime with $(q,m)=1$. Then we can see from Theorem 1 (2) that there exist $u(x),v(x)\in\mathbb{Z}[x]$ such that

$$\Phi_m(x)u(x)+\Phi_{mq}(x)v(x)=p.$$

However, the next proposition shows that p is the smallest positive integer satisfying the above equation.

Proposition 1. *There exist no $s(x),t(x)\in\mathbb{Z}[x]$ such that*

$$\Phi_m(x)s(x)+\Phi_{mq}(x)t(x)=1$$

for a natural number m and a power q of a prime p with $(q,m)=1$.

Proof. Let Δ be the set of roots of $\Phi_m(x)$. Using $\prod_{d|m}\Phi_{dq}(x)=\Phi_q(x^m)$, we have the next

$$\prod_{d|m}\Phi_{dq}(\eta)=\Phi_q(\eta^m)=\Phi_q(1)=p$$

where $\eta\in\Delta$. Thus

$$p^{|\Delta|}=\prod_{\eta\in\Delta}\prod_{d|m}\Phi_{dq}(\eta)=\prod_{d|m}\prod_{\eta\in\Delta}\Phi_{dq}(\eta).$$

We set $a_d=\prod_{\eta\in\Delta}\Phi_{dq}(\eta)$. Then a_d is an integer because a_d is a symmetric polynomial in $\mathbb{Z}[\eta\in\Delta]$ and so $a_d\in\mathbb{Z}[\text{coefficients of }\Phi_m(x)]$. Hence we have from the above equation.

$$p^{|\Delta|}=\prod_{d|m}|a_d| \text{ and } |a_d|=p^{\alpha(d)}$$

where $\alpha(d)$ is a nonnegative integer. Therefore we have

$$\varphi(m)=|\Delta|=\sum_{d|m}\alpha(d).$$

Using Möbius inversion formula, we obtain

$$\alpha(m)=\sum_{d|m}\varphi(d)\mu(\frac{m}{d}).$$

For a prime r,

$$\alpha(r^e) = \varphi(r^e) - \varphi(r^{e-1}) = \begin{cases} r^{e-2}(r-1)^2 & \text{for } e \geq 2, \\ r-2 & \text{for } e = 1. \end{cases}$$

Since $\varphi(i)$ is multiplicative, $\alpha(i)$ is also multiplicative. Thus if $\alpha(i) = 0$, then $i = 2j$ and j is odd.

On the other hand, it follows from the assumption that $\Phi_{mq}(\eta)t(\eta) = 1$ for $\eta \in \Delta$ and so $a_m = \prod_{\eta\in\Delta} \Phi_{mq}(\eta) = \pm 1$. Thus $|a_m| = 1$, and so $\alpha(m) = 0$. This implies $m = 2\ell$, ℓ is odd, and $q > 2$. Hence we have a contradiction for $\ell \geq 3$ by above arguments

$$1 = \Phi_{2\ell}(-x)s(-x) + \Phi_{2\ell q}(-x)t(-x) = \Phi_\ell(x)s(-x) + \Phi_{\ell q}(x)t(-x).$$

We have also a contradiction for $\ell = 1$ by $\Phi_2(-1) = 0$

$$1 = \Phi_2(-1)s(-1) + \Phi_{2q}(-1)t(-1) = pt(-1).$$

Remark 1. It follows from $\Phi_m(x^{p^s}) = \Phi_{mp^s}(x)\Phi_m(x^{p^{s-1}})$ for $(p, m) = 1$ that

$$\Phi_{mp^s}(x) \equiv \Phi_m(x)^{p^{s-1}(p-1)} \text{ or } \Phi_m(x)^{p^s} \bmod p.$$

We can see from Theorem 1 and the above that the ideal of $\mathbb{Z}[x]$ generated by $\Phi_m(x), \Phi_n(x)$ $(m < n)$ can be calculated as follows:

$$(\Phi_m(x), \Phi_n(x)) = \begin{cases} (p, \Phi_m(x)) & \text{if } m|n \text{ and } \frac{n}{m} \text{ is a power of a prime } p, \\ \mathbb{Z}[x] & \text{otherwise.} \end{cases}$$

The first part is an alternative proof of Proposition 1.

In the remainder of this paper, we consider our problem about $x^n - 1$ and $\Phi_m(x)$.

Theorem 2. *Let m_0 be the product of distinct prime divisors of m. If m_0 is not a divisor of n, then there exist $u(x), v(x) \in \mathbb{Z}[x]$ such that*

$$(x^n - 1)u(x) + \Phi_m(x)v(x) = \prod_{d|(m_0,n)} \Phi_{\frac{m_0}{d}}(1).$$

Proof. We may assume that $m = m_0$ from

$$\Phi_m(x) = \Phi_{m_0}(x^{\frac{m}{m_0}}) \text{ and } (x^{\frac{m}{m_0}})^n - 1 = (x^n - 1) \cdot \frac{(x^n)^{\frac{m}{m_0}} - 1}{x^n - 1}.$$

We assume d is a divisor of n. If d is not a divisor of m, there exist $u_d(x), v_d(x) \in \mathbb{Z}[x]$ from Theorem 1 (1) such that

$$\Phi_d(x)u_d(x) + \Phi_m(x)v_d(x) = 1.$$

If d is a divisor of m, there exist $u_d(x), v_d(x) \in \mathbb{Z}[x]$ from Theorem 1 (2) such that

$$\Phi_d(x)u_d(x) + \Phi_m(x)v_d(x) = \Phi_{\frac{m}{d}}(1).$$

Thus we have from $x^n - 1 = \prod_{d|n} \Phi_d(x)$,

$$(x^n - 1)u(x) + \Phi_m(x)v(x) = \prod_{d|(m,n)} \Phi_{\frac{m}{d}}(1).$$

Theorem 3 (Marciniak and Sehgal [3]). *Let* m_0 *be the product of distinct prime divisors of* m. *If* $t = \frac{m_0}{(n,m_0)} > 1$ *is not a prime, there exist integral polynomials* $u(x),\ v(x) \in \mathbb{Z}[x]$ *such that*

$$\Phi_m(x)u(x) + (x^n - 1)v(x) = 1.$$

Proof. We may assume $m = m_0$ from the same reason in Theorem 2. If t is not a prime, we have $\Phi_{\frac{m}{d}}(1) = 1$ for all $d|(m,n)$ because $\frac{m}{d} = \frac{m}{(m,d)}$ is not a prime since $t = \frac{m}{(m,n)}$ is a divisor of $\frac{m}{(m,d)} = \frac{m}{d}$.

Remark 2. If t is a prime p, then we have

$$\Phi_m(x)u(x) + (x^n - 1)v(x) = \Phi_t(1) = p.$$

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HOCHSCHILD COHOMOLOGY OF STRATIFIED ALGEBRAS

HIROSHI NAGASE

1. Introduction

When studying Hochschild cohomology it is natural to try relating cohomology of an algebra B to that of an 'easier' or 'smaller' algebra A. One such situation is that of B being a one-point extension of A, which has been studied by Happel[5]. More recently, Happel's long exact sequence has been generalized to triangular matrix algebras, for example by Michelena and Platzeck [6], C.Cibils, E.Marcos, M.J.Redondo and A.Solotar [2] and E.L.Green and O.Solberg [4].

We would like to suggest to try further generalizing these results. Natural generalizations of directed or triangular algebras are stratified algebras (when just keeping good homological connections between B and its quotient A).

2. Preliminaries

Let k be a field. Throughout this paper, all algebras are finite dimensional k-algebras and all modules are left modules unless otherwise stated. For any algebra A, we denote by A^e the enveloping algebra $A \otimes_k A^{\mathrm{op}}$. We prepare the following lemma for the next section.

Lemma 1. *Let X be a A-B-bimodule, Y a B-C-bimodule and Z a A-C-bimodule. We have the following isomorphisms:*

(1) *If $\mathrm{Tor}_i^B(X,Y)=0$ and $\mathrm{Ext}_C^i(Y,Z)=0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{A-B}(X, \mathrm{Hom}_C(Y,Z)).$$

(2) *If $\mathrm{Tor}_i^B(X,Y)=0$ and $\mathrm{Ext}_A^i(X,Z)=0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{B-C}(Y, Hom_A(X,Z)).$$

Proof. See Cartan-Eilenberg's book [1]. □

Lemma 2. *Let I be an ideal of an algebra B. If $\mathrm{Ext}^i_{B^e}(I, B/I)=0$ for all $i \geq 0$, then we have the following two long exact sequences:*

(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B,I) \to \mathrm{Ext}^n_{B^e}(B,B) \to \mathrm{Ext}^n_{B^e}(B/I,B/I) \to \mathrm{Ext}^{n+1}_{B^e}(B,I) \to \cdots$;

(2) $\cdots \to \mathrm{Ext}^n_{B^e}(B/I,I) \to \mathrm{Ext}^n_{B^e}(B,B) \to \mathrm{Ext}^n_{B^e}(B/I,B/I) \oplus \mathrm{Ext}^n_{B^e}(I,I)$
$\to \mathrm{Ext}^{n+1}_{B^e}(B/I,I) \to \cdots$.

The detailed version of this paper will be submitted for publication elsewhere.

Proof. We denote B/I by A, the inclusion $I \to B$ by f and the surjection $B \to A$ by g. We consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
\operatorname{Ext}^{i-2}_{B^e}(I,A) & \longrightarrow & \operatorname{Ext}^{i-1}_{B^e}(I,I) & \xrightarrow{f^I_{i-1}} & \operatorname{Ext}^{i-1}_{B^e}(I,B) & \longrightarrow & \operatorname{Ext}^{i-1}_{B^e}(I,A) \\
\downarrow & & \downarrow{\scriptstyle \delta_I^{i-1}} & & \downarrow & & \downarrow \\
\operatorname{Ext}^{i-1}_{B^e}(A,A) & \xrightarrow{\delta^A_{i-1}} & \operatorname{Ext}^{i}_{B^e}(A,I) & \longrightarrow & \operatorname{Ext}^{i}_{B^e}(A,B) & \longrightarrow & \operatorname{Ext}^{i}_{B^e}(A,A) \\
\downarrow{\scriptstyle g_A^{i-1}} & & \downarrow{\scriptstyle g_I^{i}} & & \downarrow & & \downarrow{\scriptstyle g_A^{i}} \\
\operatorname{Ext}^{i-1}_{B^e}(B,A) & \xrightarrow{\delta^B_{i-1}} & \operatorname{Ext}^{i}_{B^e}(B,I) & \xrightarrow{f^B_{i}} & \operatorname{Ext}^{i}_{B^e}(B,B) & \xrightarrow{g^B_{i}} & \operatorname{Ext}^{i}_{B^e}(B,A) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f_B^{i}} & & \downarrow \\
\operatorname{Ext}^{i-1}_{B^e}(I,A) & \longrightarrow & \operatorname{Ext}^{i}_{B^e}(I,I) & \xrightarrow{f^I_{i}} & \operatorname{Ext}^{i}_{B^e}(I,B) & \longrightarrow & \operatorname{Ext}^{i}_{B^e}(I,A).
\end{array}$$

Since $\operatorname{Ext}^n(I,A) = 0$ for all $n \geq 0$, we have that f^I_n and g^n_A are isomorphic for all $n \geq 0$. It is not diffecult to show that the followig two sequences are exact:

$$\operatorname{Ext}^{i-1}_{B^e}(A,A) \xrightarrow{\delta^B_{i-1} g_A^{i-1}} \operatorname{Ext}^{i}_{B^e}(B,I) \xrightarrow{f^B_i} \operatorname{Ext}^{i}_{B^e}(B,B) \xrightarrow{(g^i_A)^{-1} g^B_i} \operatorname{Ext}^{i}_{B^e}(A,A)$$

and

$$\operatorname{Ext}^{i-1}_{B^e}(A,A) \oplus \operatorname{Ext}^{i-1}_{B^e}(I,I) \xrightarrow{(-\delta^A_{i-1}, \delta_I^{i-1})} \operatorname{Ext}^{i}_{B^e}(A,I) \xrightarrow{f^B_i g^i_I} \operatorname{Ext}^{i}_{B^e}(B,B)$$

$$\xrightarrow{\binom{(g^i_A)^{-1} g^B_i}{(f^I_i)^{-1} f^i_B}} \operatorname{Ext}^{i}_{B^e}(A,A) \oplus \operatorname{Ext}^{i}_{B^e}(I,I) \xrightarrow{(-\delta^A_i, \delta^i_I)} \operatorname{Ext}^{i+1}_{B^e}(A,I).$$

□

3. Stratifying ideals

Definition 3 (Cline, Parshall and Scott [3], 2.1.1 and 2.1.2)**.** Let B be a finite dimensional algebra and $e = e^2$ an idempotent. The two-sided ideal $J = BeB$ generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.
(b) For all $n > 0$: $\operatorname{Tor}^{eBe}_n(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\operatorname{Ext}^*_A(X,Y) \simeq \operatorname{Ext}^*_B(X,Y)$$

for all A-modules X and Y.

Example 4. (1) Any ideal generated by a central idempotent is a stratifying ideal.
(2) If an algebra A has an idempotent e such that $eA(1-e) = 0$, then AeA and $A(1-e)A$ are both stratifying ideals, namely, triangular matrix algebras have stratifying ideals.
(3) Heredity ideals (used to define quasi-hereditary algebras) are examples of stratifying ideals.

Proposition 5. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. For any $i \geq 0$ and finite dimensional A^e-module M, the induced morphism $g^i_M : \mathrm{Ext}^i_{A^e}(A, M) \to \mathrm{Ext}^i_{B^e}(B, M)$ is isomorphic.*

Proof. It is enough to show that the induced morphisms g^i_M is isomorphic for any simple A^e-module M. For any A-modules X and Y, there exists the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X, Y)) & \xrightarrow{\sim} & \mathrm{Ext}^i_A(X, Y) \\ \downarrow{\scriptstyle g^i_{\mathrm{Hom}_k(X,Y)}} & & \downarrow \\ \mathrm{Ext}^i_{B^e}(B, \mathrm{Hom}_k(X, Y)) & \xrightarrow{\sim} & \mathrm{Ext}^i_B(X, Y), \end{array}$$

where these isomorphisms in rows are induced from Lemma1. Any simple A^e-module has the form of $\mathrm{Hom}_k(X, Y)$ for some simple A-modules X and Y. Hence, by using the condition (B) of stratifying ideals, it is shown that g^i_M is isomorphic for any simple A^e-module M. □

Proposition 6. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. For any $i \geq 0$, the following hold:*

(1) $\mathrm{Ext}^i_{B^e}(BeB, A) = 0$;
(2) $\mathrm{Ext}^i_{B^e}(BeB, BeB) \cong \mathrm{Ext}^i_{eBe^e}(eBe, eBe)$;
(3) $\mathrm{Ext}^i_{B^e}(A, A) \cong \mathrm{Ext}^i_{A^e}(A, A)$.

Proof. By Lemma1 and the condition (A) of stratifying ideals, for any B^e-module X, we have that

$$\begin{aligned} \mathrm{Ext}^i_{B^e}(BeB, X) &\cong \mathrm{Ext}^i_{B^e}(Be \otimes_{eBe} eB, X) \\ &\cong \mathrm{Ext}^i_{B-eBe}(Be, \mathrm{Hom}_B(eB, X)) \\ &\cong \mathrm{Ext}^i_{B-eBe}(Be \otimes_{eBe} eBe, Xe) \\ &\cong \mathrm{Ext}^i_{eBe^e}(eBe, \mathrm{Hom}_B(Be, Xe)) \\ &\cong \mathrm{Ext}^i_{eBe^e}(eBe, eXe). \end{aligned}$$

Hence (1) and (2) hold.

By Proposition5, $\mathrm{Ext}^i_{A^e}(A, A) \cong \mathrm{Ext}^i_{B^e}(B, A)$. By (1) above, $\mathrm{Ext}^i_{B^e}(B, A) \cong \mathrm{Ext}^i_{B^e}(A, A)$. Hence (3) holds. □

Theorem 7. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. There exist the following two long exact sequences:*

(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

Proof. By Lemma2 and Proposition6 (1), we have the following two long exact sequence:

$$\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{B^e}(A, A) \to \cdots;$$

and

$$\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{B^e}(A, A) \oplus \mathrm{Ext}^n_{B^e}(BeB, BeB) \to \cdots.$$

By Proposition6 (2) and (3), we have the following two long exact sequences:

$$\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{A^e}(A, A) \to \cdots ;$$

and

$$\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{A^e}(A, A) \oplus \mathrm{Ext}^n_{eBe^e}(eBe, eBe) \to \cdots .$$

□

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LINKAGE AND DUALITY OF MODULES

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Abstract. Martsinkovsky and Strooker [5] recently introduced module theoretic linkage using syzygy and transpose. This generalization brings possibility of much application of linkage, especially, to homological theory of modules. In the present paper, we connect linkage of modules to certain duality of modules. We deal with invariance of Gorenstein dimension, characterization of Cohen-Macaulay modules over a Gorenstein local ring using linkage and their generalization to non-commutative algebras.

