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序

この報告集は１９９６年１１月１３日－１６日の日程で，平成８年度文部省科学研究費（基盤研究（A）、代表：秋葉知温氏（京都大学）および代表：小池正夫氏（九州大学））の援助により、大阪商工会議所賢島研修センターで開催された「第２９回環論および表現論シンポジウム」の講演に基づき組織委員会の責任のもとに編集されたものです。

今回からシンポジウム名は「環論および表現論シンポジウム」と変更されました。１９７１年に、遠藤静男（都立大）、太刀川弘幸（東京教育大）、富永久雄（岡山大）、原田 学（大阪市大）を組織委員として、主として非可換環論に関する研究交流を目的とした研究集会「環論セミナー」が開催されました。これは毎年一回開催され７９年まで続き翌８０年の第１０回から「環論シンポジウム」(Symposium on Ring Theory) と名称を変え、報告集も英文で作成されるようになりました。７１年の第１回開催から９５年まで２８回続いたことは、組織委員と国内研究者の努力の賜ですが、特に富永先生のご尽力に負いました。先生の他界後シンポジウムの運営は主として津島行男（大阪市大）が担当し、この間、他の委員の方々が定年退官されました。一方７０年頃より数学の多くの分野において「表現論」の研究が盛んになり分野の境界も曖昧な傾向になってきています。「環論」の分野においても表現論の研究が盛んになり、８０年代には「代数の表現論」(Representasion Theory of Algebra) が「代数」(Algebra) から独立した分野として扱われるようになりました（例、'Algebras and representation theory'（ロンドン数学会，1985））。このような現状を鑑みこれまでのシンポジウムを継承発展させることを願って新たに組織委員会を設け、シンポジウム名も「環論および表現論シンポジウム」と改めました。この組織委員会は、岩永恭雄（信州大）、大代紀代市（山口大）、津島行男（大阪市大）、丸林英俊（鳴門教育大）、山形邦夫（筑波大）の５名で発足し、報告集は日本語での編集を原則としますが特に使用言語に制限を加えず、編集は組織委員会が責任を負うことになりました。

原稿の多くは TEX を利用しており併せて提出された TEX ファイルに基づき印刷し直しました。dvi ファイルの作成については山梨大学の佐藤眞久氏と筑波大学大学院（数学研究科）の竹田 馨氏にお願いしました。特に dvi ファイル作成の困難回避は佐藤氏に依りました。また会場の準備および運営に関しては大阪市立大学の加戸次郎氏が担当しました。関係諸氏に厚くお礼申し上げます。

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１９９７年１月

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$*$-MODULES OVER RING EXTENSIONS

Kent R. Fuller

A module ${}_RV$ with endomorphism ring $S = \mathrm{End}({}_RV)$ is called a *$*$-module* in case the functors induced by ${}_RV_S$ define an equivalence of categories

$$\mathrm{Hom}_R(V, \_) : \mathrm{Gen}({}_RV) \leftrightarrows \mathrm{Cogen}({}_SK) : (V \otimes_S \_)$$

where $K = \mathrm{Hom}_R(V, C)$ for an R-cogenerator ${}_RC$. Equivalently, the classes of R and S modules such that the natural maps

$$V \otimes_S \mathrm{Hom}_R(V, M) \overset{\varepsilon_M}{\to} M \text{ and } N \overset{\varphi_N}{\to} \mathrm{Hom}_R(V, (V \otimes_S N))$$

are isomorphisms are closed under epimorphic images and direct sums in R-Mod and, respectively, submodules in S-Mod. Thus tilting modules and quasi-progenerators are $*$-modules. (See [12], [5] and [6].) Here we shall outline our results in [10], after presenting a sketch of a modest improvement of the main theorem in [10].

$*$-FACTS: (1) $*$-modules are finitely generated [17].
(2) A $*$-module ${}_RV$ is a quasi-progenerator iff $\mathrm{Gen}({}_RV)$ is closed under submodules [6].
(3) A $*$-module ${}_RV$ is a tilting module iff $\mathrm{Gen}({}_RV)$ contains all injectives [6].
(4) If R is left artinian any faithful $*$-module is a tilting module [Corollary to (3)].

We say R is a *ring extension* of A if there is a ring homomorphism $\xi : A \to R$. Then given an R-module ${}_RM$, the induced A-module is ${}_AM$ with $am = \xi(a)m$, so $\mathrm{Hom}_R(M_1, M_2) \subseteq \mathrm{Hom}_A(M_1, M_2)$. A ring R is a *split extension* ($R = A \propto Q$) of a ring A by an ideal $Q \subseteq R$ in case $R = A \oplus Q$; if $Q^2 = 0$, such an extension is called a *trivial extension* ($R = A \ltimes Q$).

Miyachi [13] proved for an artin algebra $R = A \ltimes Q$ that $R \otimes_A U$ is an R-tilting module iff ${}_AU$ is a tilting module and ${}_AU$ generates both $Q \otimes_A U$ and $D(Q_A)$. Assem & Marmaridis [1] extended this to $R = A \propto Q$ with Q nilpotent. Miyashita [15] proved that if R is a ring extension of A, ${}_AU$ is a (generalized) tilting module, and $V = R \otimes_A U$, then ${}_RV$ is a (generalized) tilting module whenever $\mathrm{Ext}_A(U, V) = 0 = \mathrm{Tor}^A(R, U)$.

1. $*$-MODULES

Theorem 1.1. *Let $\xi : A \to R$ be a ring homomorphism and let ${}_RV = R \otimes_A U$. If ${}_AU$ is a $*$-module and ${}_AU$ generates ${}_AV$ then ${}_RV$ is a $*$-module.*

The detailed version of this paper has been submitted for publication elsewhere.

Proof. (Sketch.) Let $B = \text{End}({}_AU)$ and $\text{End}({}_RV) = S$. Then

$$\text{Hom}_R(V, M) \cong \text{Hom}_A(U, \text{Hom}_R({}_RR_A, M)) \cong \text{Hom}_A(U, M). \tag{1}$$

Thus

$$U \otimes_B S \cong U \otimes_B \text{Hom}_A(U, V) \cong V \tag{2}$$

since by hypothesis, $\varepsilon_{{}_AV}$ is an isomorphism. Now if ${}_RM \in R$-Mod, we have

$$\begin{aligned} V \otimes_S \text{Hom}_R(V, M) &\cong (U \otimes_B S) \otimes_S \text{Hom}_R(V, M) \\ &\cong U \otimes_B \text{Hom}_R(V, M) \\ &\cong U \otimes_B \text{Hom}_A(U, M) \end{aligned}$$

and it follows that $\varepsilon_{{}_RM}$ is an isomorphism iff $\varepsilon_{{}_AM}$ is an isomorphism. Also

$$\begin{aligned} \text{Hom}_R(V, (V \otimes_S N)) &\cong \text{Hom}_A(U, (V \otimes_S N)) \\ &\cong \text{Hom}_A(U, (U \otimes_B N)) \end{aligned}$$

so $\varphi_{{}_SN}$ is an isomorphism iff so is $\varphi_{{}_BN}$. Now the desired closure properties of

$$\{{}_RM \mid \varepsilon_{{}_RM} \text{ is an isomorphism}\} \text{ and } \{{}_SN \mid \varphi_{{}_SN} \text{ is an isomorphism}\}$$

are inherited from the corresponding subcategories of A-Mod and B-Mod. □

Corollary 1.2. *Let ${}_AU$ be a $*$-module with End$({}_AU) = B$ inducing an equivalence*

$$\text{Hom}_A(U, \_) : \mathcal{C}_0 \rightleftarrows \mathcal{D}_0 : (U \otimes_B \_)$$

and suppose that there is a ring homomorphism $\xi : A \to R$. If ${}_RV = R \otimes_A U$ and ${}_AV \in$ Gen$({}_AU)$, then there is a ring homomorphism $\chi : B \to$ End$({}_RV)$; and if S is a ring with ideal I such that $S/I \cong$ End$({}_RV)$, then ${}_RV_S$ induces an equivalence

$$\text{Hom}_R(V, \_) : \mathcal{C} \rightleftarrows \mathcal{D} : (V \otimes_S \_)$$

with

$$\mathcal{C} = \{{}_RM \mid {}_AM \in \mathcal{C}_0\} \text{ and } \mathcal{D} = \{{}_SN \mid IN = 0 \text{ and } {}_BN \in \mathcal{D}_0\}.$$

Here $\mathcal{C}$ closed under direct sums and epimorphic images, $\mathcal{D}$ closed under direct products and submodules.

Corollary 1.3. *Given a ring homomorphism $\xi : A \to R$, a quasi-progenerator ${}_AU$, and ${}_RV = R \otimes_A U$. If ${}_AU$ generates ${}_AV$, then ${}_RV$ is a quasi-progenerator.*

Proof. $*$-fact (2) applies. □

In a split extension $R = A \propto Q$, $r = a + q$ denotes a typical element of R. (Thus $(a+q)(a'+q') = aa' + (aq' + qa' + qq')$.) In this case we have ring homomorphisms $\xi : A \to R$ and $\zeta : R \to A$ with ξ injective and ζ surjective with kernel Q. If ${}_RM \in R$-Mod, then we have the ξ induced A-module by ${}_AM$, and if ${}_AX \in A$-Mod ζ induces ${}_RX$ via $(a+q)x = ax$ to obtain an R-module that is annihilated by Q. Also, we may identify $R \otimes_A U \cong U \oplus (Q \otimes_A U)$ where R operates on $U \oplus (Q \otimes_A U)$ via

$$(a+p) \cdot (u + (q \otimes w)) = au + [(aq \otimes w) + (p \otimes u) + (pq \otimes w)].$$

Thus $Q \otimes_A U \leq {}_RV$ and $V/(Q \otimes_A U) \cong {}_RU$.

Theorem 1.4. *Let $R = A \propto Q$ be a split extension of A by Q, and let ${}_AU$ be an A-module. If ${}_AU$ is a $*$-module that generates $Q \otimes_A U$, then ${}_RR \otimes_A U$ is a $*$-module. Conversely, if ${}_RV = R \otimes_A U$ is a $*$-module, then ${}_AU$ is a $*$-module.*

Proof. The first statement is by Theorem 1.1. For the second we can apply Theorem 1.1 to $\zeta : R \to A$ since ${}_RV$ generates ${}_RU$ and

$$A \otimes_R V = A \otimes_R (R \otimes_A U) \cong (A \otimes_R R) \otimes_A U \cong A \otimes_A U \cong {}_AU. \qquad \square$$

Corollary 1.5. *If ${}_AU$ is a $*$-module or a quasi-progenerator, then so is ${}_{A[x]}V = A[x] \otimes_A U$.*

Theorem 1.6. *Let $R = A \propto Q$ and $V = R \otimes_A U$, and suppose that R is noetherian and Q is nilpotent. Then ${}_RV$ is a $*$-module if and only if ${}_AU$ is a $*$-module and ${}_AU$ generates $Q \otimes_A U$.*

2. Tilting Modules

Regarding tilting modules we have

Theorem 2.1. *Let $R = A \propto Q$ be left artinian with nilpotent ideal Q, and let U be a left A-module. Then ${}_RV = R \otimes_A U$ is a tilting module if and only if ${}_AU$ is a tilting module, ${}_AU$ generates $Q \otimes_A U$ and $Ann_{Q_A}(U) = 0$.*

Proof. Uses Theorem 1.6 and $*$-Fact (4). $\square$

Corollary 2.2 (Assem & Marmaridis). *Let $R = A \propto Q$ be a finite dimensional algebra with Q nilpotent. Then $R \otimes_A U$ is a tilting R-module iff ${}_AU$ is a tilting module and ${}_AU$ generates both $Q \otimes_A U$ and $D(Q_A)$.*

Proof. By Theorem 2.1, in either case ${}_AU_B$ is a tilting bimodule where $B = \mathrm{End}({}_AU)$. Thus $D(Q_A) \in \mathrm{Gen}({}_AU)$ iff $0 = \mathrm{Ext}^1_A(U, D(Q_A)) \cong D(\mathrm{Tor}^A_1(Q, U))$ iff Q_A is torsion free in Mod-A iff $0 = \mathrm{Ker}[Q_A \to \mathrm{Hom}_B(U, (Q \otimes_A U))] = \mathrm{Ann}_{Q_A}(U)$. $\square$

In [14] Miyachi presents a proof of M. Hoshino's observation that if $R = A \propto Q$, $\mathrm{Tor}^A(R, U) = 0$ and $V = R \otimes_A U$ is a (generalized) tilting module, then ${}_AU$ is tilting and $\mathrm{Ext}_A(U, V) = 0$. It seems likely that, as in Theorem 2.1, some finiteness conditions on $R = A \propto Q$, are required in order to characterize "$R \otimes_A U$ tilting" with no prior condition on ${}_AU$.