Let Λ be a left and right Noetherian ring. Let modΛ (respectively, mod$\Lambda^{\rm op}$) be the category of all finitely generated left (respectively, right) Λ-modules. Throughout the paper, all modules are finitely generated and left modules (if the ring is non-commutatie) and right modules are considered as $\Lambda^{\rm op}$-modules. We denote the stable category by $\underline{\rm mod}\Lambda$, the syzygy functor by $\Omega : \underline{\rm mod}\Lambda \to \underline{\rm mod}\Lambda$, and the transpose functor by $\mathrm{Tr} : \underline{\rm mod}\Lambda \to \underline{\rm mod}\Lambda^{\rm op}$.

Let $T_k := \mathrm{Tr}\Omega^{k-1}$ for $k > 0$ and $\lambda := \Omega\mathrm{Tr}$. Following [5], we define

Definition. A finitely generated Λ-module M and a $\Lambda^{\rm op}$-module N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$, in other words, M is horizontally linked (to λM) if and only if $M \cong \lambda^2 M$.

A rather different definition of linkage of modules is proposed by Yoshino and Isogawa [9]. However, both definitions coincide for Cohen-Macaulay modules over a commutative Gorenstein ring (see [5], section 3).

Let us start with the following simple observation which connect duality with linkage.

Proposition 1. *T_kM is horizontally linked if and only if* $\mathrm{Ext}^k_\Lambda(M, \Lambda) = 0$.

We prepare the facts about Gorenstein dimension from [1]. A Λ-module M is said to have G-dimension zero, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\rm op}}(M^*, \Lambda) = 0$ for $k > 0$. This is equivalent to '$\mathrm{Ext}^k_\Lambda(M, \Lambda) = \mathrm{Ext}^k_{\Lambda^{\rm op}}(\mathrm{Tr}M, \Lambda) = 0$ for $k > 0$' ([1], Proposition 3.8). For a positive integer k, we say that M has G-dimension less than or equal to k, denoted by $\text{G-dim}_\Lambda M \le k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim}_\Lambda G_i = 0$ for $(0 \le i \le k)$. It follows from [1], Theorem 3.13 that $\text{G-dim}_\Lambda M \le k$ if and only if $\text{G-dim}_\Lambda \Omega^k M = 0$. If $\text{G-dim}_\Lambda M < \infty$, then $\text{G-dim}_\Lambda M = \sup\{k : \mathrm{Ext}^k_\Lambda(M, \Lambda) \neq 0\}$([1], p. 95).

In the following, the proofs are seen in [6].

Invariance of G-dimension under linkage is studied in [5].

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THEOREM 2. ([5], Theorem 1) *Let Λ be a semiperfect right and left Noetherian ring and M a Λ-module without projective direct summand. Then the following conditions are equivalent.*

(1) G-dim$_\Lambda M = 0$,

(2) G-dim$_{\Lambda^{\mathrm{op}}} \lambda M = 0$ and M is horizontally linked.

In the rest, we consider a commutative ring case and apply the above results to Cohen-Macaulay modules over a commutative Gorenstein local ring. See [2] for Cohen-Macaulay rings and modules and Gorenstein rings.

Let R be a (commutative) Gorenstein local ring and M a finitely generated R-module. Then there are the following useful equations:

$$(1)\quad \text{G-dim}_R M + \text{depth} M = \dim R$$

$$(2)\quad \text{grade}_R M + \dim M = \dim R,$$

where $\text{grade}_R M := \inf\{k \geq 0 : \text{Ext}^k_R(M, R) \neq 0\}$. The first equality is due to [1], Theorems 4.13 and 4.20 and the second one to [7], Lemma 4.8.

The combination of linkage and duality produces the following characterization of a maximal Cohen-Macaulay module which improves [5], Proposition 8.

THEOREM 3. *Let R be a Gorenstein local ring and M a finitely generated R-module without a projective direct summand. Then the following are equivalent*

(1) M is a maximal Cohen-Macaulay module,

(2) $T_k M$ is horizontally linked for $k > 0$,

(3) λM is a maximal Cohen-Macaulay module and M is horizontally linked.

G-dimension is also described using linkage.

PROPOSITION 4.
Let R be a Gorenstein local ring and M a finitely generated module. Then G-dim$_R M \leq k$ if and only if $T_{i+k} M$ is horizontally linked for $i > 0$.

We apply duality theory on a non-commutative Noetherian ring due to Iyama [4] to the category of Cohen-Macaulay modules. Suppose that Λ is a right and left Noetherian ring. Then the functor T_k gives a duality between the categories $\{X \in \underline{\text{mod}}\Lambda : \text{grade}_\Lambda X \geq k\}$ and $\{Y \in \underline{\text{mod}}\Lambda^{\mathrm{op}} : \text{rgrade}_{\Lambda^{\mathrm{op}}} Y \geq k \text{ and } \text{pd}_{\Lambda^{\mathrm{op}}} Y \leq k\}$ [4], 2.1.(1), where we denote a reduced grade of M by $\text{rgrade}_R M := \{k > 0 : \text{Ext}^k_R(M, R) \neq 0\}$ [3]. Returning to our case, we consider a commutative local ring R and a finitely generated R-module M. Then it holds that $\text{G-dim}_R M \geq \text{grade}_R M$, in general. Moreover, if $\text{G-dim}_R M < \infty$, then M is a Cohen-Macaulay module if and only if $\text{G-dim}_R M = \text{grade}_R M$ by the equations (1) and (2). Thus we can apply the above duality to the category of Cohen-Macaulay R-modules.

A finitely generated module M over a Cohen-Macaulay local ring R is called a Cohen-Macaulay module of codimension k, if $\text{depth} M = \dim M = \dim R - k$. Put the full subcategory

$$\mathcal{C}_k := \{M \in \text{mod} R : M \text{ is a Cohen-Macaulay } R\text{-module of codimension } k\}.$$

In order to give a duality, we need a counterpart of the category $\mathcal{C}_k$. Let M be a finitely generated R-module. We put the full subcategory

$\mathcal{C}'_k :=$
$\{N \in \text{mod}R : \text{rgrade}_R N = \text{pd}_R N = k$ and $\lambda^2 N$ is a maximal Cohen-Macaulay module$\}$,
where $\text{pd}_R N$ stands for a projective dimension of N. Then we have

THEOREM 5. *Let R be a Gorenstein local ring. Let $k > 0$. Then the functor T_k gives a duality between full subcategories $\mathcal{C}_k$ and $\mathcal{C}'_k$.*

Using the above duality, we can characterize a Cohen-Macaulay module of codimension $k > 0$.

COROLLARY 6. *Let M be a finitely generated R-module of $\text{grade}_R M = k > 0$. Then the following are equivalent*

(1) M is a Cohen-Macaulay module of codimension k,

(2) $\text{rgrade}_R T_k M = \text{pd}_R T_k M = k$ and $\lambda^2 T_k M$ is a maximal Cohen-Macaulay module.

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STABLE EQUIVALENCES RELATED WITH SYZYGY FUNCTORS

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Abstract. Let $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence between finite dimensional self-injective algebras over a field. Then Φ preserves triangles in the triangulated category $\underline{\mathrm{mod}}\,\Lambda$ if and only if Φ commutes with syzygy functors. As an application, we study some stable equivalence induced by socle equivalence.

Key Words: Stable equivalence, Nakayama automorphism.

1. Introduction

Throughout this paper K will be a fixed field, and all algebras will be basic finite dimensional self-injective K-algebras without simple algebra summands. By a module we mean a finite dimensional left module unless otherwise stated, and by $\mathrm{mod}\,\Lambda$ we denote the category of finite dimensional left modules over an algebra Λ. In order to distinguish an equivalence between triangulated categories from an equivalence between the additive categories, we say that a functor is a triangle equivalence if it is an equivalence between triangulated categories.

Happel proved in [2] that the stable category of a self-injective algebra is a triangulated category whose translation is the inverse of the syzygy functor. Keller-Vossieck [4] and Rickard [8] proved that the stable category of a self-injective algebra is triangle equivalent to the quotient category of the bounded derived category by its subcategory consisting of perfect complexes. These results give the motivation which develops invariants (of stable equivalence) arisen from a derived equivalence, or properties of triangulated categories. Pogorzały [7] and Xi [9] proved that the Hochschild cohomology and the representation dimension are invariants under a stable equivalence of Morita type, respectively. Our aim also develops an invariant in order to clear up the difference between a stable equivalence of Morita type, a stable equivalence not of Morita type and a stable equivalence induced by a derived equivalence. We shall show that for self-injective algebras Λ, Λ' and an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$, Φ is a triangle functor if and only if Φ commutes with the syzygy functors. As an application, we shall show that the symmetry is an invariant for some stable equivalence [5] induced by a socle equivalence.

2. A homotopy category

We shall prepare some notations related to homotopy categories. Basic notations and definitions are referred to [3] For an abelian category $\mathcal{A}$, we denote by $X^\bullet = (X^n, d_X^n)$ the

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(cochain) complex

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

with $d_X^n d_X^{n-1} = 0$ for all integers n. The shift functor T of the category of complexes is defined by $T(X^\bullet)^n = X^{n+1}$.

We denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, that is, the residue category of the category of complexes by the homotopy relation. We denote by $\mathrm{K}^-(\mathcal{A})$ or $\mathrm{K}^b(\mathcal{A})$ the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of bounded above complexes or bounded complexes, respectively.

For $* = -$ or b, a homotopy category $\mathrm{K}^*(\mathcal{A})$ is regarded as a triangulated category whose translation functor is the shift functor T, and for any morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathrm{K}^*(\mathcal{A})$ it induces the triangle in $\mathrm{K}^*(\mathcal{A})$

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\begin{pmatrix} 1_Y \\ 0 \end{pmatrix}} C(f^\bullet) \xrightarrow{\begin{pmatrix} 0 & 1_{TX} \end{pmatrix}} TX^\bullet.$$

Here $C(f^\bullet) := \left(Y^\bullet \oplus TX^\bullet, \begin{pmatrix} d_Y & Tf \\ 0 & d_{TX} \end{pmatrix}\right)$ is the mapping cone. We will consider the case $\mathcal{A} = \mathcal{P}_\Lambda$, where $\mathcal{P}_\Lambda$ is the full subcategory of $\mathrm{mod}\,\Lambda$ consisting of projective Λ-modules.

3. A stable equivalence which commutes with syzygy functors

Let Λ be a self-injective algebra. For a Λ-module X, we denote by $\iota_X : X \to I_X$ the injective hull of X. The stable category $\underline{\mathrm{mod}}\,\Lambda$ of Λ has the same objects as $\mathrm{mod}\,\Lambda$, and a morphism from X to Y in $\underline{\mathrm{mod}}\,\Lambda$ is by definition a residue class in $\mathrm{Hom}_\Lambda(X,Y)/\operatorname{proj}_\Lambda(X,Y)$, where $\operatorname{proj}_\Lambda(X,Y)$ is the subspace of $\mathrm{Hom}_\Lambda(X,Y)$ consisting of morphisms which factor through projective Λ-modules. The syzygy functor $\Omega_\Lambda : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$ is a functor naturally defined by the correspondence of an object X to the kernel of the projective cover of X. Note that the syzygy functor induces the stable equivalence functor $\underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda$ which is also called the syzygy functor, denoted by Ω_Λ or Ω again.

Happel showed that the stable category of a self-injective algebra is regarded as a triangulated category [2]. In fact, the translation functor of $\underline{\mathrm{mod}}\,\Lambda$ is given by the inverse Ω_Λ^{-1} of Ω_Λ. For each morphism $\underline{f} : X \to Y$ in $\underline{\mathrm{mod}}\,\Lambda$, the standard triangle $X \xrightarrow{\underline{f}} Y \to Z \to X[1]$ is given by the sequence $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} \Omega^{-1}X$ in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota_X} & I_X & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow[h]{} & \Omega^{-1}X & \longrightarrow & 0.
\end{array}$$

We denote Z by $C(\underline{f})$ and call it the mapping cone of $\underline{f}$ in $\underline{\mathrm{mod}}\,\Lambda$.

We denote by $\mathrm{K}^{-,b}(\mathcal{P}_\Lambda)$ the full subcategory of $\mathrm{K}^{-}(\mathcal{P}_\Lambda)$ consisting of complexes $X^\bullet$ with bounded cohomology i.e., $\mathrm{H}^n(X^\bullet) = 0$ for $n \ll 0$. Keller-Vossieck and Rickard proved the following result.

Proposition 1. [4][8] *For a self-injective algebra Λ, the stable category of Λ is triangle equivalent to the quotient category $\mathrm{K}^{-,b}(\mathcal{P}_\Lambda)/\,\mathrm{K}^b(\mathcal{P}_\Lambda)$.*

Using the similar correspondence on Proposition 1, we show the our main theorem.

Theorem 2. *Assume that there is an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ for self-injective algebras Λ and Λ'. Then the following conditions are equivalent.*

(1) Φ *is a triangle functor.*

(2) Φ *commutes with the syzygy functors i.e., $\Omega_{\Lambda'}\Phi \simeq \Phi\Omega_\Lambda$.*

In [1, Chapter X], Auslander-Reiten-Smalø proved that an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ commutes with syzygy functors if Λ and Λ' are symmetric algebras. Therefore the following corollary follows from Theorem 2.

Corollary 3. *If Λ and Λ' are symmetric algebras, then any stable equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ is a triangle functor.*

4. Nakayama automorphisms and some application

We recall the definitions and some basic properties of the Nakayama automorphism and the Nakayama functor for a self-injective algebra. Refer to [10] in detail. Let M be a Λ-module and f an automorphism of Λ. The Λ-module ${}_fM$ is the K-space M with the Λ-module structure: $a \cdot m = f(a)m$ for $a \in \Lambda$, $m \in M$. Similarly we define N_f for a right Λ-module N. For a self-injective algebra Λ, there is an automorphism ν of Λ such that Λ and $(D\Lambda)_\nu$ are isomorphic as Λ-bimodules, where $D = \mathrm{Hom}_K(-, K)$. Such an automorphism ν is uniquely determined up to inner automorphisms, and called the Nakayama automorphism of Λ. The Nakayama functor is defined by $\mathcal{N} = D\,\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$, and therefore $\mathcal{N}$ is isomorphic to ${}_\nu\Lambda \otimes_\Lambda -$. Note that $\mathcal{N}$ naturally induces the equivalence $\mathrm{mod}(\Lambda/\operatorname{soc}\Lambda) \xrightarrow{\sim} \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)$, and the stable equivalence $\underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda$ of Morita type, which are also denoted by $\mathcal{N}$.

A K-linear map $\lambda : \Lambda \to K$ is associated to a Λ-bimodule isomorphism $\varphi : \Lambda \xrightarrow{\sim} (D\Lambda)_\nu$ if $\lambda = \varphi(1_\Lambda)$. Then λ is non-degenerate and satisfies $\lambda(\nu(x)y) = \lambda(yx)$ for any $x, y \in \Lambda$.

We denote $\overline{a}$ the residue class of $a \in \Lambda$ in $\Lambda/\operatorname{soc}\Lambda$. A Λ-bimodule isomorphism $\Lambda \xrightarrow{\sim} (D\Lambda)_\nu$ induces $\Lambda/\operatorname{soc}\Lambda$-bimodule isomorphism $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$, where $\overline{\nu}$ is an algebra automorphism of $\Lambda/\operatorname{soc}\Lambda$ given by $\overline{\nu}(\overline{a}) = \overline{\nu(a)}$. Therefore we have a K-bilinear map $\overline{\lambda} : \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \times \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \to K, (\overline{a}, \overline{b}) \mapsto \lambda(ab)$ associated with $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$.

Lemma 4. *Let $\overline{\lambda} : \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \times \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \to K$ be a K-bilinear map associated with $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$. Then $\overline{\lambda}(\overline{a}, \operatorname{rad}\Lambda/\operatorname{soc}\Lambda) \neq 0$ for any non-zero $\overline{a}$ in $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda$.*

For a Λ-module homomorphism $f : X \to Y$, we define a Λ-module homomorphism between ${}_\rho X$ and ${}_\rho Y$ for an algebra automorphism ρ of Λ as follows

$$ {}_\rho\Lambda \otimes f : {}_\rho X \to {}_\rho Y, x \mapsto f(x). $$

Lemma 5. *Let A be a (not necessarily self-injective) algebra, and ρ an algebra automorphism of A. Then the following conditions are equivalent.*

(1) *ρ is an inner automorphism.*
(2) *There is an A-module isomorphism $\psi : A \xrightarrow{\sim} {}_\rho A$ such that $\psi f = ({}_\rho A \otimes f)\psi$ for any A-module endomorphism f of A.*

Lemma 5 gives the essential condition in order to characterize symmetry for a self-injective algebra Λ. Therefore we have the following lemma on a notion of module category.

Proposition 6. *The following conditions are equivalent for a self-injective algebra Λ over an algebraically closed field K.*

(1) *Λ is symmetric.*
(2) *The Nakayama functor $\mathcal{N} : \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda) \xrightarrow{\sim} \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)$ is isomorphic to the identity functor $id_{\mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)}$.*
(3) *$\overline{\nu} : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda/\operatorname{soc}\Lambda$ is an inner automorphism.*

Proposition 6 is not true if K is not algebraically closed field. In [6], we can see the counter-example.

Let $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence for self-injective algebras. We have that $\Phi\tau \simeq \tau'\Phi$, where τ and τ' are stable equivalences induced by Auslander-Reiten translations of Λ and Λ', respectively. By [1], it follows that $\tau \simeq \mathcal{N}\Omega_\Lambda^2$. If Φ is a triangle functor, then we have

$$\Phi\mathcal{N}\Omega_\Lambda^2 \simeq \mathcal{N}'\Omega_{\Lambda'}^2\Phi \simeq \mathcal{N}'\Phi\Omega_\Lambda^2,$$

therefore it follows that $\Phi\mathcal{N} \simeq \mathcal{N}'\Phi$.

In order to preserve the symmetry for triangle stable equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ between symmetric algebra Λ and self-injective algebra Λ', we consider the problem whether $\Phi \simeq \mathcal{N}'\Phi$, equivalent to $id_{\underline{\mathrm{mod}}\,\Lambda} \simeq \mathcal{N}'$, implies that Λ' is also symmetric. However, this is open in general.

Theorem 7. [5] *Let Λ and Λ' be socle equivalent self-injective algebras, say $p : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$. Assume that there are non-degenerate K-linear maps $\lambda : \Lambda \to K$ and $\lambda' : \Lambda' \to K$ such that $\lambda(ab) = \lambda'(a'b')$ for all $a, b \in \operatorname{rad}\Lambda$ and $a', b' \in \operatorname{rad}\Lambda'$ with $\bar{a}' = p(\bar{a})$ and $\bar{b}' = p(\bar{b})$. Then the stable categories $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda'$ are equivalent.*

In the case of Theorem 7, we will show that this problem is true if K is an algebraically closed field.

Theorem 8. *Let Λ and Λ' be self-injective algebras over algebraically closed field, and $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence defined in Theorem 7. If Λ is symmetric, then so is Λ'.*

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QF RINGS AND QF ASSOCIATED GRADED RINGS

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Abstract. The associated graded ring of QF (quasi-Frobenius, generally not commutative) ring R is not necessarily QF. We shall prove that the associated graded ring of R is QF if and only if R is QF and for any primitive idempotent e the upper Loewy series of Re and eR is coincident with the lower Loewy series of Re and eR respectively.

In connection with the above result we consider for any pair of positive integers t, n a ring $\Lambda = K[x_0, x_1, \cdots, x_n]/(x_i^t - \frac{1}{x_i}\prod_{j=0}^{n} x_j \,|\, i = 0, 1, \cdots, n)$, because for $t \neq n$, the associated graded ring of Λ is different from Λ but they are both QF (=0-dimensional Gorenstein). So we expect that for $t = n$, Λ is Gorenstein even if Krull dimension > 0. We pointed out however that if $t = n = 2$, Λ is not Gorenstein, but Cohen-Macauley. Further if $n = t = 3$, Λ is neither Cohen-Macauley nor toric, of course not Gorenstein.

1. A characterization of QF associated graded rings

For an Artinian ring R having the Jacobson radical J with $J^{n+1} = 0$, the series : $R \supset J \supset J^2 \supset \cdots \supset J^n \supset J^{n+1} = 0$ is called the upper Loewy series of ${}_R R$ (resp. R_R). If we put $A_i = J^i/J^{i+1}$, we can naturally define the multiplication of elements $a + J^{i+1} \in A_i$ and $b + J^{j+1} \in A_j$ to be $ab + J^{i+j+1} \in A_{i+j}$. Then by using this multiplication we make the (formal) direct sum $A_0 \oplus A_1 \oplus \cdots \oplus A_n$ into a ring R_G. Clearly this ring R_G is positive $Z-$graded and A_1 generates the radical of R_G. R_G is called the associated graded ring of R. Cf.[3]. R and R_G may be not isomorphic to each other. Cf. Example 2.1

By Morita equivalence [8] we can assume without loss of generality that rings are basic. Let e be a primitive idempotent of ring R. Then $e + J \in A_0$ is a primitive idemotent of R_G which we shall denote by e_G for short. If we denote the right (resp. left) annihilator of a subset M of R by $r(M)$ (resp. $l(M)$), then $Soc\,(Re) = r(J)e$ (resp. $Soc\,(eR) = e\,l(J)$). At first we have

Proposition 1.1. *If ${}_{R_G}Soc(R_G e_G)$ is simple for a primitive idempotent e_G, then the ${}_R Soc(Re)$ is simple. And if $Soc(R_G e_G) \simeq R_G f_G/Rad(R_G) f_G$ for a primitive idempotent f, then $Soc(Re) \simeq Rf/Jf$.*

Proof. Let $J^\rho e \neq 0$ and $J^{\rho+1} e = 0$. Then $A_\rho e_G \neq 0$. Let us denote the set $\{\alpha \in R_G | A_1 \alpha = 0\}$ by $r(A_1)$. Since A_1 generates the radical of R_G, by the assumption $Soc(R_G e_G) = r(Rad(R_G)) e_G = r(A_1) e_G$ is a unique simple R_G-submodule of $R_G e_G$. Hence $r(A_1) e_G \subseteq A_\rho e_G$. On the other hand $r(A_1) e_G \supseteq A_\rho e_G$ by $A_1 A_\rho e_G = 0$. Hence $r(A_1) e_G = A_\rho e_G$.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

Now take $u \in r(J)e$. Then there is a unique positive integer j such that $u \in J^j e \setminus J^{j+1}e$. As $au = 0$ for any $a \in J$ we have that $(a + J^2)(u + J^{j+1}) = au + J^{(j+1)+1} = 0 + J^{(j+1)+1} \in A_{j+1}$ for $a + J^2 \in A_1$. Therefore $u + J^{j+1} \in r(A_1)e_G = A_\rho e_G$ and it follows that $j = \rho$. This implies that $r(J)e \subseteq J^\rho e$. On the other hand $r(J)e \supseteq J^\rho e$ by $J(J^\rho e) = 0$. Hence $J^\rho e = r(J)e$. Further $J^\rho e$ can be identified with $A_\rho e_G$ because $J^{\rho+1}e = 0$.

Now R/J can be identified with $R_G/Rad(R_G)$. From $Rad(R_G)^\rho e_G = A_\rho e_G$ is simple as a left R_G-module it follows that $r(J)e$ is a simple R-module.

The latter statement follows from that $f_G A_\rho e_G \neq 0$ if and only if $fJ^\rho e \neq 0$. This completes the proof. □

Following Thrall [12] a ring R is said to be left QF-2 if the socle of Re is simple for every primitive idempotent e. Then we have immediately

Corollary 1.2. *R is left QF-2 if R_G is left QF-2.*

Now we shall prove

Theorem 1.3. *If R_G is QF, then R is QF.*

Proof. By the assumption that R is basic there is a set of primitive idempotents e_i such that $1 = \sum_{i=1}^{n} e_i$ and $Re_i \not\simeq Re_j$ for $i \neq j$.

Since R_G is QF, for all e_i, $Soc(R_G e_{i_G}) = r(Rad(R_G))e_{i_G} = r(A_1)e_{i_G}$ (resp. $Soc(e_{i_G} R_G) = e_{i_G} l(Rad(R_G)) = e_{i_G} l(A_1)$) is a simple left (resp. right) R_G -module.

Hence by Proposition 1.1 $r(J)e_i$ (resp. $e_i l(J)$) is a simple left (resp. right) R -module.

On the other hand it holds that ${}_R l(J)e_i \simeq {}_R Hom_R((e_i R/e_i J)_R, {}_R R_R)$ and as is quoted above ${}_R l(J)e_i$ is simple. Similarly $e_i r(J)_R \simeq Hom_R((Re_i/Je_i, {}_R R_R)$ is simple.

Therefore by [6, Theorem 2.1] it holds the duality $Hom_R(-, {}_R R_R)$ between the categories of finitely generated left R-modules and right R-modules. and hence R is QF. Cf. also [5]. □

In Example 2.1 we shall show that both the converses of Corollary 1.2 and Theorm 1.3 do not hold.

Now it needs to give a characterization of QF ring R for which the associated graded ring R_G is QF.

We say that the series : $Re = r(J^{\rho+1})e \supset r(J^\rho)e \supset r(J^{\rho-1})e \supset \cdots \supset r(J)e \supset r(R)e = 0$ is the lower Loewy series of Re.

In their book [2] Artin-Nesbitt-Thrall proved that subquotient modules $J^k e/J^{k+1}e$ and $r(J^{\rho+1-k})e/r(J^{\rho-k})e$ have non-zero isomorphic constituents for every $0 \leqq k \leqq \rho$. Then we have the following question:

Which kind of rings do satisfy the coincidence of every non-zero isomorphic constituent of $J^k e/J^{k+1}$ with $J^k e/J^{k+1}e$ itself ?

We can provide Proposition 1.4 as an answer to the question.

A positive Z-graded ring $R = A_0 \oplus A_1 \oplus \cdots \oplus A_n$ is called to be standard if A_1 generates the radical of R.

Then we have

Proposition 1.4. *If R is a standard positive Z-graded QF ring , then the upper Loewy series of Re coincides with the lower Loewy series of Re for any primitive idempotent e.*

Proof. For a primitive idempotent e let $Re = A_0e \oplus A_1e \oplus A_2e \oplus \cdots \oplus A_\rho e, \quad \rho \leqq n$, be a grading of Re. Then $A_iA_je \subseteq A_{i+j}e$ and the rad$(Re)=$ $A_1e \oplus A_2e \oplus \cdots \oplus A_\rho e$.

Then from the assumption that R is QF it holds that ${}_Rl(J)e = r(J)e = Soc(Re) = J^\rho e = A_\rho e$.

Assume that $r(J^s)e = J^{\rho+1-s}e$ for an integer $s \geq 1$ (as pointed out above this assumption is satisfied for $s = 1$), and suppose that $r(J^{s+1})e \neq J^{\rho-s}e$ for $r(J^{s+1})e \supseteq J^{\rho-s}e$.

Then there is $0 \neq y = \sum_{l \leqq j < \rho - s} y_j \in r(J^{s+1})e$ such that $0 \neq y_j \in A_je$. From $0 = J^{s+1}y = J^s(Jy)$ it follows $Jy \in r(J^s)e = J^{\rho+1-s}e = \oplus_{\rho+1-s \leqq k} A_ke$.

On the other hand $Jy = \sum_{j<\rho-s} Jy_j \in \oplus_{l \leqq j < \rho - s} A_{j+1}e = \oplus_{l+1 \leqq k < \rho+1-s} A_ke$.

Hence $Jy = 0$. Then $A_1y_l = 0$ since A_1 generates J and $y_j \in A_je$ for $l \leqq j < \rho - s$. This implies $y_l \in A_\rho e$ and this is a contradiction because $l \neq \rho$.

Therefore we conclude that $r(J^{s+1})e = J^{\rho-s}e$.

Now by induction on s we complete the proof. □

Corollary 1.5. *If R_G is QF then for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively.*

Proof. Let $J^\rho e \neq 0$ but $J^{\rho+1}e = 0$. Then by Proposition 1.4 it follows that
$(Rad(R_G))^{\rho+1-k}e_G = Soc^k(R_Ge_G) = r((Rad(R_G))^k)e_G$ for $k = 1, 2, \cdots, \rho$.

Now we want to prove that $J^{\rho+1-k}e = Soc^k(Re) = r(J^k)e$ for $k = 1, 2, \cdots, \rho$.

Suppose $x \in r(J^k)e \setminus J^{\rho+1-k}e$ since $J^{\rho+1-k}e \subseteq r(J^k)e$. Let j be the maximal integer such that $x \in J^j \setminus J^{j+1}e$. Then $j < \rho + 1 - k$. For $x + J^{j+1} \in A_je_G$ it holds that $A_k(x + J^{j+1}e) = (J^k/J^{k+1})(x + J^{j+1}e = 0 + J^{j+1+k}e = 0$. This implies that $x + J^{j+1}e \in r(A_k)e_G = r(A_1^k)e_G = r((Rad(R_G))^k)e_G = Rad(R_G)^{\rho+1-k}e_G = (A_{\rho+1-k} \oplus A_{\rho+2-k} \oplus \cdots)e_G$. Thus we have $j \geq \rho + 1 - k$. But this contradicts to $j < \rho + 1 - k$.

We can prove similarly that $eJ^{\sigma+1-k} = Soc^k(eR) = e\,l(J^k)$ for $k = 1, 2, \cdots, \sigma$, where $eJ^\sigma \neq 0$ but $eJ^{\sigma+1} = 0$. This completes the proof. □

Proposition 1.6. *If R is QF and for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively, then the associated graded ring R_G is QF.*

Proof. Let $J^\rho e \neq 0$ but $J^{\rho+1}e = 0$. From the coincidence of series of the upper Loewy series and the lower Loewy series of Re it follows that $r(J^i)e = Soc^i({}_RRe) = J^{\rho+1-i}e$, $i = 1, 2, \cdots, \rho$. And especially $Soc(Re) = J^\rho e$ is a simple left R-module since R is QF.

For $x \in J^k e \setminus J^{k+1}e$ and $k \leqq \rho - 1$ suppose $Jx \in J^{k+2}e = Soc^{\{\rho+1-(k+2)\}}(Re)$. Then $J^{\{\rho+2-(k+2)\}}x = 0$ and $x \in r(J^{\rho-k})e = J^{k+1}e$. But this is a contradiction.

Therefore if $k \leqq \rho - 1$ and if $x + J^{k+1}e \neq 0 \in A_k e_G = (J^k/J^{k+1})e_G$ it holds that $0 \neq A_1(x + J^{k+1}e) \in A_{k+1}e_G$. Therefore $Soc(R_G e_G) = r(Rad(R_G))e_G \subseteq J^\rho e = Rad(R_G)^\rho e_G$.

As $r(Rad(R_G))e_G \supseteq Rad(R_G)^\rho e_G$ we have $Soc(R_G e_G) = r(Rad(R_G))e_G = J^\rho e$, which can be considered as a simple left R_G-module because $J^\rho e$ is a simple left R-module.

Now it is clear that $Hom_{R_G}(e_G R_G/e_G Rad(R_G), {}_{R_G}R_{G R_G}) \simeq r(Rad(R_G))e_G$.
This implies that the dual module $Hom_{R_G}(e_G R_G/e_G Rad(R_G), {}_{R_G}R_{G R_G})$ of a simple right R_G-module $e_G R_G/e_G Rad(R_G)$ is a simple left R_G-module.

Similarly we have that the dual module $Hom_{R_G}(R_G e_G/Rad(R_G)e_G, {}_{R_G}R_{G R_G})$ of a simple left R_G-module $R_G e_G/Rad(R_G)e_G$ is a simple right R_G-module.

Therefore by [6, Theorem 2.1] it holds the duality $Hom_{R_G}(-, {}_{R_G}R_{G R_G})$ between the categories of finitely generated left R_G-modules and finitely generated right R_G-modules. Hence R_G is QF. □

Now by Propositions 1.5 and 1.6 we have the following characterization of QF associated graded rings:

Theorem 1.7. *The following conditions $(i), (ii)$ and (iii) are equivalent to each other:*

(i) The associated graded ring R_G is QF,

(ii) R is QF and for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively,

(iii) R is QF and for any primitive idempotent e_i and integer $0 \leqq k \leqq \rho_i$ it holds that ${}_R J^k e_i/J^{k+1}e_i \simeq {}_R Hom_R(e_i J^{\rho-k}/e_i J^{\rho-k+1}{}_R, {}_R R_R)$ (resp. $e_i J^k/e_i J^{k+1} \simeq Hom_R({}_R J^{\sigma-k}e_i/J^{\sigma-k+1}e_i, {}_R R_R)$, where $J^{\rho_i}e_i \neq 0$ but $J^{\rho_i+1}e_i = 0$ (resp. $e_i J^{\sigma_i} \neq 0$ but $e_i J^{\sigma_i+1} = 0$).

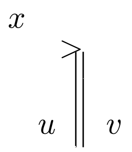
Let π be a Nakayama permutation of QF ring R on the set of all non isomorphic primitive idempotents e_i, $i = 1, 2, \cdots, n$.