3. Torsion Theory Counter Equivalences

In [3] and [4] we considered pairs of category equivalences $\mathcal{T} \rightleftarrows \mathcal{E}$ and $\mathcal{F} \rightleftarrows \mathcal{S}$ between the members of torsion theories $(\mathcal{T}, \mathcal{F})$ in R-Mod and $(\mathcal{S}, \mathcal{E})$ in S-Mod. Such a pair of equivalences is called a *torsion theory counter equivalence*; it is induced by a pair of bimodules ${}_RV_S$ and ${}_SV'_R$. In particular, ${}_RV$ and ${}_SV'$ are $*$-modules.

Theorem 3.1. *Suppose that ${}_AU_B$ and ${}_BU'_A$ induce a torsion theory counter equivalence between $(\mathcal{T}_0, \mathcal{F}_0)$ in A-Mod and $(\mathcal{S}_0, \mathcal{E}_0)$ in B-Mod. Let $A \to R$ and $B \to S$ be ring extensions, and let $V = R \otimes_A U$ and $V' = S \otimes_B U'$. If there are bimodule structures ${}_RV_S$ and ${}_SV'_R$ that admit isomorphisms*

$$\psi : U \otimes_B S \to {}_AV_S \quad \textit{and} \; \psi' : U' \otimes_A R \to {}_BV'_R$$

such that $\psi(u \otimes_B 1) = 1 \otimes_A u$ *and* $\psi'(u' \otimes_A 1) = 1 \otimes_B u'$ *for all* $u \in U$, $u' \in U'$, *then* ${}_RV_S$ *and* ${}_SV'_R$ *induce a torsion theory counter equivalence between* $(\mathcal{T}, \mathcal{F})$ *in R-Mod and* $(\mathcal{S}, \mathcal{E})$ *in S-Mod where*

$$\mathcal{T} = \{{}_RM \mid {}_AM \in \mathcal{T}_0\}, \quad \mathcal{F} = \{{}_RM \mid {}_AM \in \mathcal{F}_0\},$$

$$\mathcal{S} = \{{}_SN \mid {}_BN \in \mathcal{S}_0\}, \quad \mathcal{E} = \{{}_SN \mid {}_BN \in \mathcal{E}_0\}.$$

Corollary 3.2. *Suppose that* ${}_AU_B$ *and* ${}_BU'_A$ *induce a torsion theory counter equivalence. If* $S = B \propto Z$, $U \otimes_B Z = 0$ *and* $Z \otimes_B U' = 0$, *then* ${}_AU_S$ *and* ${}_SU'_A$ *induce a torsion theory counter equivalence.*

Proof. ${}_AU \otimes_B S \cong (U \otimes_B B) \oplus (U \otimes_B Z) \cong U \cong A \otimes_A U$, etc. □

Tachikawa [16] and Yamagata [18] (for a larger class of Frobenius trivial extensions of A) proved that if A is a hereditary artin algebra and $S = A \ltimes D({}_AA_A)$, *the number of indecomposable S-modules is exactly twice the number of indecomposable A-modules.* (This is obvious if A is indecomposable with a non-zero projective injective module, for then A is serial.) To do so, they were in effect employing a torsion theory counter equivalence. Indeed, from the last two results we obtain

Proposition 3.3. *Let A be a hereditary artin algebra with no injective projective modules, let* $U = D({}_AA_A)$ *and let* $S = A \ltimes Z$ *with be the trivial extension (QF) algebra with* $Z \cong {}_AU_A$. *Then* ${}_AU_S$ *and* ${}_SU'_A = Ext^1_A({}_AU_S, A)$ *define a torsion theory counter equivalence between* $(\mathcal{T}, \mathcal{F})$ *in A-Mod and* $(\mathcal{S}, \mathcal{E})$ *in S-Mod. Here,*

$$\mathcal{T} = Gen({}_AU) = A\text{-}Injectives$$

$$\mathcal{F} = \{{}_AM \text{ with no injective summands}\}$$

$$\mathcal{S} = \{{}_SN \mid {}_AN \text{ has no projective summands}\}$$

$$\mathcal{E} = Cogen(S/Z) = \{{}_SN \mid {}_AN \text{ is projective}\}.$$

In Proposition 3.3 $(\mathcal{T}, \mathcal{F})$ splits so the torsion theory counter equivalence yields a bijection

$$\text{Ind}(A\text{-Mod}) \longleftrightarrow \text{Ind}(\mathcal{E}) \cup \text{Ind}(\mathcal{S})$$

between the indecomposable A-modules and the indecomposable S-modules that are either torsion or torsion free, while the main parts of Tachikawa's and Yamagata's proofs establish a bijection

$$\text{Ind}(A\text{-Mod}) \longleftrightarrow \mathcal{I}$$

where $\mathcal{I}$ is the set of $N \in \text{Ind}(S\text{-Mod})$ such that

$$0 \to H'T'N \longrightarrow N \longrightarrow HTN \to 0$$

is non-trivial.

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INJECTIVE DIMENSION IN NOETHERIAN ALGEBRAS

SHIRO GOTO[1]

1. INTRODUCTION.

What I wish to perform in this note (and in the forthcoming paper [G] as well) is to trace the theory of injective dimension in Noetherian algebras to its source, following the flavor of development in commutative ring theory. Let R be a commutative ring and let A be an R-algebra with the structure morphism $f : R \to A$. I assume the algebra A is left Noetherian. Let $\mathrm{J}(R)$ and $\mathrm{J}(A)$ denote the Jacobson radicals of R and A, respectively. Otherwise specified, all modules stand for *left* modules.

I would like to choose a result (2.1) of Bass [B1, Corollary 1.3] which says any localization of the ground ring R be compatible with the A-injective envelopes to be the starting point of my rewriting. As is well-known, thanks to it and Matlis' decomposition theorem of injective modules, you can handle the injective dimension under the base changes modulo the reduction (and the localization as well) by A-regular elements in R ([B1, Theorem 2.2]). The primary decomposition theorem inside of finitely generated A-modules also readily follows (Corollary (2.3)).

In Section 3 and the subsequent sections also, I assume that R is Noetherian and the algebra A is finitely generated as an R-module. Firstly I will show the result [B2, (3.1) Lemma] of Bass which estimates the prime ideals associated to a minim al injective resolution for a given fininitely generated module over a *commutative* Noetherian ring remains true in our context. Namely

Theorem (3.2). *Let M be a finitely generated A-module. Let let $P, Q \in \operatorname{Spec} R$ and assume $P \subseteq Q$ with $\dim R_Q/PR_Q = 1$. Then $Q \in \operatorname{Ass}_R \mathrm{E}_A^{i+1}(M)$ if $P \in \operatorname{Ass}_R \mathrm{E}_A^i(M)$.*

Secondly I will explore in Section 4, using the above theorem (3.2) and some consequences of it, the structure of minimal injective resolutions of finitely generated A-modules possessing finite injective dimension. The result (4.1) is a natural generalization of the main theorem in [IS] due to Iwanaga and Sato.

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Key words and phrases. Injective dimension, minimal injective resolutions, Cohen-Macaulay algebras, Gorenstein rings.

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Theorem (4.1). *Suppose R is local and let* $\operatorname{depth}_R A = t$. *Let M be a non-zero finitely generated A-module with* $\operatorname{id}_A M = n < \infty$. *Then*

(1) $t \leq n$.

(2) *If $t < n$, the A-modules $\mathrm{E}^t_A(A)$ and $\mathrm{E}^n_A(M)$ have no common direct summand.*

I will show that from Theorem (4.1) the results [R, Corollary 2.15] and [V, Theorem 3.1] of Ramras and Vasconcelos on quasi-local algebras A (that is the algebras A with $A/\mathrm{J}(A)$ simple rings) of finite self-injective dimension readily follow.

I will give in Section 5 some equivalent conditions for quasi-local algebras A to have finite self-injective dimension (Theorem (5.1)). The conditions are entirely parallel to the fundamental characterizations [B2, (4.1) Theorem] of Bass of *commutative* Gorenstein local rings.

In the final section the algebra A is assumed to be a free R-module. Let me note the following criterion for the algebras A to have self-injective dimension equal to $\dim R$:

Theorem (6.2). *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of* $\dim R = d$ *and assume that A is a finitely generated free R-module. Let $\overline{A} = A/\mathfrak{m}A$. Let*

$$0 \to R \to E^0 \to E^1 \to \dots \to E^d \to 0$$

be a minimal injective resolution of R. Then the following five conditions are equivalent.

(1) $\operatorname{id}_A A = d$.

(2) $\operatorname{id}_{\overline{A}} \overline{A} = 0$.

(3) $\operatorname{id}_{A_Q/QA_Q} A_Q/QA_Q = 0$ *for any* $Q \in \operatorname{Spec} R$.

(4) *The A-module $A \otimes_R E$ is injective for any injective R-module E.*

(5) *The sequence*

$$0 \to A = A \otimes_R R \to A \otimes_R E^0 \to A \otimes_R E^1 \to \dots \to A \otimes_R E^d \to 0$$

is a minimal injective resolution for A.

When this is the case, for any integer $0 \leq i \leq d$ *the equality* $\operatorname{fd}_A \mathrm{E}^i_A(A) = i$ *holds true and one has an isomorphism*

$$\mathrm{E}^i_A(A) \cong \oplus_{Q \in \operatorname{Spec} R \text{ with } \operatorname{ht}_R Q = i} A \otimes_R \mathrm{E}_R(R/Q)$$

of A-modules.

Throughout this note let R be a commutative ring and A an R-algebra with $f : R \to A$ the structure map. Let $\mathrm{J}(R)$ and $\mathrm{J}(A)$ denote the Jacobson radicals of R and

A, respectively. Otherwise specified, all modules stand for *left* modules. In Section 2 the ring A is assumed to be left Noetherian. In Section 3 and the subsequent sections we assume that the ring R is Noetherian and A is a finitely generated R-module. In Section 6 A is furthermore assumed to be a free R-mo dule.

2. Preliminaries on localization.

Let S be a multiplicative system in R. Let me give a brief proof for the next lemma.

Lemma (2.1) ([B1, Corollary 1.3]). *Let $I^\bullet : 0 \to M \to I^0 \to I^1 \to \dots \to I^i \to \dots$ be a minimal injective resolution of an A-module M. Then the localization*

$$S^{-1}I^\bullet : 0 \to S^{-1}M \to S^{-1}I^0 \to S^{-1}I^1 \to \dots \to S^{-1}I^i \to \dots$$

gives a minimal injective resolution of the $S^{-1}A$-module $S^{-1}M$.

Proof. The exact functor S^{-1} preserves injectives. Let $g : X \to Y$ be an essential monomorphism of A-modules and let $0 \neq y/1 \in S^{-1}Y$. Let $\Sigma = \{(0) :_A z \mid z \in Ay, [(0) :_A z] \cap f(S) = \phi\}$. Then $(0) :_A y \in \Sigma$ so that $\Sigma \neq \phi$. Let $(0) :_A z$ be a maximal element $\in \Sigma$. Take $a \in A$ and $x \in X$ with $az = g(x) \neq 0$. Then the maximality of the left ideal $(0) :_A z$ in Σ claims $a/1 \cdot z/1 = (S^{-1}g)(x/1) \neq 0$. As $a/1 \cdot z/1 \in S^{-1}A \cdot y/1$, this shows the morphism $S^{-1}g$ is essential. //

This result (2.1) is no longer true unless A is left Noetherian. For example, let $(R, \mathfrak{m})$ be a Noetherian complete local integral domain of positive dimension. Let $E = \mathrm{E}_R(R/\mathfrak{m})$ be the injective envelope of $R/\mathfrak{m}$ and let $S = R \setminus \{0\}$. Take the trivial extension A of E over R. Then $A \cong \mathrm{E}_A(E)$ but $\mathrm{E}_{S^{-1}A}(S^{-1}E) = (0)$.