Then it holds that ${}_R Re_{\pi(j)}/Je_{\pi(j)} \simeq {}_R Hom_R(e_j R/e_j J_R, {}_R R_R)$.

Corollary 1.8. *R_G is QF if and only if R is QF and for any primitive idempotent e_i it holds that ${}_R J^k e_i/J^{k+1}e_i \simeq \oplus_j^n n_{i,j} \times Re_{\pi(j)}/Je_{\pi(j)}$ for a direct sum decomposition : $e_i J^{\rho-k}/e_i J^{\rho-k+1}{}_R \simeq \oplus_j^n n_{i,j} \times e_j R/e_j J$, where $n_{i,j} \times e_j R/e_j J$ means the direct sum of $n_{i,j}$ copies of $e_j R/e_j J$.*

As indecomposable commutative algebras are local, for them there are no difference between QF-2, QF-3 and QF rings. Hence Theorem 1.7 and Corollary 1.8 are considered to be results for non commutative rings, though Theorem 1.7 seems to be a generalization of Iarrobino's result [4; Proposition 1.7] for 0-dimensional Gorenstein algebras.

Example 2.1. (i) Let R be an algebra over a field K defined by the following quiver.



$$x^3 = vu,\quad 0 = uv,\quad 0 = xv,\quad 0 = ux.$$

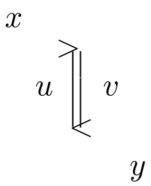
Then the K-bases : $R = \{e_1, x, x^2, x^3, u; e_2, v\}$, $J = Rad(R) = \{x, x^2, x^3, u; v\}$, $J^2 = \{x^2, x^3\}$,$J^3 = \{x^3\}$ and $J^4 = \{0\}$. By $0 \neq x^3 = vu$ and $0 = uv$, R is not commutative. As $r(J)e_1 = Kx^3 \simeq Re_1/Je_1$ and $r(J)e_2 = Kv \simeq Re_1/Je_1$, R is left QF-2.

It happens however that $(v + J^2)(u + J^2) = vu + J^3 = 0 \in G(R)$ for the contrary $vu = x^3 \neq 0 \in R$. Then $r(rad(G(R)))e_1 = K(u + J^2) + K(x^3 + J^4) \simeq G(R)e_2/Rad(G(R))e_2 \oplus G(R)e_1/Rad(G(R))e_1$ is not simple. Hence $G(R)$ is not left QF-2.

This shows that the converse of Proposition 1.1 does not hold.

The next example (ii) shows that the converse of Theorem 1.3 does not hold.

(ii)



$$x^3 = vu,\quad y^2 = uv,\quad 0 = xv,\quad 0 = ux,\quad 0 = vy,\quad 0 = yu.$$

Then the K-bases $= \{e_1, x, x^2, x^3, u, e_2, y, v\}$, where e_2 and e_2 are primitive idempotents.

By $x^3 = vu$ and $y^2 = uv$, R is not commutative. $J = Rad(R) = \{x, x^2, x^3, u; y, y^2, v\}$, $J^2 = \{x^2, x^3, y^2\}$,$J^3 = \{x^3\}$, $J^4 = 0$. $r(J) = \{x^3, y^2\} = l(J), r(J)e_1 = Kx^3 \simeq Re_1/Je_1$ and $r(J)e_2 = Ky^2 \simeq Re_2/Je_2$. Hence R is QF.

As $(v+J^2)(u+J^2) = 0+J^3$ and $(y+J^2)(u+J^2) = 0+J^3, Soc(R_G(e_1)_G) = K(u+J^2)+K(x^3 + J^3) \simeq R_G(e_2)_G/Rad(R_G)(e_2)_G \oplus R_G(e_1)/Rad(R_G)(e_1)_G)$. Hence $Soc(R_G(e_1)_G)$ is not simple. Therefore R_G is not QF.

We know that the upper Loewy series of Re_1 and Re_2 are $(1, 1+2, 1, 1)$ and $(2, 1+2, 2)$ respectively. On the other hand lower Loewy series of Re_1 and Re_2 are $(1, 1, 2+1, 1)$ and $(2, 1+2, 2)$ respetively. From Theorem 1.7 it follows also that R_G is not QF.

Example 2.2. Let Λ be a quotient ring $K[x_0, x_1, \cdots, x_n]/I$ such that the ideal I are generated by $n+1$ polynomials $x_i^t - \dfrac{1}{x_i}\prod_{j=0}^{n} x_j$, $i = 0, 1, \cdots, n$, for the pairs (t, n).

In case of $t \neq n$, for $min\{n,t\} \leqq |t-n|s < max\{n,t\}$ there is an idempotent $e \equiv \prod_{i=0}^{n} x_i^{|t-n|s}$ mod I of Λ and $\Gamma = (1-e)\Lambda \simeq K[x_0, x_1, \cdots, x_n]/(\prod_{i=0}^{n} x_i^{|t-n|s}, I)$ is an Artinian local algebra. Cf.[12] and Kikumasa-Yoshimura [6].

Let us denote the associated graded algebra $\Gamma_G = A_0 \oplus A_1 \cdots \oplus A_m$. Then if $t > n$, $m = (t+1)(n-1)$ and $dim_K(A_k) = \sharp\{(d_0, d_1, \cdots, d_i, \cdots d_j, \cdots, d_n) | \sum_{l=0}^{n} d_l = k, 0 \leqq d_l \leqq t+1$ and $d_i = d_j = 0$ for $i \neq j\}$. Hence $dim_K A_k = dim_K A_{m-k}$. It follows by Corollary 1.8 that Γ_G (and hence by Theorem 1.3 Γ) is QF.

If $t < n$, $m = (n+1)(t-1)$ and $dim_K(A_k) = \sharp\{(d_0, d_1, \cdots, d_n) | \sum_{l=0}^{n} d_l = k, 0 \leqq d_l \leqq t-1\}$. Hence we have similarly $dim_K A_k = dim_K A_{m-k}$ and Γ_G (and hence Γ) is QF.

Now we can extend our consideration for Λ to the case of $n = t$. Then as $\frac{1}{x_i}\prod_{j=0}^{n} x_j \equiv x_i^n$ mod I for $i = 1, 2, \cdots, n$, Λ is a positive Z-graded with respect to homogeneous elements $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n$ of degree 1. If we put $u = \prod_{i=1}^{n} \bar{x}_i$, then $K[u]$ is a subalgebra of Λ, which is a polynomial ring over K of a variable u . Further all $\bar{x}_i$'s satisfy the equation $X^{n+1} - u = 0 \in K[u][X]$. Hence by Noether's normalization theorem the Krull dimension of $\Lambda = 1$.

Let $t = n = 1$, then the generators of I are formally $\{x_0 - \frac{1}{x_0}x_0x_1 = x_0 - x_1, x_1 - \frac{1}{x_1}x_0x_1 = x_1 - x_0\} = \{x_0 - x_1\}$ and Λ is a polynomial ring of one variable and is obviously Gorenstein.

Let $t = n = 2$. Then $\{f_2 = x_0x_1 - x_2^2, f_0 = x_1x_2 - x_0^2, f_1 = x_2x_0 - x_1^2\}$ defines an intersection of quadratic cones. In [9] Stanley commented that the following Theorm 2.1 was proved first by Macauley.

Theorem 2.1. *If a K-algebra Λ is standard positively Z-graded and Gorenstein of Krull dimension d, then for Poincarè series $F(\Lambda, \lambda)$ it holds that $F(\Lambda, \frac{1}{\lambda}) = (-1)^d \lambda^\rho F(\Lambda, \lambda)$ (as rational functions of λ) for some integer ρ.*

By the Buchberger's algorithm we obtain the reduced Gröbner bases $\{f_0, f_1, f_2, f_3 = S(f_0, f_1) = -x_0^3 - x_1^3\}$ of I with respect to the degree- lexicographical order $x_0 < x_1 < x_2$.

As the leading terms are $Lt(f_0) = x_0^0x_1^1x_2^1, Lt(f_1) = x_0^1x_1^0x_2^1, Lt(f_2) = x_0^0x_1^0x_2^2, Lt(f_3) = x_0^0x_1^3x_2^0$ it holds $\alpha_1 < 3, \alpha_2 < 2$ for the standard bases $\bar{x}_0^{\alpha_0}\bar{x}_1^{\alpha_1}\bar{x}_2^{\alpha_2} \in \Lambda = K[x_0, x_1, x_2]/I$.

Therefore we know that

$$\{\bar{1}, \bar{x}_0, \bar{x}_0^2, \bar{x}_0^3, \cdots, \bar{x}_1, \bar{x}_0\bar{x}_1, \bar{x}_0^2\bar{x}_1, \bar{x}_0^3\bar{x}_1, \cdots, \bar{x}_1^2, \bar{x}_0\bar{x}_1^2, \bar{x}_0^2\bar{x}_1^2, \bar{x}_0^3\bar{x}_1^2, \cdots, \bar{x}_2\}$$

are the K-bases of Λ. Cf. [1:Theorem 1.7.4 and Proposition 2.1.6].

$A_0 = K\,\bar{1}$, $A_1 = K\,\bar{x}_0 + K\,\bar{x}_1 + K\,\bar{x}_2$, $A_2 = K\,\bar{x}_0^2 + K\,\bar{x}_1\bar{x}_0 + K\,\bar{x}_1^2$,

$A_n = K\,\bar{x}_0^n + K\,\bar{x}_1\bar{x}_0^{n-1} + K\,\bar{x}_1^2\bar{x}_0^{n-2}$ for $n \geq 3$,

are $Z^+ \cup \{0\}$-grading of Λ and the set of homogeneous generators is $\{\bar{x}_0, \bar{x}_1, \bar{x}_2\}$ with degree 1.

Thus the Poincarė series $F(\Lambda, \lambda) = 1 + \sum_{n=1}^{\infty} 3\lambda^n = \frac{3}{1-\lambda} - 2 = \frac{2\lambda+1}{1-\lambda}$.

Now there is no ρ which satisfies $(-1)^1\lambda^\rho F(\Lambda, \lambda) = (-1)\lambda^\rho \frac{2\lambda+1}{(1-\lambda)} =$

$= \frac{\lambda+2}{(\lambda-1)} = \frac{2\frac{1}{\lambda}+1}{1-\frac{1}{\lambda}} = F(\Lambda, \frac{1}{\lambda})$. Therefore by Theorem 2.1 Λ is not Gorenstein.

By the way we notice that in this case Λ is Cohen-Macaulay because
$\Lambda = K[\bar{x}_0] \oplus K[\bar{x}_0]\ \bar{x}_1 \oplus K[\bar{x}_0]\ \bar{x}_2$ is $K[\bar{x}_0]$-free module. Here we notice that
$K[\bar{x}_0]\ \bar{x}_1^2 \subset K[\bar{x}_0]\ \bar{x}_2$ by $x_0x_2 \equiv x_1^2 \bmod I$.

This arises a new question whether Λ is Cohen-Macaulay.

In order to answer the question let us consider Λ for $n = t = 3$. In this case the binomials $\{f_3 = x_0x_1x_2 - x_3^3, f_0 = x_1x_2x_3 - x_0^3, f_1 = x_2x_3x_0 - x_1^3, f_2 = x_3x_0x_1 - x_2^3\}$ generates I and by using the Buchberger's algorithm we obtain the following Gröbner bases
$Gr = \{f_0, f_1, f_2, f_3, f_4 = S(f_0, f_1) = x_0^4 - x_1^4, f_5 = S(f_1, f_2) = x_1^4 - x_2^4,$
$f_6 = S(f_0, f_3) = x_0x_1^2x_2^2 - x_0^3x_3^2, f_7 = S(f_1, f_3) = x_0^2x_1x_2^2 - x_1^3x_3^2,$
$f_8 = S(f_2, f_3) = x_0^2x_1^2x_2 - x_2^3x_3^2, f_9 = S(f_2, S(f_0, f_1)) = x_1^3x_2^3 - x_0^5x_3\}$ of I
and the leading terms $\{Lt(f_0) = x_1x_2x_3, Lt(f_1) = x_2x_3x_0, Lt(f_2) = x_3x_0x_1, Lt(f_3) = x_3^3,$ $Lt(f_4) = x_1^4, Lt(f_5) = x_2^4, Lt(f_6) = x_0^3x_3^2, Lt(f_7) = x_1^3x_3^2, Lt(f_8) = x_2^3x_3^2, Lt(f_9) = x_0^5x_3\}$ with respect to the degree-lexicographical order $x_0 < x_1 < x_2 < x_3$.

Now there is no positive integer n such that $x_0^n \xrightarrow{Gr}_+ 0$. Therefore $K[\bar{x}_0]$ is a polynomial ring in the variable $\bar{x}_0$. Further $f_6 = x_0x_1^2x_2^2 - x_0^3x_3^2 = x_0(x_1^2x_2^2 - x_0^2x_3^2) \in I$ and $(x_1^2x_2^2 - x_0^2x_3^2) \notin I$ because the terms $x_1^2x_2^2$ and $x_0^2x_3^2$ are not reduced by any Gröbner base. Hence $\bar{x}_0\ (\bar{x}_1^2\bar{x}_2^2 - \bar{x}_0^2\bar{x}_3^2) = \bar{0}$, but $(\bar{x}_1^2\bar{x}_2^2 - \bar{x}_0^2\bar{x}_3^2) \neq \bar{0}$.

Hence Λ is not a $K[\bar{x}_0]$-free module. This implies Λ is not Cohen-Macaulay.

As all generators of I are binomials, Λ may be a toric variety. Cf. [10].
Toric varieties are defined to be Noetherian integral domains. However as we prove just now Λ has a zero divisor we cannot expect that Λ is toric.

Proposition 2.2. *If $n = t = 3$, then Λ is neither Cohen-Macaulay nor toric. Of course Λ is not Gorenstein.*

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A GENERALIZATION OF n-TORSIONFREE MODULES

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Abstract. We consider in this paper two approximation theorems for finitely generated modules over a commutative noetherian ring; one is due to Auslander and Bridger, and the other is due to Auslander and Buchweitz. We shall give a result which implies both of these two theorems.

Key Words: Torsionfree, Semidualizing, Cohen-Macaulay approximation, Contravariantly finite.

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1. Introduction

In the late 1960s, Auslander and Bridger [2] constructed the notion of a certain approximation, which we call in this paper a spherical approximation. This notion says that each of the modules whose nth syzygies are n-torsionfree is described by using an n-spherical module and a module of projective dimension less than n. On the other hand, about two decades later, the notion of a Cohen-Macaulay approximation was introduced and developed by Auslander and Buchweitz [3]. This notion says that over a Cohen-Macaulay local ring with a canonical module, the category of finitely generated modules is obtained by glueing together the subcategory of maximal Cohen-Macaulay modules and the subcategory of modules of finite injective dimension. Cohen-Macaulay approximations have been playing an important role in commutative algebra. In this paper, we set our sight on these two notions. More precisely, we shall consider and generalize the following two theorems.

Theorem 1 (Auslander-Bridger). *The following are equivalent for a finitely generated module M over a commutative noetherian ring R:*

1. $\Omega^n M$ *is n-torsionfree;*
2. *There exists an exact sequence* $0 \to Y \to X \to M \to 0$ *of finitely generated R-modules such that* $\mathrm{Ext}_R^i(X, R) = 0$ *for* $1 \leq i \leq n$ *and* $\mathrm{pd}\, Y < n$.

Theorem 2 (Auslander-Buchweitz). *Let R be a Cohen-Macaulay local ring with a canonical module. Then for every finitely generated R-module M there exists an exact sequence* $0 \to Y \to X \to M \to 0$ *of finitely generated R-modules such that X is maximal Cohen-Macaulay and* $\mathrm{id}\, Y < \infty$.

The detailed version of this paper has been submitted for publication elsewhere.

2. The existence of n-C-spherical approximations

Throughout the present paper, R is always a commutative noetherian ring, and all R-modules are finitely generated. Auslander and Bridger [2] introduced the notion of an n-torsionfree module.

Definition 3. Let n be an integer. An R-module M is called *n-torsionfree* if $\mathrm{Ext}_R^i(\mathrm{Tr}M, R) = 0$ for $1 \le i \le n$.

In this paper, unless otherwise specified, we always denote by n a positive integer, by C an R-module, by $(-)^\dagger$ the C-dual functor $\mathrm{Hom}_R(-, C)$ and by λ_M the natural homomorphism $M \to M^{\dagger\dagger}$ for an R-module M. Note that λ_R can be identified with the homothety map $R \to \mathrm{Hom}_R(C, C)$. We can generalize the notion of an n-torsionfree module as follows.

Definition 4. Let M be an R-module. We say that M is *1-C-torsionfree* if λ_M is a monomorphism. We say that M is *n-C-torsionfree*, where $n \ge 2$, if λ_M is an isomorphism and $\mathrm{Ext}_R^i(M^\dagger, C) = 0$ for all $1 \le i \le n-2$.

We denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}\, R$. An R-homomorphism $f : X \to M$ is called a *right $\mathcal{X}$-approximation* of M if X belongs to $\mathcal{X}$ and the sequence $\mathrm{Hom}_R(-, X) \xrightarrow{(-,f)} \mathrm{Hom}_R(-, M) \to 0$, where $(-, f) = \mathrm{Hom}_R(-, f)$, is exact on $\mathcal{X}$. We say that $\mathcal{X}$ is *contravariantly finite* if any $X \in \mathcal{X}$ has a right $\mathcal{X}$-approximation. For an R-module X, we denote by $\mathrm{add}\, X$ the full subcategory of $\mathrm{mod}\, R$ consisting of all direct summands of finite direct sums of copies of X.

To develop the notion of an n-C-torsionfree module to the utmost extent, we establish the following definition.

Definition 5. We say that C is *1-semidualizing* if λ_R is a monomorphism and $\mathrm{Ext}_R^1(C, C) = 0$. We say that C is *n-semidualizing*, where $n \ge 2$, if λ_R is an isomorphism and $\mathrm{Ext}^i(C, C) = 0$ for all $1 \le i \le n$.

The following proposition, which is essentially proved in [6, Proposition 2.5.1], says that there are a lot of n-semidualizing modules.

Proposition 6. *Let R be a Cohen-Macaulay local ring of dimension $d \ge 2$ with an isolated singularity. Let I be an ideal of R which is a maximal Cohen-Macaulay R-module. Then λ_R is an isomorphism and $\mathrm{Ext}_R^i(I, I) = 0$ for every $1 \le i \le d-2$. Hence R is d-I-torsionfree, and I is $(d-2)$-semidualizing.*

For an R-module M, we define $C\mathrm{dim}_R\, M$, the add$\,C$*-resolution dimension* of M, to be the infimum of nonnegative integers n such that there exists an exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$ with each C_i being in $\mathrm{add}\, C$. Note that add$\,R$-resolution dimension is the same as projective dimension. We make the following definition.

Definition 7. Let M be an R-module.
(1) We say that M is *n-spherical* if $\mathrm{Ext}^i(M, R) = 0$ for all $1 \le i \le n$. We call an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules an *n-spherical approximation* if X is n-spherical and $\mathrm{pd}\, Y < n$.

(2) We say that M is *n-C-spherical* if $\mathrm{Ext}^i(M, C) = 0$ for all $1 \le i \le n$. We call an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules an *n-C-spherical approximation* if X is n-C-spherical and $C\dim Y < n$.

We notice that an n-C-spherical approximation gives a right approximation:

Proposition 8. *Define two full subcategories of* $\mathrm{mod}\, R$ *as follows:*

$$\mathcal{X} = \{\, X \in \mathrm{mod}\, R \mid X \text{ is } n\text{-}C\text{-spherical} \},$$
$$\mathcal{Y} = \{\, Y \in \mathrm{mod}\, R \mid C\dim Y < n \,\}.$$

Let $0 \to Y \to X \xrightarrow{f} M \to 0$ be an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then the homomorphism f is a right $\mathcal{X}$-approximation of M.

We give here a lemma.

Lemma 9. *Suppose that $\mathrm{Ext}^1(C, C) = 0$. An R-module M is 1-C-torsionfree if and only if there is an exact sequence $0 \to M \to C_0 \to N \to 0$ such that $C_0 \in \mathrm{add}\, C$ and $\mathrm{Ext}^1(N, C) = 0$.*

Now, we can state and prove the main result of this section.

Theorem 10. *Let C be an n-semidualizing R-module. The following are equivalent for an R-module M:*

(1) *$\Omega^n M$ is n-C-torsionfree;*
(2) *M admits an n-C-spherical approximation.*

Proof. Let $P_\bullet$ be a projective resolution of M.

(1) $\Rightarrow$ (2): We have an exact sequence $0 \to \Omega^{i+1} M \to P_i \to \Omega^i M \to 0$ for each i. Set $X_0 = \Omega^n M$. Note that X_0 is n-C-torsionfree. Lemma 9 implies that there exists an exact sequence $0 \to X_0 \to C_0 \to Z_1 \to 0$ such that $C_0 \in \mathrm{add}\, C$ and $\mathrm{Ext}^1(Z_1, C) = 0$. We make the pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_0 & \longrightarrow & C_0 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P_{n-1} & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{n-1} M & = & \Omega^{n-1} M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Ext}^1(Z_1, C) = 0 = \mathrm{Ext}^1(P_{n-1}, C)$, we have $\mathrm{Ext}^1(X_1, C) = 0$. If $n = 1$, then the middle column is a desired exact sequence.

Let $n \geq 2$. We can easily check that Z_1 is $(n-1)$-C-torsionfree, and that so is X_1. According to Lemma 9, there is an exact sequence $0 \to X_1 \to C_1 \to Z_2 \to 0$ with $C_1 \in \operatorname{add} C$ and $\operatorname{Ext}^1(Z_2, C) = 0$. We make the pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & C_0 & = & C_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_1 & \longrightarrow & C_1 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{n-1}M & \longrightarrow & Y_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Using the bottom row of the above diagram, we make the pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^{n-1}M & \longrightarrow & Y_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P_{n-2} & \longrightarrow & X_2 & \longrightarrow & Z_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{n-2}M & = & \Omega^{n-2}M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

From the first diagram, we immediately get $C\dim Y_2 < 2$, and $\operatorname{Ext}^2(Z_2, C) = 0$ because $\operatorname{Ext}^1(X_1, C) = 0 = \operatorname{Ext}^2(C_1, C)$. Hence $\operatorname{Ext}^i(Z_2, C) = 0$ for $i = 1, 2$, and we see from the middle row of the second diagram that $\operatorname{Ext}^i(X_2, C) = 0$ for $i = 1, 2$. Thus, if $n = 2$, then the middle column of the second diagram is a desired exact sequence.

Let $n \geq 3$. Then similar arguments to the above claims show that both Z_2 and X_2 are $(n-2)$-C-torsionfree, and Lemma 9 yields an exact sequence $0 \to X_2 \to C_2 \to Z_3 \to 0$

such that $\mathrm{Ext}^1(Z_3, C) = 0$. Similarly to the above, we make two pushout diagrams:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_2 & = & Y_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_2 & \longrightarrow & C_2 & \longrightarrow & Z_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega^{n-2}M & \longrightarrow & Y_3 & \longrightarrow & Z_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

and

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega^{n-2}M & \longrightarrow & Y_3 & \longrightarrow & Z_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P_{n-3} & \longrightarrow & X_3 & \longrightarrow & Z_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{n-3}M & = & \Omega^{n-3}M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

If $n = 3$, then the middle column of the second diagram is a desired exact sequence. If $n \geq 4$, then iterating this procedure, we eventually obtain an exact sequence $0 \to Y_n \to X_n \to M \to 0$ such that $\mathrm{Ext}^i(X_n, C) = 0$ for $1 \leq i \leq n$ and $C\dim Y_n < n$.

(2) $\Rightarrow$ (1): Let $0 \to Y \to X \to M \to 0$ be an n-C-spherical approximation of M. Since $C\dim Y < n$, there exists an exact sequence $0 \to C_{n-1} \xrightarrow{d_{n-1}} C_{n-1} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} C_0 \xrightarrow{d_0} Y \to 0$. Put $Y_i = \mathrm{Im}\, d_i$ for each i. We have exact sequences $0 \to Y_{i+1} \to C_i \to Y_i \to 0$ and

$0 \to \Omega^{i+1}M \to P_i \to \Omega^i M \to 0$ for each i. The following pullback diagram is obtained:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega M & = & \Omega M & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & L & \longrightarrow & P_0 & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The projectivity of P_0 shows that the middle row splits; we have an isomorphism $L \cong Y \oplus P_0$. Adding P_0 to the exact sequence $0 \to Y_1 \to C_0 \to Y \to 0$, we get an exact sequence $0 \to Y_1 \to C_0 \oplus P_0 \to Y \oplus P_0 \to 0$. Thus the following pullback diagram is obtained:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Y_1 & = & Y_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_1 & \longrightarrow & C_0 \oplus P_0 & \longrightarrow & X & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & Y \oplus P_0 & \longrightarrow & X & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Applying a similar argument to the left column of the above diagram, we get exact sequences $0 \to X_{i+1} \to C_i \oplus P_i \to X_i \to 0$ for $0 \leq i \leq n-1$, where $X_0 = X$ and $X_n = \Omega^n M$. The assumption yields $\mathrm{Ext}^i(X_0, C) = 0 = \mathrm{Ext}^i(C_0 \oplus P_0, C)$ for $1 \leq i \leq n$, hence we have an exact sequence $0 \to X_0^\dagger \to (C_0 \oplus P_0)^\dagger \to X_1^\dagger \to 0$ and $\mathrm{Ext}^1(X_1, C) = 0$ for $1 \leq i \leq n-1$. Inductively, for each $0 \leq i \leq n-1$ an exact sequence $0 \to X_i^\dagger \to (C_i \oplus P_i)^\dagger \to X_{i+1}^\dagger \to 0$ is obtained and $\mathrm{Ext}^j(X_i, C) = 0$ for $1 \leq j \leq n-i$. We have a commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & X_1 & \longrightarrow & C_0 \oplus P_0\\
 & & {\scriptstyle \lambda_{X_1}}\downarrow & & {\scriptstyle \lambda_{C_0 \oplus P_0}}\downarrow\\
0 & \longrightarrow & X_1^{\dagger\dagger} & \longrightarrow & (C_0 \oplus P_0)^{\dagger\dagger}
\end{array}$$

with exact rows. The assumption says that λ_R is injective, and we see that $\lambda_C = \lambda_{R^\dagger}$ is injective. Hence the map $\lambda_{C_0 \oplus P_0}$ is injective, and so is λ_{X_1}. Therefore X_1 is 1-C-torsionfree. If $n \geq 2$, then λ_R is an isomorphism, and so is λ_C. There is a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_2 & \longrightarrow & C_1 \oplus P_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
 & & {\scriptstyle \lambda_{X_2}}\downarrow & & {\scriptstyle \lambda_{C_1 \oplus P_1}}\downarrow & & {\scriptstyle \lambda_{X_1}}\downarrow & & \\
0 & \longrightarrow & X_2^{\dagger\dagger} & \longrightarrow & (C_1 \oplus P_1)^{\dagger\dagger} & \longrightarrow & X_1^{\dagger\dagger} & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

with exact rows and columns, and λ_{X_2} is an isomorphism by the snake lemma. Hence X_2 is 2-C-torsionfree. If $n \geq 3$, then we have a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & X_3 & \longrightarrow & C_2 \oplus P_2 & \longrightarrow & X_2 & \longrightarrow & 0 & & \\
 & & {\scriptstyle \lambda_{X_3}}\downarrow & & {\scriptstyle \lambda_{C_2 \oplus P_2}}\downarrow\cong & & {\scriptstyle \lambda_{X_2}}\downarrow\cong & & & & \\
0 & \longrightarrow & X_3^{\dagger\dagger} & \longrightarrow & (C_2 \oplus P_2)^{\dagger\dagger} & \longrightarrow & X_2^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}^1(X_3^\dagger, C) & \longrightarrow & 0
\end{array}$$

with exact rows. From this diagram it follows that λ_{X_3} is an isomorphism and $\mathrm{Ext}^1(X_3^\dagger, C) = 0$, which means that X_3 is 3-C-torsionfree. Repeating a similar argument, we see that X_i is i-C-torsionfree for every $1 \leq i \leq n$. Therefore $\Omega^n M = X_n$ is n-C-torsionfree, and the proof of the theorem is completed. $\square$

Theorem 1 is a direct corollary of Theorem 10. Theorem 2 is also a corollary of Theorem 10:

Proof of Theorem 2. If $d = 0$, then $0 \to 0 \to M \xrightarrow{=} M \to 0$ is a desired exact sequence. Let $d \geq 1$. Then ω is d-semidualizing, and $\Omega^d M$ is d-ω-torsionfree. Hence Theorem 10 guarantees the existence of an exact sequence $0 \to Y \to X \to M \to 0$ such that X is d-ω-spherical and $\omega\mathrm{dim}\, Y < d$. Therefore X is maximal Cohen-Macaulay. On the other hand, noting that ω is an indecomposable R-module, we have an exact sequence $0 \to \omega^{l_{d-1}} \to \omega^{l_{d-2}} \to \cdots \to \omega^{l_0} \to Y \to 0$. Decomposing this into short exact sequences and noting that ω has finite injective dimension, one sees that Y also has finite injective dimension. $\square$

3. Modules whose nth syzygies are n-C-torsionfree

We begin with stating the following lemma.

Lemma 11. *Let M be an R-module.*

(1) *If R is 1-C-torsionfree, then so is ΩM.*
(2) *If R is 2-C-torsionfree, then for each $n \geq 2$ the map $\lambda_{\Omega^n M}$ is a split monomorphism and the cokernel is isomorphic to $\mathrm{Ext}^n(M, C)^\dagger$.*

For R-modules M, N, we define $\operatorname{grade}(M,N)$ by the infimum of integers i such that $\operatorname{Ext}^i(M,N) \neq 0$. One has $\operatorname{grade}(M,N) = \inf\{\operatorname{depth} N_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp} M\}$. We state a criterion for $\Omega^i M$ to be n-C-torsionfree for $1 \leq i \leq n$ in terms of grade, which can be shown by Lemma 11 and induction on n.

Proposition 12. *Let C be an R-module such that R is $(n-1)$-C-torsionfree.*

(1) *If $\Omega^i M$ is i-C-torsionfree for every $1 \leq i \leq n$, then* $\operatorname{grade}(\operatorname{Ext}^i(M,C),C) \geq i-1$ *for every $1 \leq i \leq n$.*
(2) *The converse also holds if R is n-C-torsionfree.*

Now we want to consider the difference between this condition and the condition that $\Omega^n M$ is n-C-torsionfree.

Lemma 13. *Let C be an R-module such that λ_R is an isomorphism and* $\operatorname{Ext}^i(C,C) = 0$ *for $1 \leq i \leq n$. If M is an R-module with $C\dim M < \infty$, then* $\operatorname{grade}(\operatorname{Ext}^i(M,C),C) \geq i$ *for any $1 \leq i \leq n$.*

Using this lemma, we can show that under the assumption that C is n-semidualizing, $\Omega^i M$ is i-C-torsionfree for $1 \leq i \leq n$ if and only if $\Omega^n M$ is n-C-torsionfree.

Proposition 14. *Let C be an n-semidualizing R-module. The following are equivalent for an R-module M:*

(1) *$\Omega^n M$ is n-C-torsionfree;*
(2) *$\Omega^i M$ is i-C-torsionfree for every $1 \leq i \leq n$.*

Our next aim is to prove the main result of this section. For this, we introduce the following lemma, which will often be used later.

Lemma 15. *Let R be a local ring and r a positive integer. Suppose that λ_R is an isomorphism and* $\operatorname{Ext}^i(C,C) = 0$ *for all $1 \leq i < r$. Then the following hold.*

(1) $\operatorname{depth} R \geq r$ *if and only if* $\operatorname{depth} C \geq r$.
(2) *Let R be a Cohen-Macaulay local ring with* $\dim R < r$. *Then C is a maximal Cohen-Macaulay R-module.*

Now we can prove the main result of this section.

Theorem 16. *Suppose that R is n-C-torsionfree. Then the following are equivalent:*

(1) $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{depth} R_\mathfrak{p} \leq n-2$;
(2) *$\Omega^n M$ is n-C-torsionfree for any R-module M.*

Proof. When $n = 1$, the assertion (1) holds because there is no prime ideal $\mathfrak{p}$ of R satisfying $\operatorname{depth} R_\mathfrak{p} \leq n-2$, and the assertion (2) holds by Lemma 11(1). In the following, we consider the case where $n \geq 2$.

(1) $\Rightarrow$ (2): Fix an R-module M. Induction hypothesis shows that $\Omega^i M$ is i-C-torsionfree for $1 \leq i \leq n-1$. By Proposition 12, we have $\operatorname{grade}(\operatorname{Ext}^i(M,C),C) \geq i-1$ for $1 \leq i \leq n-1$, and it suffices to prove that the inequality $\operatorname{grade}(\operatorname{Ext}^n(M,C),C) \geq n-1$ holds. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $\operatorname{depth} C_\mathfrak{p} \leq n-2$, then $\operatorname{depth} R_\mathfrak{p} \leq n-2$ by Lemma 15(1). The assumption says that $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$, and $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} \leq n-2$. Therefore $\operatorname{Ext}^n_R(M,C)_\mathfrak{p} \cong \operatorname{Ext}^n_{R_\mathfrak{p}}(M_\mathfrak{p},C_\mathfrak{p}) = 0$. Thus we see that $\operatorname{grade}(\operatorname{Ext}^n(M,C),C) \geq n-1$, as desired.