Corollary (2.2). *Let I be an indecomposable injective A-module. Then the following assertions hold true.*

(1) $S^{-1}I = (0)$ *if* $sx = 0$ *for some* $s \in S$ *and* $0 \neq x \in I$.
(2) $\# \mathrm{Ass}_R I \leq 1$.
(3) $\# \mathrm{Ass}_R I = 1$ *if R is Noetherian.*
(4) *Assume that $(R, \mathfrak{m})$ is a Noetherian local ring and let $\mathrm{H}^i_\mathfrak{m}(*) (i \in \mathbb{Z})$ denote the i-th local cohomology functor with respect to the maximal ideal* $\mathfrak{m}$. *Then* $\mathrm{H}^i_\mathfrak{m}(I) = (0)$ *for all* $i > 0$. (Hence for any A-module M, the local cohomology module $\mathrm{H}^i_\mathfrak{m}(M)$ is given by the i-th cohomology of the complex $\mathrm{H}^0_\mathfrak{m}(I^\bullet)$:

$$\dots \to 0 \to \mathrm{H}^0_\mathfrak{m}(I^0) \to \mathrm{H}^0_\mathfrak{m}(I^1) \to \dots \to \mathrm{H}^0_\mathfrak{m}(I^i) \to \dots$$

of A-modules, where $I^\bullet : 0 \to M \to I^0 \to I^1 \to \dots \to I^i \to \dots$ denotes an A-injective resolution of M.)

Proof. (1) Let $L = Ax$. Then by (2.1) $S^{-1}I = \mathrm{E}_{S^{-1}A}(S^{-1}L)$ since $I = \mathrm{E}_A(L)$, so that $S^{-1}I = (0)$ because $S^{-1}L = (0)$. (2) Let $P, Q \in \mathrm{Ass}_R I$ and assume $P \not\subseteq Q$. Let $t \in P$ with $t \notin Q$. Let $S = R \setminus Q$ and choose $x \in I$ so that $(0) :_R x = P$. Then $tx = 0$ whence $I_Q = (0)$ by (1), which is absurd. Thus $P \subseteq Q$ and so $\# \mathrm{Ass}_R I \leq 1$. If the ring R is Noetherian, $\# \mathrm{Ass}_R I = 1$ because $\mathrm{Ass}_R I \neq \phi$. (3) Left to readers.//

Passing to the decomposition of the injective envelope $\mathrm{E}_A(M)$ of M into a finite direct sum of indecomposable injectives, the next result follows from (2.2)(3). The proof is standard and I omit it.

Corollary (2.3). *Suppose R is a Noetherian ring and let M be a finitely generated A-module. Then*

(1) $\mathrm{Ass}_R M$ *is a finite set.*

(2) *Let $\mathrm{Ass}_R M = \{Q_1, Q_2, ..., Q_n\}$ with $n = \# \mathrm{Ass}_R M$. Then M contains A-submodules $\{M_i\}_{1 \leq i \leq n}$ which satisfy the following two conditions :*

(a) $\cap_{1 \leq i \leq n} M_i = (0)$.

(b) $\mathrm{Ass}_R M/M_i = \{Q_i\}$ for all $1 \leq i \leq n$.

For an A-module M let $'M = \{m \in M \mid sm = 0 for some s \in S\}$ and $''M = M/'M$. The *S-torsion* part $'M$ is an A-submodule of M and the A-module $''M$ is *S-torsionfree*. If I is an indecomposable injective A-module, then $I = {}'I$ or $'I = (0)$ by (2.2)(1). As for a general injective A-module I, passing to the decomposition into a direct sum of indecomposable injectives, we see the submodule $'I$ of I is also A-injective and hence get a direct sum decomposition

$$I \cong {}'I \oplus {}''I$$

of I. The canonical map $''I \to S^{-1}{}''I$ $(= S^{-1}I)$ of localization is bijective, because all elements of S act on each indecomposable component of $''I$ as A-isomorphisms. Apply this observation to a minimal injective resolution

$$I^\bullet : 0 \to M \to I^0 \to I^1 \to ... \to I^i \to ...$$

of a given A-module M. Then we get a short exact sequence

$$0 \to {}'I^\bullet \to I^\bullet \to {}''I^\bullet \to 0$$

of complexes together with an isomorphism

$$''I^\bullet \cong S^{-1}{}''I^\bullet = S^{-1}I^\bullet$$

of complexes of injective A-modules. And a simple diagram chase will prove the next assertion.

Corollary (2.4). *Assume any element of S acts on M as a nonzerodivisor. Let $i \in \mathbb{Z}$. Then*

$$\mathrm{E}^i_A(M) \cong \mathrm{E}^{i-1}_A(S^{-1}M/M) \oplus \mathrm{E}^i_{S^{-1}A}(S^{-1}M)$$

so that $\mathrm{id}_A M = \max\{\mathrm{id}_A(S^{-1}M/M)+1, \mathrm{id}_{S^{-1}A} S^{-1}M\}$.

If $S = \{t^i \mid i \geq 0\}$ for some $t \in R$, the equality in (2.4) is improved as follows. Let me note a sketch of proof.

Theorem (2.5). *Let $t \in R$ and assume $S = \{t^i \mid i \geq 0\}$. Suppose t is a nonzerodivisor for both A and M. Let $\overline{A} = A/tA$ and $\overline{M} = M/tM$. Then*

(1) ([B1, Theorem 2.2]) $\mathrm{id}_A M = \max\{\mathrm{id}_{\overline{A}} \overline{M} + 1, \mathrm{id}_{S^{-1}A} S^{-1}M\}$.
(2) $\mathrm{fd}_A \mathrm{E}^i_A(M) = \max\{\mathrm{fd}_{\overline{A}} \mathrm{E}^i_{\overline{A}}(\overline{M}) + 1, \mathrm{fd}_{S^{-1}A} \mathrm{E}^i_{S^{-1}A}(S^{-1}M)\}$ for all $i \in \mathbb{Z}$.

Proof. Let $I^\bullet : 0 \to M \to I^0 \to I^1 \to \dots \to I^i \to \dots$ be a minimal injective resolution of M. Then the element t acts on each term I^i as an A-epimorphism, since t is a nonzerodivisor for A. Let $J^i = (0) :_{I^i} t$ for each $i \in \mathbb{Z}$. Then we have a complex $J^\bullet$ of injective $\overline{A}$-modules and a short exact sequence

$$0 \to J^\bullet \to I^\bullet \xrightarrow{t} I^\bullet \to 0$$

of complexes of A-modules. Note $J^\bullet$ is a subcomplex of $'I^\bullet$ and $J^i = (0)$ if and only if $'I^i = (0)$. On the other hand, since t is a nonzerodivisor for M too, we have $J^0 = (0)$. And from the long exact sequence of cohomology modules derived from the above short exact sequence $0 \to J^\bullet \to I^\bullet \xrightarrow{t} I^\bullet \to 0$ of complexes, we see that

$$0 \to \overline{M} \to J^1 \to J^2 \to \dots \to J^{i+1} \to \dots$$

is a minimal $\overline{A}$-injective resolution for $\overline{M}$. Hence we get assertion (1). The second assertion is due to the next general result (2.6). //

Proposition (2.6). *Let A be an algebra over a commutative ring R and I an A-module. Let $t \in R$. Suppose t is a nonzerodivisor for A and t acts on I as an epimorphism. Let $J = (0) :_I t$ and $\overline{A} = A/tA$. Then we have the equality*

$$\mathrm{fd}_A I = \max\{\mathrm{fd}_{\overline{A}} J + 1, \mathrm{fd}_{A_t} I_t\}.$$

3. Estimation of prime ideals associated to a minimal injective resolution.

¿From now on let me assume R is a Noetherian ring and the R-algebra A is finitely generated as an R-module. To begin with I note

Proposition (3.1). *Let I be an indecomposable injective A-module. Then the following assertions hold true.*

(1) $\mathrm{Ass}_R I = \{Q\}$ *for some* $Q \in \mathrm{Spec}\, R$.

(2) *The A-module I is a direct summand of* $\mathrm{Hom}_R(A_A, \mathrm{E}_R(R/Q))$.

(3) *The A-module I contains a unique simple A_Q-module S so that* $I = \mathrm{E}_{A_Q}(S)$.

Proof. See(2.2)(3) for assertion (1). To show assertions (2) and (3), passing to the local ring R_Q, we may assume without loss of generality that the base ring $(R, \mathfrak{m})$ is local and $Q = \mathfrak{m}$. Take $0 \neq x \in I$ so that $\mathfrak{m}x = (0)$ and put $L = L$. Then $\mathfrak{m}L = (0)$ and the injective envelope $\mathrm{E}_R(L)$ of the R-module L is a direct sum of finite copies of $\mathrm{E}_R(R/\mathfrak{m})$. Since L is an A-submodule of $\mathrm{Hom}_R(A_A, \mathrm{E}_R(L))$ and $\mathrm{Hom}_R(A_A, \mathrm{E}_R(L))$ is a direct sum of finite copies of $\mathrm{Hom}_R(A_A, \mathrm{E}_R(R/\mathfrak{m}))$, the indecomposable injective A-module $I = \mathrm{E}_A(L)$ is contained in $\mathrm{Hom}_R(A_A, \mathrm{E}_R(R/\mathfrak{m}))$. Assertion (3) follows from the fact that the A-module L has a composition series. //

In the case where $(R, \mathfrak{m})$ is a local ring, for each A-module M and $i \in \mathbb{Z}$ we have that $\mathfrak{m} \in \mathrm{Ass}_R \mathrm{E}^i_A(M)$ if and only if $\mathrm{Ext}^i_A(A/\mathrm{J}(A), M) \neq (0)$. Also

$$\mathrm{depth}_R\, M = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_A(A/\mathrm{J}(A), M) \neq (0)\}$$

if M is a finitely generated A-module.

The next result is essentially due to Bass [B2, (3.1) Lemma].

Theorem (3.2). *Let M be a finitely generated A-module. Let $P, Q \in \mathrm{Spec}\, R$ and assume $P \subsetneq Q$ with $\dim R_Q/PR_Q = 1$. Let $i \in \mathbb{Z}$. Then $Q \in \mathrm{Ass}_R \mathrm{E}^{i+1}_A(M)$ if $P \in \mathrm{Ass}_R \mathrm{E}^i_A(M)$.*

Proof. May assume $(R, \mathfrak{m})$ is local and $Q = \mathfrak{m}$. Let $L = h^{-1}(\mathrm{J}(A_P))$ where $h : A \to A_P$ denotes the canonical homomorphism of localization. Then $L \supseteq PA$ and $(A/L)_P \cong A_P/\mathrm{J}(A_P)$. Choose an element $t \in \mathfrak{m}$ so that $t \notin P$ and put $C = A/(L + tA)$. Then $\dim_R C = 0$ because $\dim R/(P + tR) = 0$ and $L \supseteq PA$. Therefore the A-module C possesses a composition series. On the other hand, from the short exact sequence $0 \to A/L \xrightarrow{t} A/L \to C \to 0$ of A-modules, we get an exact sequence $\mathrm{Ext}^i_A(A/L, M) \xrightarrow{t} \mathrm{Ext}^i_A(A/L, M) \to \mathrm{Ext}^{i+1}_A(C, M)$ of R-modules. We see $\mathrm{Ext}^i_A(A/L, M) \neq (0)$ because

$$\mathrm{Ext}^i_A(A/L, M)_P \cong \mathrm{Ext}^i_{A_P}(A_P/\mathrm{J}(A_P), M_P) \neq (0),$$

so that $\mathrm{Ext}^{i+1}_A(C, M) \neq (0)$ by Nakayama's lemma. Hence $\mathrm{Ext}^{i+1}_A(S, M) \neq (0)$ for some composition factor S of C and so $\mathfrak{m} \in \mathrm{Ass}_R \mathrm{E}^{i+1}_A(M)$. //

Thanks to the dimension theory in commutative Noetherian rings, from (3.2) we readily get the following

Corollary (3.3). *Let M ($\neq (0)$) be a finitely generated A-module.*

(1) *Let $(R, \mathfrak{m})$ be a local ring. Then $\mathfrak{m} \in \mathrm{Ass}_R \mathrm{E}_A^{i+\dim R/P}(M)$ if $P \in \mathrm{Ass}_R \mathrm{E}_A^i(M)$.*

(2) *Assume $\mathrm{id}_A M = n$ is finite. Then $\mathrm{Ass}_R \mathrm{E}_A^n(M) \subseteq \mathrm{Max}\, R$ and hence $\mathrm{E}_A^n(M)$ contains an essential socle.*

(3) (Auslander) If $(R, \mathfrak{m})$ is a local ring, then

$$\mathrm{id}_A M = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}_A^i(A/\mathrm{J}(A), M) \neq (0)\}.$$

(4) $\dim_R M \leq \mathrm{id}_A M$.

Proof. I give a proof for assertion (4) only. Let $P_0 \subset P_1 \subset \ldots \subset P_k$ be a strictly increasing chain in $\mathrm{Supp}_R M$ with $P = P_0$ is minimal in $\mathrm{Supp}_R M$ and $Q = P_k$ is maximal in $\mathrm{Supp}_R M$. Then by (1) we have $k \leq \mathrm{id}_A M$. Hence $\dim_R M \leq \mathrm{id}_A M$. //

The next result is well-known. But let me give a brief proof in our context.