(2) $\Rightarrow$ (1): When $n = 2$, Lemma 11(2) implies that $\mathrm{Ext}^2_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) = 0$ for all R-modules M and $\mathfrak{p} \in \mathrm{Ass}\, C$, because $\mathrm{Ass}(\mathrm{Ext}^2_R(M, C)^\dagger) = \mathrm{Supp}\,\mathrm{Ext}^2_R(M, C) \cap \mathrm{Ass}\, C$. The isomorphism $\lambda_R : R \to \mathrm{Hom}(C, C)$ shows that $\mathrm{Ass}\, C$ coincides with $\mathrm{Ass}\, R$. Hence, setting $M = \Omega^i_R(R/\mathfrak{p})$, one has $\mathrm{Ext}^{i+2}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), C_\mathfrak{p}) \cong \mathrm{Ext}^2_{R_\mathfrak{p}}((\Omega^i_R(R/\mathfrak{p}))_\mathfrak{p}, C_\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \mathrm{Ass}\, R$ and any $i > 0$. Therefore $\mathrm{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$ for $\mathfrak{p} \in \mathrm{Spec}\, R$ with $\mathrm{depth}\, R_\mathfrak{p} = 0$.

Let $n \geq 3$. Fix an R-module M. We have an exact sequence $0 \to \Omega^{n+1}M \to P \to \Omega^n M \to 0$ such that P is a projective R-module. From this we get another exact sequence $0 \to (\Omega^n M)^\dagger \to P^\dagger \to (\Omega^{n+1}M)^\dagger \to \mathrm{Ext}^{n+1}(M, C) \to 0$. Decompose this into short exact sequences:

$$(1)\qquad \begin{cases} 0 \to (\Omega^n M)^\dagger \to P^\dagger \to N \to 0, \\ 0 \to N \to (\Omega^{n+1}M)^\dagger \to \mathrm{Ext}^{n+1}(M, C) \to 0. \end{cases}$$

Note from the assumption that both $\Omega^n M$ and $\Omega^{n+1}M = \Omega^n(\Omega M)$ are n-C-torsionfree. Since R is n-C-torsionfree, we see from the first sequence in (1) that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{n+1}M & \longrightarrow & P & \longrightarrow & \Omega^n M & \longrightarrow & 0 \\ & & \Big\downarrow \alpha & & \Big\downarrow \lambda_P\, \cong & & \Big\downarrow \lambda_{\Omega^n M}\, \cong & & \\ 0 & \longrightarrow & N^\dagger & \longrightarrow & P^{\dagger\dagger} & \longrightarrow & (\Omega^n M)^{\dagger\dagger} & \longrightarrow & \mathrm{Ext}^1(N, C) \longrightarrow 0 \end{array}$$

with exact rows, and $\mathrm{Ext}^i(N, C) = 0$ for $2 \leq i \leq n - 2$. This diagram shows that α is an isomorphism and $\mathrm{Ext}^1(N, C) = 0$. The second sequence in (1) gives an exact sequence $0 \to \mathrm{Ext}^{n+1}(M, C)^\dagger \to (\Omega^{n+1}M)^{\dagger\dagger} \xrightarrow{\beta} N^\dagger \to \mathrm{Ext}^1(\mathrm{Ext}^{n+1}(M, C), C) \to 0$ and $\mathrm{Ext}^i(\mathrm{Ext}^{n+1}(M, C), C) = 0$ for $2 \leq i \leq n - 2$. Since the diagram

$$\begin{array}{ccc} \Omega^n M & = & \Omega^n M \\ \Big\downarrow \lambda_{\Omega^n M}\, \cong & & \Big\downarrow \alpha\, \cong \\ (\Omega^n M)^{\dagger\dagger} & \xrightarrow{\beta} & N^\dagger \end{array}$$

commutes, the map β is an isomorphism, and $\mathrm{Ext}^{n+1}(M, C)^\dagger = 0 = \mathrm{Ext}^1(\mathrm{Ext}^{n+1}(M, C), C)$. Thus we have $\mathrm{Ext}^i(\mathrm{Ext}^{n+1}(M, C), C) = 0$ for every $i \leq n - 2$, which means that the inequality $\mathrm{grade}(\mathrm{Ext}^{n+1}(M, C), C) \geq n - 1$ holds. Therefore, if $\mathfrak{p}$ is a prime ideal of R with $\mathrm{depth}\, R_\mathfrak{p} \leq n - 2$, then $\mathrm{depth}\, C_\mathfrak{p} \leq n - 2$ by Lemma 15(1), and it follows that $\mathfrak{p}$ does not belong to $\mathrm{Supp}\,\mathrm{Ext}^{n+1}_R(M, C)$, i.e., $\mathrm{Ext}^{n+1}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) = 0$. Putting $M = \Omega^i_R(R/\mathfrak{p})$, we obtain $\mathrm{Ext}^{n+1+i}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), C_\mathfrak{p}) \cong \mathrm{Ext}^{n+1}_{R_\mathfrak{p}}((\Omega^i_R(R/\mathfrak{p}))_\mathfrak{p}, C_\mathfrak{p}) = 0$ for any $i > 0$. This implies that $\mathrm{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$, and the proof is completed. $\square$

The lemma below says that over a Gorenstein local ring of dimension $d \geq 2$, any n-semidualizing module is free for $n \geq d$.

Lemma 17. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Gorenstein local ring. If λ_R is an isomorphism and $\mathrm{Ext}^i(C, C) = 0$ for $1 \leq i \leq d$, then $C \cong R$.*

Applying the above lemma, we can get a sufficient condition for R and C to satisfy Theorem 16(1).

Proposition 18. *Suppose that R is n-C-torsionfree and that $R_\mathfrak{p}$ is Gorenstein for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \le n-2$. Then $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$ for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \le n-2$. (Hence $\Omega^n M$ is n-C-torsionfree for any R-module M.)*

We have studied the case where the nth syzygies of all R-modules are n-C-torsionfree. As the last result of this paper, we give a result concerning when the nth syzygy of a given module is n-C-torsionfree.

Proposition 19. *Let M be an R-module, and let C be an R-module such that R is n-C-torsionfree. Suppose that $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \le n-2$. Then $\Omega^n M$ is n-C-torsionfree.*

This proposition can be proved by induction on n. Apply Proposition 12, Lemmas 11(1) and 15(1).

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CASTELNUOVO-MUMFORD REGULARITY FOR COMPLEXES AND RELATED TOPICS

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ABSTRACT. Let A be a noetherian AS regular Koszul quiver algebra (if A is commutative, it is essentially a polynomial ring), and $\operatorname{gr} A$ the category of finitely generated graded left A-modules. Following Jørgensen, we define the Castelnuovo-Mumford regularity $\operatorname{reg}(M^\bullet)$ of a complex $M^\bullet \in D^b(\operatorname{gr} A)$ in terms of the local cohomologies or the minimal projective resolution of $M^\bullet$. Let $A^!$ be the quadratic dual ring of A. Then $A^!$ is selfinjective and Koszul (e.g., if A is the polynomial ring $k[x_1, \dots, x_d]$, then $A^!$ is the exterior algebra $E = \bigwedge \langle y_1, \dots, y_d \rangle$). For the Koszul duality functor $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$, we have $\operatorname{reg}(M^\bullet) = \max\{ i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \}$. As an application, we refine results of Martinez-Villa and Zacharia on *weakly Koszul modules* over $A^!$ (especially, over E).

1. INTRODUCTION

Let $A := \bigoplus_{i \geq 0} A_i$ be a noetherian AS regular Koszul quiver algebra over a field k. Such a quiver algebra (with relation) has been studied by Martinez-Villa and coworkers (c.f. [6, 9, 10, 11]). And a *connected* (i.e., $A_0 = k$) AS regular algebra is very important in non-commutative algebraic geometry (c.f. [18]). If A is commutative and connected, it is a polynomial ring $k[x_1, \dots, x_d]$ with $\deg x_i = 1$ for each i.

Let $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) be the category of graded left (resp. right) A-modules, and $\operatorname{gr} A$ (resp. $\operatorname{gr} A^{\mathsf{op}}$) its full subcategory consisting of finitely generated modules. Set $\mathfrak{r} := \bigoplus_{i \geq 1} A_i$ to be the graded Jacobson radical. We have the left exact functor $\Gamma_{\mathfrak{r}} : \operatorname{Gr} A \to \operatorname{Gr} A$ defined by $\Gamma_{\mathfrak{r}}(M) = \{ x \in M \mid \mathfrak{r}^n x = 0 \text{ for } n \gg 0 \}$, and its right derived functor $R\Gamma_{\mathfrak{r}} : D^b(\operatorname{Gr} A) \to D^b(\operatorname{Gr} A)$. For $M^\bullet \in D^b(\operatorname{Gr} A)$, the i^{th} cohomology of $R\Gamma_{\mathfrak{r}}(M^\bullet)$ is denoted by $H^i_{\mathfrak{r}}(M^\bullet)$. Similarly, we have the corresponding functors $\Gamma_{\mathfrak{r}^{\mathsf{op}}}$, $R\Gamma_{\mathfrak{r}^{\mathsf{op}}}$, and $H^i_{\mathfrak{r}^{\mathsf{op}}}$ for graded *right* A-modules. When A is a polynomial ring $k[x_1, \dots, x_d]$, $H^i_{\mathfrak{r}}(-)$ is known as the *local cohomology module* with support in the graded maximal ideal $\mathfrak{r}$.

We have a bounded cochain complex $\mathcal{D}^\bullet$ of graded A-A bimodules which gives duality functors $R\operatorname{Hom}_A(-, \mathcal{D}^\bullet) : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^{\mathsf{op}})$ and $R\operatorname{Hom}_{A^{\mathsf{op}}}(-, \mathcal{D}^\bullet) : D^b(\operatorname{gr} A^{\mathsf{op}}) \to D^b(\operatorname{gr} A)$. These functors are quasi-inverse of each other. Moreover, we have "local duality theorem"

$$R\underline{\operatorname{Hom}}_A(-, \mathcal{D}^\bullet) \cong R\Gamma_{\mathfrak{r}}(-)^\vee \quad \text{and} \quad R\underline{\operatorname{Hom}}_{A^{\mathsf{op}}}(-, \mathcal{D}^\bullet) \cong R\Gamma_{\mathfrak{r}^{\mathsf{op}}}(-)^\vee,$$

where $(-)^\vee$ stands for the graded k-dual. This is a quiver algebra version of [18].

For $M^\bullet \in D^b(\operatorname{gr} A)$ and $i, j \in \mathbb{Z}$, set $\beta^i_j(M^\bullet) := \dim_k \underline{\operatorname{Ext}}_A^{-i}(M^\bullet, A/\mathfrak{r})_{-j}$. Of course, $\beta^i_j(-)$ measures the "size" of a minimal projective resolution. Using the above duality, we can generalize a well-know result of Eisenbud-Goto [5] concerning graded modules over a polynomial ring.

This note is basically a summary of [17] which has been accepted for publication in *J. Pure Appl. Algebra.*

Definition-Theorem. (c.f. Jørgensen, [8]) *For $M^\bullet \in D^b(\mathrm{gr}\, A)$, we have*

$$\mathrm{reg}(M^\bullet) := \sup\{\, i + j \mid H^i_{\mathfrak{r}}(M^\bullet)_j \neq 0 \,\} = \sup\{\, i + j \mid \beta^i_j(M^\bullet) \neq 0 \,\} < \infty.$$

We call this value the "Castelnuovo-Mumford regularity" of $M^\bullet$.

For $M^\bullet \in D^b(\mathrm{gr}\, A)$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and the differential maps are zero. Then $\mathrm{reg}(\mathcal{H}(M^\bullet)) \geq \mathrm{reg}(M^\bullet)$. The difference $\mathrm{reg}(\mathcal{H}(M^\bullet)) - \mathrm{reg}(M^\bullet)$ is a theme of the latter half of this note.

Let $A^!$ be the quadratic dual ring of A. Then $A^!$ is finite dimensional, Koszul and self-injective by [10]. (e.g., If A is the polynomial ring $k[x_1, \dots, x_d]$, then $A^!$ is the exterior algebra $\bigwedge\langle y_1, \dots, y_d\rangle$). The Koszul duality functors $\mathcal{F} : D^b(\mathrm{gr}\, A^!) \to D^b(\mathrm{gr}\, A)$ and $\mathcal{G} : D^b(\mathrm{gr}\, A) \to D^b(\mathrm{gr}\, A^!)$ give an equivalence $D^b(\mathrm{gr}\, A) \cong D^b(\mathrm{gr}\, A^!)$ (c.f. [2]). We have

$$\mathrm{reg}(M^\bullet) = \max\{\, i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \,\}.$$

For $N \in \mathrm{gr}\, A^!$ and $n \in \mathbb{Z}$, $N_{\langle n\rangle}$ denotes the submodule of N generated by the degree n component N_n. We say N is *weakly Koszul*, if $N_{\langle n\rangle}$ has an *n-linear projective resolution* (i.e., $\beta^i_j(N_{\langle n\rangle}) \neq 0 \Rightarrow i + j = n$) for all n. Martinez-Villa and Zacharia [11] proved that the i^{th} syzygy $\Omega_i(N)$ of $N \in \mathrm{gr}\, A^!$ is weakly Koszul for $i \gg 0$. Of course, the same is true for $N \in \mathrm{gr}\,(A^!)^{\mathsf{op}}$. Set $\mathrm{lpd}(N) := \min\{\, i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul} \,\}$.

Theorem. *Let $N \in \mathrm{gr}\, A^!$, and $N' := \underline{\mathrm{Hom}}_{A^!}(N, A^!) \in \mathrm{gr}\,(A^!)^{\mathsf{op}}$ its dual. Then*

$$\mathrm{lpd}(N') = \mathrm{reg}(\,\mathcal{H} \circ \mathcal{F}(N)\,).$$

If $A^!$ is the exterior algebra $E = \bigwedge\langle y_1, \dots, y_d\rangle$, we have an upper bound of $\mathrm{lpd}(N)$ depending only on $\max\{\dim_k N_i \mid i \in \mathbb{Z}\}$ and d. This bound gives huge numbers, and must be very far from optimal. On the other hand, we have $\mathrm{lpd}(E/J) \leq \min\{1, d-2\}$ for a monomial ideal J of E. This slightly improves a result of Herzog and Römer.

2. Preliminaries

First, we sketch basic properties of an algebra of a quiver with relations.

Let Q be a finite quiver. That is, $Q = (Q_0, Q_1)$ is a finite oriented graph, where Q_0 is the set of vertices and Q_1 is the set of arrows. The path algebra kQ is a positively graded algebra with grading given by the lengths of paths. Let J be the graded Jacobson radical of kQ (i.e., the ideal generated by all arrows). If $I \subset J^2$ is a graded ideal, we say $A = kQ/I$ is a *graded quiver algebra*. Of course, $A = \bigoplus_{i\geq 0} A_i$ is a graded ring. The subalgebra A_0 is a product of copies of the field k, one copy for each element of Q_0. If $A_0 = k$ (i.e., Q has only one vertex), we say A is *connected*. If a graded algebra $R = \bigoplus_{i\geq 0} R_i$ with $R_0 = k$ is generated by R_1 as a k-algebra and $\dim_k R_1 < \infty$, then it can be regarded as a graded quiver algebra. Set $\mathfrak{r} := \bigoplus_{i\geq 1} A_i$. Unless otherwise specified, we assume that A is left and right noetherian throughout this note.

Let $\mathrm{Gr}\, A$ (resp. $\mathrm{Gr}\, A^{\mathsf{op}}$) be the category of graded left (resp. right) A-modules, and $\mathrm{gr}\, A$ (resp. $\mathrm{gr}\, A^{\mathsf{op}}$) its full subcategory consisting of finitely generated modules. Since A is noetherian, $\mathrm{gr}\, A$ and $\mathrm{gr}\, A^{\mathsf{op}}$ are abelian categories. In the sequel, we will define several concepts for $\operatorname{Gr} A$, but the corresponding concepts for $\operatorname{Gr} A^{\mathsf{op}}$ can be defined in the similar way.

The n^{th} shift $M(n)$ of $M = \bigoplus_{i\in\mathbb{Z}} M_i \in \operatorname{Gr} A$ is defined by $M(n)_i = M_{n+i}$. Set $\iota(M) := \inf\{ i \mid M_i \neq 0 \}$.

For $v \in Q_0$, we have the idempotent e_v of A associated with v. Note that $1 = \sum_{v\in Q_0} e_v$. Set $P_v := Ae_v$ and ${}_vP := e_vA$. Then we have ${}_AA = \bigoplus_{v\in Q_0} P_v$ and $A_A = \bigoplus_{v\in Q_0}({}_vP)$. Each P_v and ${}_vP$ are indecomposable projectives. Conversely, any indecomposable projective in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) is isomorphic to P_v (resp. ${}_vP$) for some $v \in Q_0$ up to degree shifting. Set $k_v := P_v/(\mathfrak{r}P_v)$ and ${}_vk := {}_vP/({}_vP\,\mathfrak{r})$. Each k_v and ${}_vk$ are simple.

Let $C^b(\operatorname{Gr} A)$ be the category of bounded cochain complexes in $\operatorname{Gr} A$, and $D^b(\operatorname{Gr} A)$ its derived category. For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the p^{th} translation of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$. A module M can be regarded as a complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the 0^{th} term.

For $M, N \in \operatorname{Gr} A$, set $\underline{\operatorname{Hom}}_A(M,N) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$ to be a graded k-vector space with $\underline{\operatorname{Hom}}_A(M,N)_i = \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$. Similarly, we can also define $\underline{\operatorname{Hom}}^\bullet_A(M^\bullet, N^\bullet)$, $\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, N^\bullet)$, and $\underline{\operatorname{Ext}}^i_A(M^\bullet, N^\bullet)$ for $M^\bullet, N^\bullet \in D^b(\operatorname{Gr} A)$.

If V is a k-vector space, V^* denotes the dual space $\operatorname{Hom}_k(V,k)$. For $M \in \operatorname{Gr} A$ (resp. $M \in \operatorname{Gr} A^{\mathsf{op}}$), $M^\vee := \bigoplus_{i\in\mathbb{Z}}(M_i)^*$ has a graded *right* (resp. *left*) A-module structure given by $(fa)(x) = f(ax)$ (resp. $(af)(x) = f(xa)$) and $(M^\vee)_i = (M_{-i})^*$. If W is a graded A-A bimodule, then so is $W^\vee$. Note that $I_v := ({}_vP)^\vee$ (resp. ${}_vI := (P_v)^\vee$) is injective in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$). Moreover, I_v and ${}_vI$ are graded injective hulls of k_v and ${}_vk$ respectively. In particular, the A-A bimodule $A^\vee$ is injective both in $\operatorname{Gr} A$ and in $\operatorname{Gr} A^{\mathsf{op}}$.

Let W be a graded A-A-bimodule (we mainly concern the cases $W = A$ or $W = A^\vee$). If $M \in \operatorname{Gr} A$, we can regard $\underline{\operatorname{Hom}}_A(M, W)$ as a graded *right* A-module by $(fa)(x) = f(x)a$. We have a natural isomorphism $\underline{\operatorname{Hom}}_A(M, A^\vee) \cong M^\vee$. We can also define $\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, W) \in D^b(\operatorname{Gr} A^{\mathsf{op}})$ and $\underline{\operatorname{Ext}}^i_A(M^\bullet, W) \in \operatorname{Gr} A^{\mathsf{op}}$ for $M^\bullet \in D^b(\operatorname{Gr} A)$.

Let $P^\bullet$ be a bounded complex in $\operatorname{gr} A$ such that each P^i is projective. We say $P^\bullet$ is *minimal* if $\partial(P^i) \subset \mathfrak{r}P^{i+1}$ for all i. Any complex $M^\bullet \in C^b(\operatorname{gr} A)$ has a minimal projective resolution, which is unique up to isomorphism. We denote a graded module $A/\mathfrak{r}$ by A_0. Set $\beta^i_j(M^\bullet) := \dim_k \underline{\operatorname{Ext}}^{-i}_A(M^\bullet, A_0)_{-j}$. Let $P^\bullet$ be a minimal projective resolution of $M^\bullet$, and $P^i := \bigoplus_{l=1}^m T^{i,l}$ an indecomposable decomposition. Then we have

$$\beta^i_j(M^\bullet) = \#\{\, l \mid T^{i,l}(j) \cong P_v \text{ for some } v \,\}.$$

Definition 1. Let A be a graded quiver algebra. We say A is *Artin-Schelter regular* (AS-regular, for short), if

- A has finite global dimension d.
- $\underline{\operatorname{Ext}}^i_A(k_v, A) = \underline{\operatorname{Ext}}^i_{A^{\mathsf{op}}}({}_vk, A) = 0$ for all $i \neq d$ and all $v \in Q_0$.
- There are a permutation δ on Q_0 and an integer n_v for each $v \in Q_0$ such that $\underline{\operatorname{Ext}}^d_A(k_v, A) \cong {}_{\delta(v)}k(n_v)$ (equivalently, $\underline{\operatorname{Ext}}^d_{A^{\mathsf{op}}}({}_vk, A) \cong k_{\delta^{-1}(v)}(n_v)$) for all v.

Remark 2. The AS regularity is a very important concept in non-commutative algebraic geometry (see for example [18]). But there many authors assume the connectedness of A. We also remark that Martinez-Villa and coworkers called the rings in Definition 1 *generalized Auslander regular algebras.*

Definition 3. For an integer $l \in \mathbb{Z}$, we say $M^\bullet \in \operatorname{gr} A$ has an *l-linear (projective) resolution*, if $\beta^i_j(M^\bullet) = 0$ for all i, j with $i + j \neq l$. If $M^\bullet$ has an l-linear resolution for some l, we say $M^\bullet$ has a *linear resolution.*

Definition 4. We say A is *Koszul*, if the graded left A-module A_0 has a linear resolution. (Note that $A_0 \cong \bigoplus_{v \in Q_0} k_v$.)

In the above definition, we can regard A_0 as a right A-module (we get the equivalent definition). The next fact is easy to prove.

Lemma 5. *If A is AS-regular, Koszul, and has global dimension d, then $\underline{\operatorname{Ext}}^d_A(k_v, A) \cong {}_{\delta(v)}k(d)$ and $\underline{\operatorname{Ext}}^d_{A^{\mathsf{op}}}({}_vk, A) \cong k_{\delta^{-1}(v)}(d)$ for all $v \in Q_0$. Here δ is the permutation of Q_0 given in Definition 1.*

In the rest of this paper, A is always a noetherian AS-regular Koszul quiver algebra of global dimension d.

Example 6. (1) A polynomial ring $k[x_1, \dots, x_d]$ is clearly a noetherian AS-regular Koszul (quiver) algebra of global dimension d.

(2) Let $k\langle x_1, \dots, x_d\rangle$ be the free associative algebra, and $(q_{i,j})$ a $d \times d$ matrix with entries in k satisfying $q_{i,j}q_{j,i} = q_{i,i} = 1$ for all i, j. Then $A = k\langle x_1, \dots, x_n\rangle/(\, x_jx_i - q_{i,j}x_ix_j \mid 1 \leq i, j \leq d\,)$ is a noetherian AS-regular Koszul algebra with global dimension d. This fact must be well-known to the specialist, but we will sketch a proof here. Since $x_1, ..., x_d \in A_1$ form a regular normalizing sequence with $k = A/(x_1, \dots, x_d)$, A is a noetherian ring with a balanced dualizing complex by [12, Lemma 7.3]. We can construct a free resolution of $k = A/\mathfrak{r}$, which is a "q-analog" of the Koszul complex of a polynomial ring $k[x_1, \dots, x_d]$. So A is Koszul and has global dimension d. Since A has finite global dimension and admits a balanced dualizing complex, it is AS-regular (c.f. [12, Remark 3.6 (3)]).

(3) For examples of non-connected AS regular algebras, see [6].

For $M \in \operatorname{Gr} A$, set

$$\Gamma_{\mathfrak{r}}(M) = \lim_{\rightarrow} \underline{\operatorname{Hom}}_A(A/\mathfrak{r}^n, M) = \{\, x \in M \mid A_n\, x = 0 \text{ for } n \gg 0 \,\} \in \operatorname{Gr} A.$$

Then $\Gamma_{\mathfrak{r}}(-)$ gives a left exact functor from $\operatorname{Gr} A$ to itself. So we have a right derived functor $\mathbf{R}\Gamma_{\mathfrak{r}} : D^b(\operatorname{Gr} A) \to D^b(\operatorname{Gr} A)$. For $M^\bullet \in D^b(\operatorname{Gr} A)$, $H^i_{\mathfrak{r}}(M^\bullet)$ denotes the i^{th} cohomology of $\mathbf{R}\Gamma_{\mathfrak{r}}(M^\bullet)$. Similarly, we can define $\mathbf{R}\Gamma_{\mathfrak{r}^{\mathsf{op}}}$ and $H^i_{\mathfrak{r}^{\mathsf{op}}}$ for $D^b(\operatorname{Gr} A^{\mathsf{op}})$ in the same way. If M is an A-A bimodule, $H^i_{\mathfrak{r}}(M)$ and $H^i_{\mathfrak{r}^{\mathsf{op}}}(M)$ are also.

Since A is AS regular, we have $\mathbf{R}\Gamma_{\mathfrak{r}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A)$. By the same argument as [18, Proposition 4.4], we also have $\mathbf{R}\Gamma_{\mathfrak{r}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A^{\mathsf{op}})$. It does not mean that $H^d_{\mathfrak{r}}(A) \cong A^\vee(d)$ as A-A bimodules. But there is an A-A bimodule L such that $L \otimes_A H^d_{\mathfrak{r}}(A) \cong A^\vee(d)$ as A-A bimodules. Here the underlying graded additive group of L is A, but the bimodule structure is give by $A \times L \times A \ni (a, l, b) \mapsto \phi(a)lb \in A = L$ for a (fixed) graded k-algebra automorphism ϕ of A. In particular, $L \cong A$ as left A-modules and as right A-modules (separately). If A is commutative, then ϕ is the identity map.

Set $L' \cong \underline{\operatorname{Hom}}_A(L, A)$ and $\mathcal{D}^\bullet := L'(-d)[d]$. Note that $\mathcal{D}^\bullet$ belongs both $D^b(\operatorname{gr} A)$ and $D^b(\operatorname{gr} A^{\mathsf{op}})$. We have $H^i_{\mathfrak{r}}(\mathcal{D}^\bullet) = H^i_{\mathfrak{r}^{\mathsf{op}}}(\mathcal{D}^\bullet) = 0$ for all $i \neq 0$ and $H^0_{\mathfrak{r}}(\mathcal{D}^\bullet) \cong H^0_{\mathfrak{r}^{\mathsf{op}}}(\mathcal{D}^\bullet) \cong A^\vee$ as A-A bimodules by the same argument as [18, §4]. Thus (an injective resolution of) $\mathcal{D}^\bullet$ is a *balanced dualizing complex* of A in the sense of [18]. It is easy to check that $\mathbf{R}\underline{\operatorname{Hom}}_A(-, \mathcal{D}^\bullet)$

and $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathsf{op}}}(-,\mathcal{D}^\bullet)$ give duality functors between $D^b(\mathrm{gr}\,A)$ and $D^b(\mathrm{gr}\,A^{\mathsf{op}})$, which are quasi-inverse of each other.

Theorem 7 (c.f. Yekutieli [18] and Martinez-Villa [9]). *For $M^\bullet \in D^b(\mathrm{gr}\,A)$, we have*

$$\mathbf{R}\Gamma_{\mathfrak{r}}(M^\bullet)^\vee \cong \mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet,\mathcal{D}^\bullet). \quad \textit{In particular,} \quad (H^i_{\mathfrak{r}}(M^\bullet)_j)^* \cong \underline{\mathrm{Ext}}^{-i}_A(M^\bullet,\mathcal{D}^\bullet)_{-j}.$$

The above result was proved by Yekutieli in the connected case. (In some sense, Martinez-Villa proved a more general result than ours, but he did not concern complexes.) The proof of [18, Theorem 4.18] also works in our case.

Definition 8 (Jørgensen, [8]). For $M^\bullet \in D^b(\mathrm{gr}\,A)$, we say

$$\mathrm{reg}(M^\bullet) := \sup\{\, i+j \mid H^i_{\mathfrak{r}}(M^\bullet)_j \neq 0 \,\}$$

is the *Castelnuovo-Mumford regularity* of $M^\bullet$.

By Theorem 7 and the fact that $\mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet,\mathcal{D}^\bullet) \in D^b(\mathrm{gr}\,A^{\mathsf{op}})$, we have $\mathrm{reg}(M^\bullet) < \infty$ for all $M^\bullet \in D^b(\mathrm{gr}\,A)$.

Theorem 9 (Jørgensen, [8]). *If $M^\bullet \in C^b(\mathrm{gr}\,A)$, then*

$$\mathrm{reg}(M^\bullet) = \max\{\, i+j \mid \beta^i_j(M^\bullet) \neq 0 \,\}. \tag{2.1}$$

When A is a polynomial ring and $M^\bullet$ is a module, the above theorem is a fundamental result obtained by Eisenbud and Goto [5]. In the non-commutative case, under the assumption that A is connected but not necessarily regular, this has been proved by Jørgensen [8]. (If A is not regular, we have $\mathrm{reg}(A) > 0$ in many cases. So one has to assume that $\mathrm{reg}\,A = 0$ there.) In our case (i.e., A is AS-regular), we have a much simpler proof. So we will give it here. This proof is also different from one given in [5].

Proof. Set $Q^\bullet := \underline{\mathrm{Hom}}^\bullet_A(P^\bullet, L'(-d)[d]\,)$. Here $P^\bullet$ is a minimal projective resolution of $M^\bullet$, and L' is the A-A bimodule given in the construction of the dualizing complex $\mathcal{D}^\bullet$. Note that $\underline{\mathrm{Hom}}_A(P_v, L') \cong {}_{\delta^{-1}(v)}P$ for all $v \in Q_0$. Let s be the right hand side of (2.1), and l the minimal integer with the property that $\beta^l_{s-l}(M^\bullet) \neq 0$. Then $\iota(Q^{-d-l}) = l-s+d$, and $\iota(Q^{-d-l+1}) \geq l-s+d$. Since $Q^\bullet$ is a minimal complex, we have

$$0 \neq H^{-d-l}(Q^\bullet)_{l-s+d} = \underline{\mathrm{Ext}}^{-d-l}_A(M^\bullet,\mathcal{D}^\bullet)_{l-s+d} = (H^{d+l}_{\mathfrak{r}}(M^\bullet)_{-l+s-d})^*.$$

Thus $\mathrm{reg}(M^\bullet) \geq s$. The opposite inequality can be proved similarly and more easily. □

3. Koszul duality

In this section, we study the relation between the Castelnuovo-Mumford regularity of complexes and the Koszul duality. For precise information of this duality, see [2, §2].

Recall that $A = kQ/I$ is a graded quiver algebra over a finite quiver Q. Let Q^{op} be the *opposite quiver* of Q. That is, $Q_0^{\mathsf{op}} = Q_0$ and there is a bijection from Q_1 to Q_1^{op} which sends an arrow $\alpha : v \to u$ in Q_1 to the arrow $\alpha^{\mathsf{op}} : u \to v$ in Q_1^{op}. Consider the bilinear form $\langle -,- \rangle : (kQ)_2 \times (kQ^{\mathsf{op}})_2 \to A_0$ defined by

$$\langle \alpha\beta, \gamma^{\mathsf{op}}\delta^{\mathsf{op}} \rangle = \begin{cases} e_v & \text{if } \alpha=\delta \text{ and } \beta=\gamma, \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta, \gamma, \delta \in Q_1$. Here $v \in Q_0$ is the vertex with $\beta \in Ae_v$. Let $I^\perp \subset kQ^{\mathsf{op}}$ be the ideal generated by $\{\, y \in (kQ^{\mathsf{op}})_2 \mid \langle x, y \rangle = 0 \text{ for all } x \in I_2 \,\}$. We say $kQ^{\mathsf{op}}/I^\perp$ is the *quadratic dual ring* of A, and denote it by $A^!$. Clearly, $(A^!)_0 = A_0$. Since A is Koszul, so is $A^!$. Since A is AS regular, $A^!$ is a finite dimensional selfinjective algebra with $A = \bigoplus_{i=0}^d A_i$ by [10, Theorem 5.1]. If A is a polynomial ring, then $A^!$ is the exterior algebra $\bigwedge(A_1)^*$.

Let V be a finitely generated left A_0-module. Then $\operatorname{Hom}_{A_0}(A^!, V)$ is a graded *left* $A^!$-module with $(af)(a') = f(a'a)$ and $\operatorname{Hom}_{A_0}(A^!, V)_i = \operatorname{Hom}_{A_0}((A^!)_{-i}, V)$. Since $A^!$ is selfinjective, we have $\operatorname{Hom}_{A_0}(A^!, A_0) \cong A^!(d)$. Hence $\operatorname{Hom}_{A_0}(A^!, V)$ is a projective (and injective) left $A^!$-module for all V. If V has degree i (e.g., $V = M_i$ for some $M \in \operatorname{gr} A$), then we set $\operatorname{Hom}_{A_0}(A^!, V)_j = \operatorname{Hom}_{A_0}(A^!_{-j-i}, V)$.