Corollary (3.4) (cf. [R, Theorem 2.10]). *Let M be a finitely generated A-module. Let $t \in \mathrm{J}(R)$ and assume that t is a nonzerodivisor for both A and M. Let $\overline{A} = A/tA$ and $\overline{M} = M/tM$. Then $\mathrm{id}_A M = \mathrm{id}_{\overline{A}} \overline{M} + 1$.*

Proof. If $\mathrm{id}_A M = \infty$, $Ass_R \mathrm{E}_A^i(M) \neq \phi$ for each integer $i \geq 0$. Let $P \in \mathrm{Ass}_R \mathrm{E}_A^i(M)$ and choose a maximal ideal Q in R so that $Q \supseteq P$. Then by (3.3)(1) $Q \in \mathrm{Ass}_R \mathrm{E}_A^{i+k}(M)$, where $k = \dim R_Q/PR_Q$. Let $I = \mathrm{E}_A^{i+k}(M)$. Then $(0) :_I t \neq (0)$. Hence from the proof of (2.5) we see $\mathrm{id}_{\overline{A}} \overline{M} \geq i + k - 1$ and thus $\mathrm{id}_{\overline{A}} \overline{M} = \infty$. Assume $\mathrm{id}_A M = n$ is finite. Since $\mathrm{Ass}_R \mathrm{E}_A^n(M) \subseteq \mathrm{Max}\, R$ by (3.3)(2), any element $Q \in \mathrm{Ass}_R \mathrm{E}_A^n(M)$ contains t, so that by the proof of (2.5) we get $\mathrm{id}_{\overline{A}} \overline{M} = n - 1$. //

4. A STRUCTURE THEOREM FOR MINIMAL INJECTIVE RESOLUTIONS.

The main result of this section is the following.

Theorem (4.1). *Assume that R is a local ring and let $t = \mathrm{depth}_R A$. Let M be a non-zero finitely generated A-module with $\mathrm{id}_A M = n$ finite. Then*

(1) *$t \leq n$.*

(2) *If $t < n$, the injective A-modules $\mathrm{E}_A^t(A)$ and $\mathrm{E}_A^n(M)$ have no common non-zero direct summand.*

Proof. Let $x_1, x_2, \ldots, x_t$ be a maximal A-regular sequence and let $I = (x_1, x_2, \ldots, x_t)R$. Let $K_\bullet = A \otimes_R K_\bullet(x_1, x_2, \ldots, x_t, R)$ denote the Koszul complex generated by the elements $f(x_1)$, $f(x_2)$, ..., $f(x_t)$ over A. Then $K_\bullet$ is a minimal A-free resolution of A/IA. Hence $\mathrm{hd}_A A/IA = t$. We compute $\mathrm{Ext}_A^t(A/IA, M)$ with this resolution $K_\bullet$ of A/IA and get an isomorphism $\mathrm{Ext}_A^t(A/IA, M) \cong M/IM$. Hence by Nakayama's lemma $\mathrm{Ext}^t_A(A/IA, M) \neq (0)$, so that $t \leq n = \mathrm{id}_A M$. Assume $\mathrm{E}^t_A(A)$ and $\mathrm{E}^n_A(M)$ have a common indecomposable direct summand, say I and let S denote the simple A-submodule of I (cf. (3.3)(2)). Then, applying the argument in the proof of (2.5) to $\mathrm{E}^t_A(A)$, we see S is contained in A/IA too. Let $C = (A/IA)/S$ and consider the short exact sequence $0 \to S \to A/IA \to C \to 0$ in order to get an exact sequence

$$\mathrm{Ext}^n_A(A/IA, M) \to \mathrm{Ext}^n_A(S, M) \to \mathrm{Ext}^{n+1}_A(C, M).$$

Then because $\mathrm{Ext}^{n+1}_A(C, M) = (0)$ (recall that $\mathrm{id}_A M = n$) and $\mathrm{Ext}^n_A(S, M) \neq (0)$ (note that S is an A-submodule of $\mathrm{E}^n_A(M)$ too), we get $\mathrm{Ext}^n_A(A/IA, M) \neq (0)$ so that $n \leq t = \mathrm{hd}_A A/IA$. Hence $t = n$. //

Let $\mathrm{id}_A A = n$ be finite and assume that $\mathrm{E}^0_A(A)$ and $\mathrm{E}^n_A(A)$ have a common indecomposable direct summand I. Choose $Q \in \mathrm{Ass}_R I$. Then after localizing at Q, we have that $\mathrm{depth}_{R_Q} A_Q = 0$ and that $\mathrm{E}^0_{A_Q}(A_Q)$ and $\mathrm{E}^n_{A_Q}(A_Q)$ have a common non-zero direct summand $I = I_Q$. Hence $n = 0$ by (4.1) and so we have

Corollary (4.2) ([IS, Theorem]). *Suppose* $0 < \mathrm{id}_A A = n < \infty$. *Then* $\mathrm{E}^0_A(A)$ *and* $\mathrm{E}^n_A(A)$ *have no common non-zero direct summand.*

We say that A is a quasi-local ring if the ring $A/\mathrm{J}(A)$ is a simple ring. When A is quasi-local, the algebra A has a unique simple module. Hence from (4.1) we readily have

Corollary (4.3) (cf. [R, Corollary 2.15]). *Assume that R is local and A is quasi-local. Then one has the equality*

$$\mathrm{id}_A M = \mathrm{depth}_R A$$

for any non-zero finitely generated A-module M of finite injective dimension.

Corollary (4.4) ([V, Theorem 3.1]). *Assume that R is local and A is quasi-local. If* $\mathrm{id}_A A < \infty$, *then A is a Cohen-Macaulay R-module and* $\mathrm{id}_A A = \dim_R A$.

Proof. This follows from (3.3)(4) and (4.3). //

5. Characterizations for A to have finite self-injective dimension in the case where R is local and A is quasi-local.

In this section I assume that $(R, \mathfrak{m})$ is local and A is quasi-local. The purpose is to give the following characterizations for A to have finite self-injective dimension. Let $d = \dim R$ and $n = \dim_R A$.

Theorem (5.1). *The following conditions are equivalent.*

(1) $\mathrm{id}_A A < \infty$.

(2) $\mathrm{id}_A A = n$.

(3) *A is a Cohen-Macaulay R-module and the local cohomology module* $\mathrm{H}^n_{\mathfrak{m}}(A)$ *of A is A-injective.*

(4) $\mathrm{Ext}^i_A(A/\mathrm{J}(A), A) = (0)$ *if* $i \neq n$.

(5) *A is a Cohen-Macaulay R-module and* $\mathrm{Hom}_A(A/\mathrm{J}(A), \mathrm{E}^n_A(A)) \cong A/\mathrm{J}(A)$.

(If R is a Cohen-Macaulay local ring possessing the canonical module K_R*, one may add the next condition :*

(6) A is a Cohen-Macaulay R-module and $\mathrm{Ext}^{d-n}_R({}_A A, \mathrm{K}_R) \cong A_A$.)

When this is the case, the following assertions hold true.

(a) ([R, Corollary 2.15]) The right self-injective dimension of A is finite and equals n.

(b) $\mathrm{E}^n_A(A) \cong \mathrm{H}^n_{\mathfrak{m}}(A) \cong \mathrm{E}_A(A/\mathrm{J}(A))$.

(c) ([V, Theorem 3.2]) $\mathrm{id}_{A_Q} A_Q = \dim_{R_Q} A_Q$ for all $Q \in \mathrm{Supp}_R A$.

(d) Let $Q \in \mathrm{Spec}\, R$ and $i \in \mathbb{Z}$. Then $Q \in \mathrm{Ass}_R \mathrm{E}^i_A(A)$ if and only if $Q \in \mathrm{Supp}_R A$ with $\dim_{R_Q} A_Q = i$.

(e) $\mathrm{fd}_A \mathrm{E}^i_A(A) = i$ for all $0 \leq i \leq n$.

Proof. To check the equivalence of conditions from (1) to (6), we may assume by (2.5), (3.4), and (4.4) that A is a Cohen-Macaulay R-module (recall $\mathrm{depth}_R A = \inf\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_A(A/\mathrm{J}(A), A) \neq (0)\}$). Hence by induction on $n = \dim_R A$ we may furthermore assume that $d = n = 0$. In this case the proof is fairly standard and left to readers. Among the last assertions, (a), (b) and (e) similarly follow by induction on $n = \dim_R A$. Look at assertion (c). Let $Q \in \mathrm{Supp}_R A$ and put $\mathrm{id}_{A_Q} A_Q = k$. Then $Q \in \mathrm{Ass}_R \mathrm{E}^k_A(A)$, since $QR_Q \in \mathrm{Ass}_{R_Q} \mathrm{E}^k_{A_Q}(A_Q)$ by (3.3)(2). Therefore $k + \dim R/Q \leq n$ by (3.3) (1). On the other hand, since A is a Cohen-Macaulay R-module, we have $\dim_{R_Q} A_Q + \dim R/Q = n$. Hence $k \leq \dim_{R_Q} A_Q$ so that $\dim_{R_Q} A_Q = \mathrm{id}_{A_Q} A_Q$ by (3.3)(4). The implication $\Longleftarrow$) in assertion (d) is clear and the implication $\Longrightarrow$) readily follows from the fact that A_Q is a Cohen-Macaulay R_Q-module and the equality $\mathrm{depth}_{R_Q} A_Q = \sup\{i \in \mathbb{Z} \mid \mathrm{Ext}^i_{A_Q}(A_Q/\mathrm{J}(A_Q), A_Q) \neq (0)\}$. //

Remark (5.2). Some part of Theorem (5.1) holds true for any Cohen-Macaulay R-algebra A, as I will discuss in [G].

6. Characterizations for A to have self-injective dimension equal to $\dim R$ in the case where A is a free R-module.

In this section I assume that $(R, \mathfrak{m})$ is a local ring and A is a free R-module. Let $d = \dim R$ and $\overline{A} = A/\mathfrak{m}A$. Let me briefly state two results on A, which clarifies when the algebra A has self-injective dimension equal to $\dim R$. The detail will be postponed until [G].

Lemma (6.1). *The following assertions hold true.*

(1) *Any injective A-module is R-injective.*

(2) $\mathrm{id}_A A = \mathrm{id}_R R + \mathrm{id}_{\overline{A}} \overline{A}$.

(3) *If* $\mathrm{gldim}\, A < \infty$, *$R$ is a regular local ring.*

(4) *If* $\mathrm{gldim}\, \overline{A} < \infty$ *and R is regular, then* $\mathrm{gldim}\, A = \dim R + \mathrm{gldim}\, \overline{A}$.

Theorem (6.2). *Let $(R, \mathfrak{m})$ be a Gorenstein local ring of* $\dim R = d$ *and let*

$$0 \to R \to E^0 \to E^1 \to \dots \to E^d \to 0$$

be a minimal injective resolution of R. Then the following five conditions are equivalent.

(1) $\mathrm{id}_A A = d$.

(2) $\mathrm{id}_{\overline{A}} \overline{A} = 0$.

(3) $\mathrm{id}_{A_Q/QA_Q} A_Q/QA_Q = 0$ *for any* $Q \in \mathrm{Spec}\, R$.

(4) *The A-module $A \otimes_R E$ is injective for any injective R-module E.*

(5) *The sequence*

$$0 \to A = A \otimes_R R \to A \otimes_R E^0 \to A \otimes_R E^1 \to \dots \to A \otimes_R E^d \to 0$$

is a minimal injective resolution for A.

When this is the case, for any integer $0 \le i \le d$ *the equality* $\mathrm{fd}_A\, \mathrm{E}^i_A(A) = i$ *holds true and one has an isomorphism*

$$\mathrm{E}^i_A(A) \cong \oplus_{Q \in \mathrm{Spec}\, R \text{ with } \mathrm{ht}_R Q = i} A \otimes_R \mathrm{E}_R(R/Q)$$

of A-modules.

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GOOD FILTRATIONS ON EQUIVARIANT MODULES OVER GRADED ALGEBRAS WITH REDUCTIVE GROUP ACTION

MITSUYASU HASHIMOTO

Abstract

Homological aspects of (G, A)-modules are considered, and Auslander-Buchweitz type approximation theorem which unifies Cohen-Macaulay approximation and Ringel's Δ-good approximations (in some sense) is proved. We show that determinantal rings give non-trivial examples of the theorem.

1 Homological algebra of equivariant modules

Let k be an algebraically closed field, and G' a reductive algebraic group over k. Let $S = \operatorname{Sym} Q$ be a polynomial ring over k on which G' acts k-linearly. Let I be a G'-ideal which is homogeneous and perfect of codimension h so that $A := S/I$ is a homogeneous Cohen-Macaulay k-algebra with a degree preserving k-algebra automorphism action of G'. A G'-module A-module M is called a (G', A)-module (or a G'-equivariant A-module) if $g(am) = (ga)(gm)$ holds for any $g \in G'$, $a \in A$ and $m \in M$. Let $\mathcal{A}$ be the category of graded (G', A)-modules which is A-finitely generated. The study of the category $\mathcal{A}$ is the main subject here. However, the graded conditions are sometimes nuisance, and we use the following trick.

We set $G := G' \times \mathbb{G}_m$, where $\mathbb{G}_m = GL_1(= k^\times)$ is the one-dimensional algebraic torus. For a $\mathbb{Z}$-graded G'-module $V = \bigoplus_i V_i$, we define the G-action on V by $(g, t)(v_i) := t^i \cdot (gv_i)$, where the product $\cdot$ in the right-hand side is the scalar product. Conversely, for a G-module V, we define $V_i := \{v \in V \mid (e, t)v = t^i \cdot v\}$, and it is easy to verify that $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is a graded G'-module. Thus, the category of $\mathbb{Z}$-graded G'-modules and the category of G-modules are equivalent in a canonical way. It is easy to verify that a G'-action on a k-algebra A as degree preserving k-algebra automorphisms is nothing but a G-action on A. A graded (G', A)-module is nothing but a (G, A)-module. Note that $\mathcal{A}$ is nothing but the full subcategory of the category of (G, A)-modules consisting of A-finite objects. The category $\mathcal{A}$ is abelian, but has neither enough projectives nor enough injectives in general. However, the category of (G, A)-modules is abelian with enough injectives and exact filtered inductive limits. We denote the category of (G, A)-modules by $\mathcal{B}$. As G' is reductive, G is also reductive.

A little bit more generally, if G is a given reductive group with a specified $\mathbb{G}_m \subset Z(G)$, then any G-module V admits a canonical grading $V = \bigoplus_i V_i$ with each V_i a

The detailed version of this paper will be submitted for publication elsewhere.

G-submodule. In this case, we assume that Q is a finite dimensional G-module which is *positively graded*, and $S = \operatorname{Sym} Q$, and $A = S/I$ are as above.

For example, if $G = GL_n$, then G has a $\mathbb{G}_m$ as its center (the subgroup of non-zero scalar matrices), but it is not of the form $G' \times \mathbb{G}_m$.

Let M and N be (G, A)-modules, and V a G module. The tensor product $M \otimes V$ is a (G, A)-module by $g(m \otimes v) = gm \otimes gv$ and $a(m \otimes v) = am \otimes v$. It is easy to see that $M \otimes_A N$ is a (G, A)-module so that the canonical map $M \otimes N \to M \otimes_A N$ is a (G, A)-linear map.

If $M \in \mathcal{A}$ and $N \in \mathcal{B}$, then $\operatorname{Hom}_A(M, N)$ is a (G, A)-module by $(gf)(m) := g(f(g^{-1}(m)))$ for $g \in G$, $f \in \operatorname{Hom}_A(M, N)$ and $m \in M$.

The following is a consequence of so-called the local finiteness theorem.

Lemma 1.1 *Let M be an object of $\mathcal{A}$ (resp. $\mathcal{B}$). Then, there is an epimorphism $P \to M$ in $\mathcal{A}$ (resp. $\mathcal{B}$) with P being A-free. More precisely, we can take P of the form $P = A \otimes P_0$ with P_0 a finite dimensional G-module (resp. a G-module).*

Although $\mathcal{A}$ and $\mathcal{B}$ do not have enough projectives, we can construct the derived functor of $? \otimes_A N$ in those categories, thanks to the theory of derived categories [7, Corollary 5.3.β].

Proposition 1.2 *Let $N \in \mathcal{B}$. Then, there is a derived functor*

$$\underline{L}(? \otimes_A N) : D^-(\mathcal{B}) \to D^-(\mathcal{B})$$

of $? \otimes_A N$. For any $M \in \mathcal{B}$, we have

$$L_i(? \otimes_A N)(M) \cong L_i(M \otimes_A ?)(N)$$

in $\mathcal{B}$, and both are isomorphic to $\operatorname{Tor}_i^A(M, N)$ as a graded A-module. Similarly, there is a derived functor

$$\underline{R}\operatorname{Hom}_A(?, N) : D^-(\mathcal{A}) \to D^+(\mathcal{B})$$

of $\operatorname{Hom}_A(?, N)$. For $M \in \mathcal{A}$, we have that $R^i \operatorname{Hom}_A(?, N)(M) \cong \operatorname{Ext}_A^i(M, N)$ as graded A-modules.

On the other hand, the derived functor of $\operatorname{Hom}_A(M, ?)$ exists, as $\mathcal{B}$ has enough injectives.

Theorem 1.3 *Let M and I be objects in $\mathcal{B}$. If I is $\mathcal{B}$-injective, then we have $\operatorname{Ext}_A^i(M, I) = 0$ for $i > 0$.*

However, note that I is *not* A-injective in general. In this case, there is some A-module M' (which does not have any (G, A)-module structure, by the theorem) such that $\operatorname{Ext}_A^1(M', I) \neq 0$. Thanks to the theorem, we have

$$R^i \operatorname{Hom}_A(M, ?)(N) \cong R^i \operatorname{Hom}_A(?, N)(M)$$

in $\mathcal{B}$ for $M \in \mathcal{A}$ and $N \in \mathcal{B}$.

Thus, we consider that $\operatorname{Tor}_i^A(M, N)$ is an object in $\mathcal{B}$ for $M, N \in \mathcal{B}$ and $i \geq 0$, and that $\operatorname{Ext}_A^i(M, N)$ is an object in $\mathcal{B}$ for $M \in \mathcal{A}$ and $N \in \mathcal{B}$.

Proposition 1.4 *Let M and N be objects in $\mathcal{B}$, and V a G-module. Then, the following hold.*

1 *If V is finite dimensional, then there is a canonical isomorphism*

$$\mathrm{Ext}^i_{\mathcal{B}}(M \otimes V, N) \cong \mathrm{Ext}^i_{\mathcal{B}}(M, \mathrm{Hom}(V, N)) \cong \mathrm{Ext}^i_{\mathcal{B}}(M, N \otimes V^*).$$

2 *There is a spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}^p_G(V, \mathrm{Ext}^q_{\mathcal{A}}(M, N)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{B}}(M \otimes V, N).$$

3 *If $M, N \in \mathcal{A}$, then the canonical map*

$$\mathrm{Ext}^i_{\mathcal{A}}(M, N) \to \mathrm{Ext}^i_{\mathcal{B}}(M, N) \tag{1}$$

is an isomorphism for any $i \geq 0$. Moreover, $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$ is finite dimensional for $i \geq 0$. In particular, the Krull-Schmidt theorem holds for $\mathcal{A}$.

The author does not know if the map (1) is isomorphic when we consider non-reductive groups.

2 Cohen-Macaulay approximations and Ringel's Δ-good approximations

Thanks to the construction in the last section, the canonical module $K_A := \mathrm{Ext}^h_S(A, \omega_{S/k})$ of A is endowed with a (G, A)-module structure, where h is the codimension of A, and $\omega_{S/k}$ is the (G, S)-module $\bigwedge^{\mathrm{top}} \Omega_{S/k} = S \otimes \bigwedge^{\mathrm{top}} Q$.

Let $\mathcal{X}$ be the full subcategory of $\mathcal{A}$ consisting of all M such that M is maximal Cohen-Macaulay as an A-module and that the canonical dual $\mathrm{Hom}_A(M, K_A)$ admits a good filtrations as a G-module. Let $\mathcal{Y}$ be the full subcategory of $\mathcal{A}$ consisting of all N such that the A-injective dimension of N is finite, and $\mathrm{Hom}_A(K_A, N)$ admits a good filtrations as a G-module. Here, we say that a G-module V admits a good filtration when there is a filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V, \qquad \bigcup_i V_i = V$$

of G-modules such that V_i/V_{i-1} is isomorphic to some induced module of G (or equivalently, $(V_i/V_{i-1})^*$ is isomorphic to some Weyl module of G) for any $i \geq 1$. This notion has been considered by many authors in representation theory of algebraic groups, see [3, 4, 5, 6, 9, 11, 12]. We set $\omega = \mathcal{X} \cap \mathcal{Y}$.

For a full subcategory $\mathcal{Z}$ of $\mathcal{A}$, $\hat{\mathcal{Z}}$ denotes the full subcategory of $\mathcal{A}$ consisting of all objects with a $\mathcal{Z}$-resolution of finite length. For $M \in \mathcal{A}$, we set $\mathcal{Z}\text{-resol.dim}(M) = \infty$ if $M \notin \hat{\mathcal{Z}}$, and define $\mathcal{Z}\text{-resol.dim}(M)$ to be the minimum length of the $\mathcal{Z}$-resolution of M if $M \in \hat{\mathcal{Z}}$.

Utilizing the theory of $\mathcal{X}$-approximation by M. Auslander and R. O. Buchweitz [2], we obtained the following.

Theorem 2.1 *Assume that S, A and K_A admits a good filtrations as G-modules. Then, the following holds.*

1 $\hat{\omega} = \mathcal{Y}$

2 *If $M \in \mathcal{A}$, then the following holds.*

i *($\mathcal{X}$-approximation) There is an exact sequence in $\mathcal{A}$*

$$0 \to Y \xrightarrow{i} X \xrightarrow{p} M \to 0$$

such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

ii *($\mathcal{Y}$-hull) There is an exact sequence in $\mathcal{A}$*

$$0 \to M \xrightarrow{\iota} Y \xrightarrow{\pi} X \to 0$$

such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

3 *For $M \in \mathcal{A}$, the following are equivalent.*

i $M \in \mathcal{X}$ **ii** $\mathrm{Ext}^i_{\mathcal{A}}(M, \mathcal{Y}) = 0$ $(i > 0)$

ii' $\mathrm{Ext}^1_{\mathcal{A}}(M, \mathcal{Y}) = 0$. **iii** $\mathrm{Ext}^i_{\mathcal{A}}(M, \omega) = 0$ $(i > 0)$

iv $M = 0$ *or* $\mathcal{X}_S$-resol.dim$(M) = h = \dim S - \dim A$,

where $\mathcal{X}_S$ is the full subcategory of the category of graded G-equivariant S-modules consisting of those F such that F is a finite S-free module and $\mathrm{Hom}_S(F, S)$ admits good filtrations. In particular, the map p in the exact sequence in **2**, **i** *is a right $\mathcal{X}$-approximation of M. We can make it right minimal, removing common direct summands through i if any.*

4 *For $M \in \mathcal{A}$, the following are equivalent.*

i $M \in \mathcal{Y}$ **ii** $\mathrm{Ext}^i_{\mathcal{A}}(\mathcal{X}, M) = 0$ $(i > 0)$ **ii'** $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{X}, M) = 0$

Hence, the map ι in the exact sequence in **2**, **ii** *is a left $\mathcal{Y}$-approximation of M. We can make it left minimal, after removing common direct summands through π if any.*

5 *For $M \in \mathcal{A}$, we have*

$$\begin{aligned} \mathcal{X}\text{-resol.dim}(M) &= \sup(\{i \mid \mathrm{Ext}^i_{\mathcal{A}}(M, \mathcal{Y}) \neq 0\} \cup \{0\}) \\ &= \sup(\{i \mid \mathrm{Ext}^i_{\mathcal{A}}(M, \omega) \neq 0\} \cup \{0\}) \end{aligned}$$

6 *For $Y \in \mathcal{Y}$, we have* ω-resol.dim$(Y) = \mathcal{X}$-resol.dim(Y).

7 *Any object in ω is uniquely a direct sum of indecomposable objects of the form $K_A \otimes T(\lambda)$, where $T(\lambda)$ is the indecomposable tilting G-module of high weight λ for a dominant weight λ of G (see* [5]*).*

The theorem generalizes both the graded version of Cohen-Macaulay approximations and Ringel's Δ-good approximations in the representations of reductive groups.

For the proof, as well as both Cohen-Macaulay approximations and Ringel's Δ-good approximations, the following theorem plays an important role.

Theorem 2.2 *Let V and W be G-modules with good filtrations. Then, $V \otimes W$ also admits a good filtration.*

This theorem was proved by S. Donkin for the case the characteristic of k is not two or G does not have a component of type E_7 or E_8, and O. Mathieu proved in the full generality [11].