For $M^\bullet \in C^b(\operatorname{gr} A)$, let $\mathcal{G}(M^\bullet) := \operatorname{Hom}_{A_0}(A^!, M^\bullet) \in C^b(\operatorname{gr} A^!)$ be the total complex of the double complex with $\mathcal{G}(M^\bullet)^{i,j} = \operatorname{Hom}_{A_0}(A^!, M^i_j)$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(f)(x) = \sum_{\alpha \in Q_1} \alpha f(\alpha^{\mathsf{op}} x), \qquad d''(f)(x) = \partial_{M^\bullet}(f(x))$$

for $f \in \operatorname{Hom}_{A_0}(A^!, M^i_j)$ and $x \in A^!$. The grading of $\mathcal{G}(M^\bullet)$ is given by

$$\mathcal{G}(M^\bullet)^p_q := \bigoplus_{p=i+j,\, q=-l-j} \operatorname{Hom}_{A_0}((A^!)_l, M^i_j).$$

Similarly, for a complex $N^\bullet \in C^b(\operatorname{gr} A^!)$, we can define a new complex $\mathcal{F}(N^\bullet) := A \otimes_{A_0} N^\bullet \in C^b(\operatorname{gr} A)$ as the total complex of the double complex with $\mathcal{F}(N^\bullet)^{i,j} = A \otimes_{A_0} N^i_j$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(a \otimes x) = \sum_{\alpha \in Q_1} a\alpha \otimes \alpha^{\mathsf{op}} x, \qquad d''(a \otimes x) = a \otimes \partial_{N^\bullet}(x)$$

for $a \otimes x \in A \otimes_{A_0} N^i$. The gradings of $\mathcal{F}(N^\bullet)$ is given by

$$\mathcal{F}(N^\bullet)^p_q := \bigoplus_{p=i+j,\, q=l-j} A_l \otimes_{A_0} N^i_j.$$

Clearly, each term of $\mathcal{F}(N^\bullet)$ is a projective A-module. For a module $N \in \operatorname{gr} A^!$, $\mathcal{F}(N)$ is a minimal complex. Hence we have

$$\beta^i_j(\mathcal{F}(N)) = \begin{cases} \dim_k N_i & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operations $\mathcal{F}$ and $\mathcal{G}$ define functors $\mathcal{F} : D^b(\operatorname{gr} A^!) \to D^b(\operatorname{gr} A)$ and $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$, and they give an equivalence $D^b(\operatorname{gr} A) \cong D^b(\operatorname{gr} A^!)$ of triangulated categories. This equivalence is called the *Koszul duality.* When A is a polynomial ring, this equivalence is called *Bernstein-Gel'fand-Gel'fand correspondence.* See, for example, [4].

Proposition 10 (c.f. [4, Proposition 2.3])**.** *In the above situation, we have*

$$\beta^i_j(M^\bullet) = \dim_k H^{i+j}(\mathcal{G}(M^\bullet))_{-j}.$$

Proof.

$$\begin{aligned}\underline{\mathrm{Ext}}^i_{A^!}(A_0, N^\bullet)_j &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A^!)}(A_0, N^\bullet[i](j)) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A)}(\,\mathcal{F}(A_0),\, \mathcal{F}(N^\bullet[i](j))\,) \\ &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\, A)}(\,A,\, \mathcal{F}(N^\bullet)[i+j](-j)\,) \\ &\cong H^{i+j}(\mathcal{F}(N^\bullet))_{-j}.\end{aligned}$$

□

The next result immediately follows from Theorem 9 and Proposition 10.

Corollary 11. $\mathrm{reg}(M^\bullet) = \max\{\, i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\,\}$.

For $M^\bullet \in D^b(\mathrm{gr}\, A)$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and all differential maps are zero. By a spectral sequence argument, we see that

$$\mathrm{reg}(\mathcal{H}(M^\bullet)) \geq \mathrm{reg}(M^\bullet). \tag{3.1}$$

In the next section, we will see that the difference $\mathrm{reg}(M^\bullet) - \mathrm{reg}(\mathcal{H}(M^\bullet))$ can be arbitrary large. For $N^\bullet \in D^b(\mathrm{gr}\, A^!)$, we can define $\mathcal{H}(N^\bullet)$ is the same way.

We can refine Proposition 10 using the notion of *linear strands* of projective resolutions, which was introduced by Eisenbud et. al. ([4, §3]). Let $P^\bullet$ be a *minimal* projective resolution of $M^\bullet \in D^b(\mathrm{gr}\, A)$. Consider the decomposition $P^i := \bigoplus_{j\in\mathbb{Z}} P^{i,j}$ such that any indecomposable summand of $P^{i,j}$ is isomorphic to a summand of $A(-j)$. For an integer l, we define the *l-linear strand* $\mathrm{proj.\,lin}_l(M^\bullet)$ of a projective resolution of $M^\bullet$ as follows: The term $\mathrm{proj.\,lin}_l(M^\bullet)^i$ of cohomological degree i is $P^{i,l-i}$ and the differential $P^{i,l-i} \to P^{i+1,l-i-1}$ is the corresponding component of the differential $P^i \to P^{i+1}$ of $P^\bullet$. So the differential of $\mathrm{proj.\,lin}_l(M^\bullet)$ is represented by a matrix whose entries are elements in A_1. Set $\mathrm{proj.\,lin}(M^\bullet) := \bigoplus_{l\in\mathbb{Z}} \mathrm{proj.\,lin}_l(M^\bullet)$. Clearly, $\beta^i_j(M^\bullet) = \beta^i_j(\mathrm{proj.\,lin}(M^\bullet))$ for all i, j.

Proposition 12 (c.f. [4, Corollary 3.6])**.** *For $N^\bullet \in D^b(\mathrm{gr}\, A^!)$, we have*

$$\mathrm{proj.\,lin}_l(\mathcal{F}(N^\bullet)) = \mathcal{F}(H^l(N^\bullet))[-l], \quad \textit{in particular,} \quad \mathrm{proj.\,lin}(\mathcal{F}(N^\bullet)) = \mathcal{F}(\mathcal{H}(N^\bullet)).$$

4. Weakly Koszul Modules

Let B be a noetherian Koszul algebra with the graded Jacobson radical $\mathfrak{r}$. For $M \in \mathrm{gr}\, B$ and $i \in \mathbb{Z}$, $M_{\langle i\rangle}$ denotes the submodule of M generated by its degree i component M_i. The next result naturally appears in the study of Koszul algebras, and might be a folk-theorem (see [17] for further information).

Proposition 13. *In the above situation, the following are equivalent.*

1. *$M_{\langle i\rangle}$ has a linear projective resolution for all i.*
2. *$H^i(\mathrm{proj.\,lin}(M)) = 0$ for all $i \neq 0$.*
3. *$\mathrm{gr}_{\mathfrak{r}} M := \bigoplus_{i=0}^\infty \mathfrak{r}^{i-1}M/\mathfrak{r}^i M$ has a linear resolution as a $B\,(\cong \mathrm{gr}_{\mathfrak{r}} B)$-modules.*

Definition 14 (Martinez-Villa et.al., c.f. [11])**.** We say $M \in \mathrm{gr}\, B$ is *weakly Koszul*, if it satisfies the equivalent conditions of Proposition 13.

If $M \in \operatorname{gr} B$ has a linear resolution, then it is weakly Koszul. Moreover, if M is weakly Koszul, then the i^{th} syzygy $\Omega_i(M)$ is also for all $i \geq 1$.

Let A be a noetherian AS-regular Koszul quiver algebra of global dimension d, and $A^!$ its quadratic dual, as in the previous sections.

Theorem 15 (Martinez-Villa and Zacharia, [11]). *If $N \in \operatorname{gr} A^!$ (or $N \in \operatorname{gr}(A^!)^{\mathsf{op}}$), then the i^{th} syzygy $\Omega_i(N)$ is weakly Koszul for $i \gg 0$.*

Definition 16 (Herzog et. al., [7, 14]). For $0 \neq N \in \operatorname{gr} A^!$ (or $N \in \operatorname{gr}(A^!)^{\mathsf{op}}$), set

$$\operatorname{lpd}(N) := \inf\{\, i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul} \,\}.$$

Remark 17. Herzog and Iyengar ([7]) studied the invariant lpd over noetherian commutative Koszul algebras. Among other things, they proved that $\operatorname{lpd}(M)$ is always finite over some "nice" rings (e.g., graded complete intersections which are Koszul).

The next result follows from Corollary 11 and Proposition 12.

Theorem 18. *Let $N \in \operatorname{gr} A^!$, and $N' := \operatorname{Hom}_{A^!}(N, A^!) \in \operatorname{gr}(A^!)^{\mathsf{op}}$ its dual. Then we have*

$$\begin{aligned} \operatorname{lpd}(N') &= \operatorname{reg}(\,\mathcal{H} \circ \mathcal{F}(N)\,) \\ &= \max\{\, \operatorname{reg}(H^i(\mathcal{F}(N))) + i \mid i \in \mathbb{Z} \,\}. \end{aligned}$$

Note that $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}(N)) \geq \operatorname{reg}(\mathcal{F}(N)) = \max\{\, i \mid H^i(\mathcal{G} \circ \mathcal{F}(N)) \neq 0 \,\} = 0$ by the inequality (3.1) and Corollary 11.

If $\operatorname{lpd}(N) \geq 1$ for some $N \in \operatorname{gr} A^!$, then $\sup\{\, \operatorname{lpd}(L) \mid L \in \operatorname{gr} A^! \,\} = \infty$. In fact, if $\Omega_{-i}(N)$ is the i^{th} *cosyzygy* of N (since $A^!$ is selfinjective, we can consider cosyzygies), then $\operatorname{lpd}(\Omega_{-i}(N)) > i$. But when A is the polynomial ring $S = k[x_1, \ldots, x_d]$ and $A^!$ is the exterior algebra $E = \bigwedge\langle y_1, \ldots, y_d\rangle$, we have an upper bound of $\operatorname{lpd}(N)$ for $N \in \operatorname{gr} E$ depending only on $\max\{\, \dim_k N_i \mid i \in \mathbb{Z} \,\}$ and d. But before stating this, we recall a result on a upper bound of $\operatorname{reg}(M)$ for $M \in \operatorname{gr} S$.

Theorem 19 (Brodmann and Lashgari, [3]). *Let $S == k[x_1, \ldots, x_d]$ be a polynomial ring. Assume that a graded submodule $M \subset S^{\oplus n}$ is generated by elements whose degrees are at most δ. Then we have $\operatorname{reg}(M) \leq n^{d!}(2\delta)^{(d-1)!}$.*

When $n = 1$ (i.e., when M is an ideal), the above bound is given by Bayer and Mumford [1], and sharper than it seems. In fact, for each $m \in \mathbb{N}$, there is an ideal $I \subset k[x_1, \ldots, x_{10m+1}]$ which is generated by elements of degree at most four but satisfies $\operatorname{reg}(I) \geq 2^{2^m} + 1$. For our study on $\operatorname{lpd}(N)$, the case when $\delta = 1$ (but n is general) is essential. When $n = \delta = 1$, we have $\operatorname{reg}(M) = 1$ in the situation of Theorem 19. So I believe that the bound can be largely improved (at least) when $\delta = 1$.

Theorem 20. *Let $E = \bigwedge\langle y_1, \ldots, y_d\rangle$ be an exterior algebra, and $N \in \operatorname{gr} E$. Set $n := \max\{\, \dim_k N_i \mid i \in \mathbb{Z} \,\}$. Then $\operatorname{lpd}(N) \leq n^{d!}2^{(d-1)!}$.*

Proof. Set $L := N' \in \operatorname{gr} E$. (For graded E-modules, we do not have to distinguish left modules form right ones.) By Theorem 18, it suffices to prove $\operatorname{reg}(H^i(\mathcal{F}(L))) + i \leq$

$n^{d!}2^{(d-1)!}$ for each i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}(L))$ is the cohomology of the sequence

$$S \otimes_k L_{-1} \xrightarrow{\partial_{-1}} S \otimes_k L_0 \xrightarrow{\partial_0} S \otimes_k L_1.$$

Since $\mathrm{im}(\partial_0)(-1)$ is a submodule of $S^{\oplus \dim_k L_1}$ generated by elements of degree 1, we have $\mathrm{reg}(\mathrm{im}(\partial_0)) < n^{d!}2^{(d-1)!}$ by Theorem 19. Consider the short exact sequence $0 \to \ker(\partial_0) \to S \otimes_k L_0 \to \mathrm{im}(\partial_0) \to 0$. Since $\mathrm{reg}(S \otimes_k L_0) = 0$, we have $\mathrm{reg}(\ker(\partial_0)) \leq n^{d!}2^{(d-1)!}$. Similarly, we have $\mathrm{reg}(\mathrm{im}(\partial_{-1})) \leq n^{d!}2^{(d-1)!}$ by Theorem 19. By the short exact sequence $0 \to \mathrm{im}(\partial_{-1}) \to \ker(\partial_0) \to H^0(\mathcal{F}(L)) \to 0$, we have $\mathrm{reg}(H^0(\mathcal{F}(L))) \leq n^{d!}2^{(d-1)!}$. □

In a special case, there is much more reasonable bound for $\mathrm{lpd}(N)$.

Definition 21 (Römer, [13]). We say an integer vector $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ is *squarefree*, if $a_i = 0, 1$ for all i. Let $N = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} N_{\mathbf{a}}$ be a finitely generated $\mathbb{Z}^d$-graded modules over the exterior algebra $E = \bigwedge \langle y_1, \dots, y_d \rangle$. We say N is *squarefree*, if $N_{\mathbf{a}} \neq 0$ implies that $\mathbf{a}$ is squarefree.

This concept naturally appears in the study of combinatorial commutative algebra (c.f. [14, 16]). For example, all monomial ideals of E are squarefree.

Proposition 22 (Herzog and Römer, [14]). *If N is a squarefree E-module, then we have $\mathrm{lpd}(N) \leq d - 1$.*

In [17], we describe $\mathrm{lpd}(N)$ for a squarefree E-module N in terms of combinatorial commutative algebra. We will show it below in the case when N is a monomial ideal. We also remark that there is a squarefree E-module N with $\mathrm{lpd}(N) = d - 1$.

Set $[d] := \{1, \dots, d\}$. Let $\Delta \subset 2^{[d]}$ be an (abstract) simplicial complex (i.e., $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$). It is easy to see that $\Delta^\vee := \{ F \subset [d] \mid [d] \setminus F \notin \Delta \}$ is a simplicial complex again. We also have $\Delta^{\vee\vee} = \Delta$. Set $J_\Delta = (\prod_{i \in F} y_i \mid F \subset [d], F \notin \Delta)$ to be a monomial ideal of E. Any monomial ideal of E is given in this way. Similarly, set $I_\Delta = (\prod_{i \in F} x_i \mid F \subset [d], F \notin \Delta)$ to be a monomial ideal of S, and call it the *Stanley-Reisner ideal* of Δ. Any squarefree monomial ideal of S is given in this way.

Proposition 23. *For a simplicial complex $\Delta \subset 2^{[d]}$, we have*

$$\mathrm{lpd}(J_\Delta) = \max\{ i - \mathrm{depth}_S(\mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S)) \mid 0 \leq i \leq d \}. \quad (4.1)$$

Here we set the depth of the 0 module to be $+\infty$.

If $\mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S) \neq 0$, then we have $i - \mathrm{depth}_S(\mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S)) \geq 0$. One might think the right side of the equality (4.1) is strange. But the right side of (4.1) equals 0 if and only if $S/I_{\Delta^\vee}$ is *sequentially Cohen-Macaulay* (see [15]). In this sense, $\mathrm{lpd}(J_\Delta)$ measures "how is $S/I_{\Delta^\vee}$ far from sequentially Cohen-Macaulay?".

Corollary 24 (Römer, [13]). *For a simplicial complex $\Delta \subset 2^{[d]}$, the following are equivalent.*

1. *$J_\Delta \subset E$ is weakly Koszul.*
2. *$I_\Delta \subset S$ is weakly Koszul.*
3. *$S/I_{\Delta^\vee}$ is sequentially Cohen-Macaulay.*

We remark that there are many examples of Stanley-Reisner ideals $I_\Delta \subset S$ which are weakly Koszul (dually, Stanley-Reisner rings S/I_Δ which are sequentially Cohen-Macaulay).

Corollary 25. *If $d \geq 3$, then we have* $\mathrm{lpd}(E/J_\Delta) \leq d-2$.

In this moment, I have no idea whether the above bound is (nearly) sharp for large d.

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