3 Examples of determinantal rings

Furthermore, the determinantal ring satisfies the assumption of the theorem. Namely, if $1 \leq t \leq m \leq n$, $V = k^m$, $W = k^n$, $G = GL(V) \times GL(W)$, $S = \mathrm{Sym}(V \otimes W)$, and I is the ideal of S generated by all t-minors of a generic $m \times n$ matrix, then the assumption of the theorem is satisfied. We have the following.

0 There is a canonical inclusion $\mathbb{G}_m \to Z(G)$ given by $t \mapsto (t \cdot 1_V, t \cdot 1_W)$, and $Q = V \otimes W$ is of degree one. It is easy to see that I is a (homogeneous) G-ideal.

1 M. Hochster and J. A. Eagon proved that I is perfect of codimension $(m-t+1)(n-t+1)$, so A is Cohen-Macaulay [8].

2 The straightening formula (see e.g., [1]) tells that both S and I are good, so $A = S/I$ is also good.

Thus, the only thing to be checked is the following.

Theorem 3.1 *The canonical module K_A is good.*

The proof consists in Kempf's construction, with some detailed observation on G-actions.

In what follows, a scheme mean a k-scheme separated of finite type. For two schemes X and Y, we denote the product $X \times_{\mathrm{Spec}\, k} Y$ by $X \times Y$. For a k-morphism $f : X \to Y$ and a vector bundle $\mathcal{V}$ over Y, we sometimes denote $f^*\mathcal{V}$ simply by $\mathcal{V}$, by abuse of notation.

Let X be a scheme, and $\mathcal{V}$ and $\mathcal{W}$ vector bundles over X of rank m and n, respectively. Then, there is a canonical functor F from the category of G-modules to the category of $GL(\mathcal{V}) \times GL(\mathcal{W})$-modules. For a G-module M, we denote $F(M)$ by $M(\mathcal{V}, \mathcal{W})$. If X is a G-scheme and $\mathcal{V}$ and $\mathcal{W}$ are homogeneous vector bundles, then $M(\mathcal{V}, \mathcal{W})$ is a G-linearlizable quasi-coherent $\mathcal{O}_X$-module in a natural way.

We also need the notion of universal functors. Let A be a commutative k-algebra. We denote the category of A-modules by ${}_A\mathbb{M}$. The full subcategory of finite free A-modules is denoted by $F(A)$. A *universal functor* of type (r, s) is a pair family of functors $M = ((M_A), (\rho_f))$, where

1 A runs through all k-algebras.

2 M_A is a functor from $F(A)^r \times (F(A)^{\mathrm{op}})^s$ to ${}_A\mathbb{M}$ for each A.

3 f runs through all k-algebra maps.

4 For each $f : A \to B$, ρ_f is a natural isomorphism from $(B\otimes ?)\circ M_A$ to $M_B \circ ((B\otimes ?)^r \times (B\otimes ?)^s)$, which satisfies the condition: For any maps of commutative k-algebras $f : A \to B$ and $g : B \to C$, the following diagram is commutative:

$$\begin{array}{ccc}
C\otimes_B B\otimes_A M_A & \xrightarrow{\quad C\otimes_B \rho_f \quad} & C\otimes_B M_B(B\otimes_A ?)^r,(B\otimes_A ?)^s) \\
 & & \Big\downarrow \rho_g((B\otimes_A ?)^r,(B\otimes_A ?)^s) \\
\Big\downarrow \alpha_{f,g} M_A & & M_C((C\otimes_B (B\otimes_A ?))^r,(C\otimes_B (B\otimes_A ?))^s) \\
 & & \Big\downarrow M_C(\alpha^r,\alpha^s) \\
C\otimes_A M_A & \xrightarrow{\quad \rho_{gf} \quad} & M_C((C\otimes_A ?)^r,(C\otimes_A ?)^s)
\end{array}$$

where $\alpha : C\otimes_B (B\otimes_A ?) \to C\otimes_A ?$ is the usual identification.

For a universal functor M of type (r,s) and k-vector spaces $V_1,\ldots,V_r,W_1,\ldots,W_s$, there is a canonical representation $M(V_1,\ldots,V_r,W_1,\ldots,W_s)$ of $GL(V_1)\times\cdots\times GL(V_r)\times GL(W_1)\times\cdots\times GL(W_s)$, restricting each M_A to the groupoid $GL(V_1\otimes A)\times\cdots\times GL(V_r\times A)\times GL(W_1\otimes A)^{op}\times\cdots GL(W_s\otimes A)^{op}$.

A morphism of universal functors $\phi : M \to N$ is a collection $\phi=(\phi_A)$ such that for each k-algebra A, $\phi_A : M_A \to N_A$ is a natural transformation, and the compatibility $B\otimes_A \phi_A = \phi_B((B\otimes_A ?)^r,(B\otimes_A ?)^s)$ is satisfied for any k-algebra map $A\to B$. The category of universal functors (of type (r,s)) is obtained. This category is abelian.

If $M_1,\ldots,M_{r+s}$ are universal functors of type $(r_1,s_1),\ldots,(r_{r+s},s_{r+s})$, respectively, each $(M_i)_k$ is a functor to the category of *finite dimensional* k-vector spaces, and N is a universal functor of type (r,s), then $N(M_1(?),\ldots,M_{r+s}(?))$ is also universal of type $(\sum_{i=1}^r r_i+\sum_{i=r+1}^{s+1} s_i,\sum_{i=1}^r s_i+\sum_{i=r+1}^{s+1} r_i)$, as can be easily seen.

For example, $(V_1,\ldots,V_r)\mapsto V_1\otimes\cdots\otimes V_r$ is universal of type $(r,0)$. The symmetric power S_i, the exterior power $\bigwedge^i$, and the divided power D_i are universal of type $(1,0)$ for any $i\geq 0$. The hom-group $\mathrm{Hom}(?_2,?_1)$ is universal of type $(1,1)$.

We consider the case $(r,s)=(1,0)$, the simplest case. In this case, we call a universal functor a polynomial representation of GL. There is a remarkable family of polynomial representations of GL, parameterized by partitions. A sequence of non-negative integers $\lambda=(\lambda_1,\lambda_2,\ldots)$ is called a *partition* if it is weakly decreasing and $\lambda_i=0$ for $i\gg 0$. For each partition λ, a polynomial representation $\nabla(\lambda)$ of GL corresponds (the Schur functor $L_{\tilde\lambda}$ of Akin-Buchsbaum-Weyman, see [1]). Let V be an n-dimensional vector space. Then, $\nabla(\lambda)(V)=0$ if $\lambda_{n+1}\neq 0$, while $\nabla(\lambda)(V)$ is the induced module of highest weight $(\lambda_1,\ldots,\lambda_n)$ of $GL(V)$, if $\lambda_{n+1}=0$.

A universal functor M of type $(2,0)$ is said to be a functor with good filtrations, if M admits a filtration

$$0=M_0\subset M_1\subset M_2\subset\cdots$$

of M such that $\varinjlim M_i=M$ and $M_i/M_{i-1}\cong\nabla(\lambda)(?_1)\otimes\nabla(\mu)(?_2)$ for $i\geq 1$. Good filtrations for universal functors with more variables are defined similarly.

The following is a restatement of the theorem of Akin-Buchsbaum-Weyman.

Proposition 3.2 *The symmetric algebra* $S(?_1,?_2)=\mathrm{Sym}(?_1\otimes ?_2)$ *admits a good filtrations as a universal functor of type* $(2,0)$. *The determinantal ideal generated by* *t-minors in $S(?_1,?_2)$ is also a universal functor, which we denote by $I(?_1,?_2)$, also admits good filtrations.*

Now we consider $\mathbf{X} := \mathrm{Hom}(V, W^*) \cong \mathrm{Spec}\, S$, and we denote by Y the closed subscheme $\mathrm{Spec}\, A = \mathrm{Spec}\, S/I$. We denote the Grassmann variety of $(t-1)$-quotients of V by $\mathbf{G}$. Note that $\mathbf{G} = G/P$ for some appropriate maximal parabolic subgroup of P, and $\mathbf{G}$ is a k-smooth projective G-variety (on which $GL(W)$ acts trivially). Let

$$0 \to \mathcal{R} \to V \to \mathcal{Q} \to 0$$

be the tautological exact sequence in $\mathbf{G}$. Note that $\mathcal{Q}$ is of rank $t-1$, and the exact sequence is that of homogeneous G-bundles.

We set $\mathbf{Z} := \mathcal{R} \otimes W^*$, which is also a vector bundle over $\mathbf{G}$. Obviously, there is an exact sequence of homogeneous bundles

$$0 \to \mathbf{Z} \xrightarrow{i} \mathbf{X} \times \mathbf{G} \to \mathcal{Q} \otimes W^* \to 0.$$

We denote the first (resp. second) projection from $\mathbf{X} \times \mathbf{G}$ to $\mathbf{X}$ (resp. $\mathbf{G}$) by p_1 (resp. p_2).

The following is a variation of Kempf's vanishing.

Proposition 3.3 *Let M be a universal functor of type $(3,0)$ with good filtrations. Then, we have $H^i(\mathbf{G}, M(V, \mathcal{Q}, W)) = 0$ for $i > 0$, and $H^0(\mathbf{G}, M(V, \mathcal{Q}, W))$ is a G-module with good filtrations. Moreover, the map of G-modules*

$$M(V, V, W) \cong H^0(\mathbf{G}, M(V, V, W)) \to H^0(\mathbf{G}, M(V, \mathcal{Q}, W))$$

induced by the canonical map $M(V, V, W) \to M(V, \mathcal{Q}, W)$ is surjective.

This proposition is essentially proved and used by Roberts-Weyman [13].

As p_2 is affine and $(p_2)_*\mathcal{O}_\mathbf{Z} = \mathrm{Sym}(\mathcal{Q} \otimes W) = S(\mathcal{Q}, W)$, we have that $R^i\pi_*\mathcal{O}_\mathbf{Z} = 0$ for $i > 0$, and $H^0(\mathbf{Z}, \mathcal{O}_\mathbf{Z})$ admits good filtrations. Moreover, as we have $I(\mathcal{Q}, W) = 0$, we have $S(\mathcal{Q}, W) \cong S/I(\mathcal{Q}, W)$. Some detailed observation yields that the canonical map $S/I = S/I(V, W) \to H^0(\mathbf{G}, S/I(\mathcal{Q}, W))$ is isomorphic. With some additional observation, we have the following.

Proposition 3.4 (Kempf) *The composite morphism $p_1 i : \mathbf{Z} \to \mathbf{X}$ factors through $\mathbf{Y}$, and induces a morphism $\pi : \mathbf{Z} \to \mathbf{Y}$. The morphism π is a resolution of singularities. Moreover, we have that $R^i\pi_*\mathcal{O}_\mathbf{Z} = 0$ for $i > 0$ and $\pi_*\mathcal{O}_\mathbf{Z} \cong \mathcal{O}_Y$. Hence, Y is a normal variety.*

Now we calculate the canonical module K_A.

Proposition 3.5 *The canonical sheaf $\omega_Z := \bigwedge^{\mathrm{top}} \Omega_{Z/k}$ is isomorphic to*

$$(\textstyle\bigwedge^{\mathrm{top}} V)^{\otimes(t-1)} \otimes (\bigwedge^{\mathrm{top}} W)^{\otimes(t-1)} \otimes (\bigwedge^{\mathrm{top}} \mathcal{Q})^{\otimes(n-m)}$$

as a G-equivariant $\mathcal{O}_Z$-module.

Hence, we have that $R^i\pi_*\omega_Z = 0$ for $i > 0$. Now we may invoke the theory of rational singularity [10], and we have that Y is Cohen-Macaulay (already proved by Hochster-Eagon), and $\pi_*\omega_Z \cong K_A$. This shows that K_A admits good filtrations, and Theorem 3.1 is proved.

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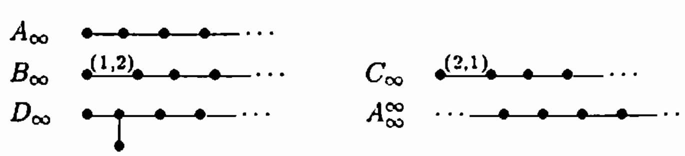
AUSLANDER-REITEN COMPONENTS AND TRIVIAL MODULES FOR INTEGRAL GROUP RINGS

TAKAYUKI INOUE AND SHIGETO KAWATA

ABSTRACT. Let G be a non-cyclic p-group and $\mathcal{O}$ a complete discrete valuation ring of characteristic 0 with the maximal ideal (π) and the residue field $\mathcal{O}/(\pi)$ of characteristic $p > 0$. Let Δ be the Auslander-Reiten component containing the trivial $\mathcal{O}G$-module $\mathcal{O}_G$. Suppose furthermore, if $p = 2$ and G is the Klein four group, $(\pi) \neq (2)$. Then the tree class of Δ is A_∞.

G を有限群とする. $\mathcal{O}$ は標数 0 の完備離散付値環, (π) は $\mathcal{O}$ の極大イデアルで, 剰余体 $k = \mathcal{O}/(\pi)$ の標数は $p > 0$（p は $|G|$ を割り切るある素数）であるとする. R で $\mathcal{O}$ または k を表すものとする. 群環 RG の Auslander-Reiten component (AR-component) Θ の tree class について, Webb は次を示した : Θ が属するブロックの不足群が巡回群でなければ, Θ の tree class は



かまたは Euclidean diagrams のどれかである [13, Theorem A]. さらに Erdmann は $R = k$ の場合には, Θ が属するブロックが wild ならば Θ の tree class は A_∞ であることを示した [7]. ここでは, $R = \mathcal{O}$ の場合に, 次の事実が得られたことを報告したい.

定理 G を巡回群ではない p-群とし, Δ を自明な $\mathcal{O}G$ 加群 $\mathcal{O}_G$ を含む AR-component とする. さらに, もし $p = 2$ で G がクラインの 4 元群のときには, $(\pi) \neq (2)$ も仮定する. このとき Δ の tree class は A_∞ である.

G が p-群のときは, RG 自身がブロックになっている. 定理の仮定のもとでは $\mathcal{O}G$ は wild であることが知られている [5]. また G がクラインの 4 元群で $(\pi) = (2)$ のと

The detailed version of this paper will be submitted for publication elsewhere.

きは，自明な加群 $\mathcal{O}_G$ を含む AR-component の tree class は $\tilde{D}_4$ となることが知られている [4, Proposition 3.4].

ここで考える RG-加群はすべて R上有限生成で R-自由とする．射影的でない直既約 RG-加群 W に対して $\mathcal{A}(W)$ と書いて Auslander-Reiten 列 (AR-列) $0 \to \tau W \to M(W) \to W \to 0$ を表すことにする ($M(W)$ で $\mathcal{A}(W)$ の中間項を表す). $R = \mathcal{O}$ のときは $\tau = \Omega$ であり，$R = k$ のときは $\tau = \Omega^2$ である (ここで Ω は Heller operator) (例えば [10], [1] 参照). また $\mathcal{O}G$-加群 W に対して $\overline{W}$ と書いて kG-加群 $W/\pi W$ を表すこととする．群環の Auslander-Reiten の理論の基本事項については，[2], [6] を参照する．

1. 射影的 $\mathcal{O}G$-準同型写像

RG-準同型写像 $f : M \to N$ が射影加群を経由するとき，f を射影的とよぶ．この節では，G は p-群として，射影的 $\mathcal{O}G$-準同型写像について考える．$\mathcal{O}G$-加群 M, N に対して $\underline{\mathrm{Hom}}_{\mathcal{O}G}(M,N) := \mathrm{Hom}_{\mathcal{O}G}(M,N)/\mathrm{Proj}(\mathrm{Hom}_{\mathcal{O}G}(M,N))$ とおく．ここで $\mathrm{Proj}(\mathrm{Hom}_{\mathcal{O}G}(M,N))$ は M から N への射影的 $\mathcal{O}G$-準同型写像の全体である．また $\mathcal{O}G$-加群 M に対して id_M と書いて M の恒等写像を表すことにする．

ところで，自明な $\mathcal{O}G$-加群 $\mathcal{O}_G$ と，$\mathcal{O}$ の元 α について，$\alpha\mathrm{id}_{\mathcal{O}_G}$ が射影的であるためには

$$\alpha\mathrm{id}_{\mathcal{O}_G} = \mathrm{Tr}_1^G(\lambda\mathrm{id}_{\mathcal{O}}) \quad (\exists\lambda \in \mathcal{O})$$

とかけることが必要十分なので次がわかる．

補題 1.1 α を $\mathcal{O}$ の元とする．このとき，$\alpha\mathrm{id}_{\mathcal{O}_G}$ が射影的 $\Longleftrightarrow \alpha \in (|G|)$. とくに，$\underline{\mathrm{End}}_{\mathcal{O}G}(\mathcal{O}_G) \cong \mathcal{O}/(|G|)$.

補題 1.1 から任意の $\mathcal{O}G$-加群 M について $|G|\mathrm{id}_M$ （$= |G|\mathrm{id}_{\mathcal{O}_G} \otimes_{\mathcal{O}} \mathrm{id}_M$）は射影的である．とくに任意の $\mathcal{O}G$-加群 M, N について $\underline{\mathrm{Hom}}_{\mathcal{O}G}(M,N)$ はトーション $\mathcal{O}$-加群である．これから $\ell(\underline{\mathrm{Hom}}_{\mathcal{O}G}(M,N))$ と書いて $\underline{\mathrm{Hom}}_{\mathcal{O}G}(M,N)$ の $\mathcal{O}$-加群としての長さを表すことにする．

次の補題 1.2 は次節で "additive function" を構成するときに用いる．

補題 1.2 G は p-群とし，$\mathcal{O}G$-加群 M は projective-free とする．また M の射影被覆を P_M とおく．このとき次の不等式が成立する:

$$\mathrm{rank}_{\mathcal{O}} P_M \leq |G|(\ell(\underline{\mathrm{Hom}}_{\mathcal{O}G}(M,\mathcal{O}_G)) + \ell(\underline{\mathrm{Hom}}_{\mathcal{O}G}(\Omega M,\mathcal{O}_G))).$$

2. Additive functions

G は一般の有限群とし，Θ を群環 RG の AR-component とする．$f:\Theta\to\mathbf{N}$ が Θ に属する任意の W に対して $f(W)+f(\tau W)=\sum a_{VW}f(V)$ (ここで a_{VW} は AR-列 $\mathcal{A}(W)$ の中間項 $M(W)$ における V の重複度) が成り立つとき，f を Θ 上の additive function という．さらに任意の W に対して $f(W)=f(\tau W)$ が成り立つとき，f を τ-periodic additive function とよぶことにする．

Okuyama は，[9] で群環 kG の AR-component 上に additive function を構成した．この節では，そこでの議論を工夫することによって，$\mathcal{O}G$ の AR-component においても additive function が構成できることを示したい．(奥山氏から教えていただいた.)

$\mathcal{O}G$-加群 X, W に対して，

$$d_X(W)=\ell(\underline{\mathrm{Hom}}_{\mathcal{O}G}(X,W))+\ell(\underline{\mathrm{Hom}}_{\mathcal{O}G}(\Omega^{-1}X,W))$$

とおく．

Θ から直既約 $\mathcal{O}G$-加群 V を一つとってくる．P を，$V\downarrow_P$ が射影的にならないような G の p-部分群のなかで極小なものとし，U を $V\downarrow_P$ の直和因子で射影的でない $\mathcal{O}P$-加群とする．このとき [3, Lemma 2.5] から $\Omega^2U\cong U$ が成り立つ．[9, Lemma 3] と同じ議論によって，Θ に属する任意の W に対し $U^*\otimes W\downarrow_P$ が射影的ではないことがわかる (ここで，$U^*=\mathrm{Hom}_{\mathcal{O}}(U,\mathcal{O})$: 双対加群)．これらの事実と，補題 1.2 から次のことが導かれる．

命題 2.1 $d=d_{U\uparrow^G}$ とおく ($U\uparrow^G=U\otimes_{\mathcal{O}P}\mathcal{O}G$: 誘導加群).

(1) Θ に属する任意の W に対し $d(W)>0$.

(2) Θ に属する任意の W に対し $d(W)=d(\Omega W)$.

(3) $\mathcal{A}(W):0\to\Omega W\to(\oplus a_{VW}V)\oplus(\text{projectives})\to W\to 0$ を AR-列とする (ただし V は $M(W)$ の射影的ではない直既約因子で，a_{VW} は $M(W)$ における V の重複度)．このとき $2d(W)\geq\sum a_{VW}d(V)$.

(4) もし Θ が Ω-periodic な直既約加群を含まなければ，(3) で等号が成り立つ．

この命題と，AR-列の性質から次の事実が得られる．

系 2.2 Θ が periodic な直既約加群を含まなければ，d は Ω-periodic additive function である．とくに d は Θ の tree class 上の additive function を誘導する．

系 2.3 X は Θ に属さない直既約加群で，Θ に属する任意の直既約加群 W に対し $X^*\otimes W$ は射影的ではないとする．このとき d_X は Θ 上の (Ω-periodic とは限らない) additive function である．

3. $\mathcal{A}(\mathcal{O}_G)$ の中間項 $M(\mathcal{O}_G)$

まず Thévenaz [12, Section 6] に従って，射影的でない直既約 $\mathcal{O}G$-加群 W に対して AR-列 $\mathcal{A}(W)$ を構成する．

$\underline{\mathrm{End}}_{\mathcal{O}G}(W) := \mathrm{End}_{\mathcal{O}G}(W)/\{$ 射影的自己準同型写像 $\}$ は simple socle を持つ．ρ をその simple socle の生成元とすると $\mathcal{A}(W)$ は次のような pull-back として得られる：

$$\begin{array}{ccccccccc} 0 & \to & \Omega W & \to & M(W) & \to & W & \to & 0 \\ & & \| & & \downarrow & \text{P.B.} & \downarrow \rho & & \\ 0 & \to & \Omega W & \to & P_W & \to & W & \to & 0 \end{array}$$

(ただし，P_W は W の射影被覆)．

さて，G は p-群とし，$\mathcal{O}_G$ の AR-列 $\mathcal{A}(\mathcal{O}_G)$ を考える．補題 1.1 より $\underline{\mathrm{End}}_{\mathcal{O}G}(\mathcal{O}_G) \cong \mathcal{O}/(|G|)$ なので $|G|\pi^{-1}\mathrm{id}_{\mathcal{O}_G}$ が $\underline{\mathrm{End}}_{\mathcal{O}G}(\mathcal{O}_G)$ の simple socle の生成元となる．$\mathcal{O}_G$ の射影被覆 $\mathcal{O}G \to \hat{G}\mathcal{O}G = \mathcal{O}_G$ ($\mathcal{O}G \ni x \mapsto \hat{G}x \in \hat{G}\mathcal{O}G$, ここで，$\hat{G} = \sum_{x \in G} x$) と $|G|\pi^{-1}\mathrm{id}_{\mathcal{O}_G}$ による pull-back を見ることにより，AR-列 $\mathcal{A}(\mathcal{O}_G)$ の中間項 $M(\mathcal{O}_G)$ は $\mathcal{O}G \oplus \mathcal{O}G$ の部分加群として次のようなものであることがわかる：

$$M(\mathcal{O}_G) = (|G|\pi^{-1}1, \hat{G})\mathcal{O}G + \sum_{x \in G}(x-1, 0)\mathcal{O}G \subset \mathcal{O}G \oplus \mathcal{O}G.$$

このことから次を得る．

補題 3.1([8, Proposition 3.2]) $M(\mathcal{O}_G)$ は直既約である．

補題 3.2 $\mathrm{End}_{\mathcal{O}G}(M(\mathcal{O}_G))/J(\mathrm{End}_{\mathcal{O}G}(M(\mathcal{O}_G))) \cong k = \mathcal{O}/(\pi)$．とくに $\mathrm{End}_{\mathcal{O}G}(M(\mathcal{O}_G))/J(\mathrm{End}_{\mathcal{O}G}(M(\mathcal{O}_G))) \cong \mathrm{End}_{\mathcal{O}G}(\mathcal{O}_G)/J(\mathrm{End}_{\mathcal{O}G}(\mathcal{O}_G))$.

補題 3.1 から $\mathcal{O}_G$ は Δ の中で '端' に位置していて，Δ の tree class は A_∞^∞ ではないことがわかる．また補題 3.2 から，

$$\mathrm{rad}(\mathrm{Hom}_{\mathcal{O}G}((M(\mathcal{O}_G), \mathcal{O}_G))/\mathrm{rad}^2(\mathrm{Hom}_{\mathcal{O}G}((M(\mathcal{O}_G), \mathcal{O}_G))$$

の $\mathrm{End}_{\mathcal{O}G}(M(\mathcal{O}_G))$-加群としての長さと $\mathrm{End}_{\mathcal{O}G}(\mathcal{O}_G)$-加群としての長さが（共に k-次元となって）等しくなるが，これは AR-列 $\mathcal{A}(\mathcal{O}_G)$ の中間項における $M(\mathcal{O}_G)$ の重複度 $(=1)$ と，AR-列 $\mathcal{A}(\Omega^{-1}M(\mathcal{O}_G))$ の中間項における $\mathcal{O}_G$ の重複度が等しいことを意味しているので，Δ の tree class は B_∞, C_∞ ではないことがわかる．

Webb の結果 [13, Theorem A] から，Δ の tree class は $A_\infty, B_\infty, C_\infty, D_\infty, A_\infty^\infty$ かまたは Euclidean diagram であるので，定理を証明するためにはあとは，Δ の tree class

は D_∞ でも Euclidean diagram でもないことを示せばよい.

そのために場合分けをする.

Case I. $|G| \geq p^3$ または $(\pi) \neq (p)$.

Case II. $G = C_p \times C_p$ (p:奇数) かつ $(\pi) = (p)$.

4 節で Case I を考え, 5 節で Case II を考えることにする.

4. Case I

この節では Case I の仮定

$$|G| \geq p^3 \text{または} (\pi) \neq (p)$$

のもとで定理が成り立つことを示したい.

次の補題 4.1 を利用することによって, Case I のときは Δ の tree class は D_∞ ではないことが導ける.

補題 4.1 Case I の仮定のもとでは, Δ に属する任意の (射影的でない) 直既約 $\mathcal{O}G$-加群 W に対して, $\overline{\mathcal{A}(W)} : 0 \to \overline{\Omega W} \to \overline{M(W)} \to \overline{W} \to 0$ は分裂する.

また, [13,Theorem A] の証明の中の議論と同様な考察により, Δ の tree class は Euclidean diagram でないことが示される. 実際 :

Δ に属する直既約加群 W に対して, $r(W) = \mathrm{rank}_{\mathcal{O}} W$ とおく. $\mathcal{O}G$-加群 W が projective-free ならば $\overline{W}$ も projective-free である. よって補題 4.1 から Case I のときは Δ は射影加群を含まない. このことから r は Δ 上の additive function となる. いま, Δ の tree class が Euclidean diagram と仮定すると, r は bounded となり ([13, Corollary 2.4]), とくに $\{\dim_k \Omega^n k_G\}$ が bounded だから $\mathcal{O}_G$ が Ω-periodic となる. しかしこれは $\mathcal{O}G$ が有限表現型であることになり矛盾である. よって Δ の tree class は Euclidean diagram でないこともわかるので, Δ の tree class は A_∞ である.

5. Case II

この節では Case II の仮定

$$G = C_p \times C_p \ \ (p : \text{奇数}) \text{かつ} (\pi) = (p)$$

のもとで定理が成り立つことを示す. Case II のときは次の補題 5.1 から, Δ の tree class は D_∞ ではないことが導ける.

補題 5.1　Case II の仮定のもとでは,
$\overline{\mathcal{A}(M(\mathcal{O}_G))} : 0 \to \overline{\Omega M(\mathcal{O}_G)} \to \overline{M(M(\mathcal{O}_G))} \to \overline{M(\mathcal{O}_G)} \to 0$ は, AR-列 $\mathcal{A}(k_G) : 0 \to \Omega^2 k_G \to M(k_G) \to k_G \to 0$ と split 列 $0 \to \Omega k_G \to \Omega k_G \oplus \Omega k_G \to \Omega k_G \to 0$ の直和である.

最後に, Δ の tree class は Euclidean diagram でないことを示したい.

$G = \langle x \rangle \times \langle y \rangle$ とおく. $\mathcal{O}\langle x \rangle$-加群 $\Omega \mathcal{O}_{\langle x \rangle}$ に $\langle y \rangle$ を自明に作用させることにより, これを $\mathcal{O}G$-加群とみなす. この $\mathcal{O}G$-加群を X とおく. X について次がわかる.

補題 5.2　(1) Δ は X を含まない.
(2) Θ に属する任意の直既約加群 W に対し $X^* \otimes W$ は射影的ではない.
(3) $\overline{X}$ は Ω-periodic ではない.

さて, 補題 5.2 と系 2.3 から d_X は Θ 上の (Ω-periodic とは限らない) additive function である. いま, Δ の tree class が Euclidean diagram と仮定すると, d_X は bounded となり ([13, Corollary 2.4]), 補題 1.2 からとくに $\{\dim_k \Omega^n X\}$ が bounded で $\mathcal{O}_G$ が Ω-periodic となる. しかしこれは補題 5.2(3) に矛盾する. よってΔ の tree class は Euclidean diagram でないことがわかった.

以上のことから, Δ の tree class は A_∞ であると結論できる.

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A GENERALIZATION OF REJECTION LEMMA OF DROZD-KIRICHENKO

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Abstract

Let R be a complete discrete valuation ring and Λ be an R-order. Rejection Lemma of Drozd-Kirichenko **[HN-1][DK]** asserts that if projective injective Λ-lattice L satisfies $L \not\simeq \operatorname{rad} L$, there is an overorder Γ of Λ such that any indecomposable Λ-lattice except L is also Γ-lattice. In this paper, by using the language of Auslander-Reiten quiver of Λ, we characterize every finite subset $\mathcal{S}$ of $\operatorname{Ind}\Lambda$ such that there is an overorder Γ of Λ which satisfies $\operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma = \mathcal{S}$.

1 Introduction

In this paper, let $(R, \Lambda, {}_\Lambda\mathfrak{M})$ be one of the followings.

(1) R: complete discrete valuation ring, Λ: R-order in semisimple $Q(R)$-algebra, ${}_\Lambda\mathfrak{M}$: category of left Λ-lattices **[CR]**.

(2) R: commutative artin ring, Λ: R-algebra which is finitely generated as R-module, ${}_\Lambda\mathfrak{M}$: category of finitely generated left Λ-modules.

We call Λ in (1) an order, and Λ in (2) an algebra. In both cases, Krull-Schmidt Thorem and Existence Theorem of almost split sequences hold in ${}_\Lambda\mathfrak{M}$.

$$\begin{aligned} \operatorname{Ind}\Lambda &:= \{ L \in {}_\Lambda\mathfrak{M} \mid L \text{ is indecomposable } \} \\ \operatorname{proj}\Lambda &:= \{ L \in \operatorname{Ind}\Lambda \mid L \text{ is projective } \} \\ \operatorname{inj}\Lambda &:= \{ L \in \operatorname{Ind}\Lambda \mid L \text{ is injective } \} \end{aligned}$$

For an overorder (resp. quotient algebra) Γ of Λ, we may naturally consider $\operatorname{Ind}\Gamma$ as a subset of $\operatorname{Ind}\Lambda$.

1.1 Definition A subset $\mathcal{S}$ of $\operatorname{Ind}\Lambda$ will be called

(1) *rejectable* if there is an overorder (resp. quotient algebra) Γ such that $\mathcal{S} = \operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma$.

(2) *bounded* if $\sup_{L\in\mathcal{S}} l(L) < \infty$.

Here, $l(L) := \begin{cases} \operatorname{length}_{(Q(R)\otimes_R\Lambda)}(Q(R)\otimes_R L). & (\ \Lambda \text{ is an order.}\ ) \\ \operatorname{length}_\Lambda(L) & (\ \Lambda \text{ is an algebra.}\ ) \end{cases}$

(3) *trivial* if $\Lambda(\mathcal{S}) = \Lambda$.

Here, $\Lambda(\mathcal{S}) := \begin{cases} \bigcap_{L\in\operatorname{Ind}\Lambda-\mathcal{S}}\{x \in Q(R)\otimes_R\Lambda \mid xL \subseteq L\} & (\ \Lambda \text{ is an order.}\ ) \\ \Lambda/(\bigcap_{L\in\operatorname{Ind}\Lambda-\mathcal{S}}\operatorname{Ann} L) & (\ \Lambda \text{ is an algebra.}\ ) \end{cases}$

(4) *cofaithful* if $\begin{cases} \bigoplus_{L\in\operatorname{Ind}\Lambda-\mathcal{S}} L \text{ is a faithful } \Lambda\text{-module.} & (\ \Lambda \text{ is an order.}\ ) \\ \text{always} & (\ \Lambda \text{ is an algebra.}\ ) \end{cases}$

The detailed version of this paper will be submitted for publication elsewhere.

The map $\Gamma \mapsto (\operatorname{Ind}\Lambda - \operatorname{Ind}\Gamma)$ defines a bijection from the set of all overorders (resp. quotient algebras) of Λ onto the set of all rejectable subsets of $\operatorname{Ind}\Lambda$.

1.2 Our Results In this paper, we give a criterion for bounded $\mathcal{S}$ to be rejectable (=Rejection Lemma). Because $\mathcal{S}$ is minimal rejectable if and only if $\mathcal{S}$ is minimal non-trivial cofaithful, it suffices to give the followings.

(A) A criterion for $\mathcal{S}$ to be cofaithful (2.2).

(B) A criterion for $\mathcal{S}$ to be trivial (2.3).

(C) An algorithm to describe the Auslander-Reiten quiver $\mathfrak{A}(\Lambda(\mathcal{S}))$ of $\Lambda(\mathcal{S})$ from $\mathfrak{A}(\Lambda)$ for rejectable $\mathcal{S}$ (2.4).

A remarkable fact is that the criterion for $\mathcal{S}$ to be rejectable depends, as in the case of D-K Rejection Lemma, only on the structure of $\mathcal{S}$, but not on the structure of the whole $\mathfrak{A}(\Lambda)$. To be precise, the information we need is the following.

(I1) Structure of $\mathcal{S}$ as valued translation quiver.

(I2) Preassignment of the subset $\mathcal{S}_p$ (resp. $\mathcal{S}_i$) consisting of projective (resp. injective) vertices in $\mathfrak{A}(\Lambda)$ contained in $\mathcal{S}$.

It turns out that for any valued translation quiver $\mathcal{S}$ with the information above (I1) and (I2), rejectableness of $\mathcal{S}$ coincides in order cases and algebra cases.

1.3 Notations

1.3.1 Let $\mathbb{Z}(\operatorname{Ind}\Lambda)$ be the free $\mathbb{Z}$-module generated by the base set $\operatorname{Ind}\Lambda$. By Krull-Schmidt theorem, we can identify ${}_\Lambda\mathfrak{M}$ with the submonoid $\mathbb{N}(\operatorname{Ind}\Lambda)$. On $\mathbb{Z}(\operatorname{Ind}\Lambda)$, we introduce the *inner product* $\langle\ ,\ \rangle$ taking $\operatorname{Ind}\Lambda$ as an orthonormal base. We shall also introduce an *ordering* in $\mathbb{Z}(\operatorname{Ind}\Lambda)$ by

$$X \leq Y \Leftrightarrow \langle X, L\rangle \leq \langle Y, L\rangle \text{ for any } L \in \operatorname{Ind}\Lambda$$

1.3.2 Let $\mathcal{S}$ be a subset of $\operatorname{Ind}\Lambda$. The inclusion $\mathcal{S} \subseteq \operatorname{Ind}\Lambda$ induces a $\mathbb{Z}$-mononorphism $i_{\mathcal{S}} : \mathbb{Z}\mathcal{S} \to \mathbb{Z}(\operatorname{Ind}\Lambda)$, by which we often identify as $\mathbb{Z}\mathcal{S} \subseteq \mathbb{Z}(\operatorname{Ind}\Lambda)$. Then $\mathbb{Z}(\operatorname{Ind}\Lambda - \mathcal{S})$ is the orthogonal complement of $\mathbb{Z}\mathcal{S}$ with respect to the inner product $\langle\ ,\ \rangle$. Let $|_{\mathcal{S}} : \mathbb{Z}(\operatorname{Ind}\Lambda) \to \mathbb{Z}\mathcal{S}$ denote the orthogonal projection.

1.3.3 The source map from $L \in \operatorname{Ind}\Lambda$ is written as $L \to \theta^- L$, the sink map to L is written as $\theta^+ L \to L$. The Auslander (resp. inverse Auslander) translate of L is written as $\tau^+ L$ (resp. $\tau^- L$). We put $\tau^- L$ (resp. $\tau^+ L$) := 0 if L is injective (resp. projective). $\theta^+, \theta^-, \tau^+$ and τ^- uniquely extend to elements of $\operatorname{End}_{\mathbb{Z}}(\mathbb{Z}(\operatorname{Ind}\Lambda))$. Moreover, we define $\phi^+, \phi^- \in \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}(\operatorname{Ind}\Lambda))$ and $\phi^+_{\mathcal{S}}, \phi^-_{\mathcal{S}} \in \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}\mathcal{S})$ as follows.

$$\phi^+ := 1 - \theta^+ + \tau^+, \qquad \phi^- := 1 - \theta^- + \tau^-$$
$$\phi^+_{\mathcal{S}} := |_{\mathcal{S}} \circ \phi^+ \circ i_{\mathcal{S}}, \qquad \phi^-_{\mathcal{S}} := |_{\mathcal{S}} \circ \phi^- \circ i_{\mathcal{S}}$$

1.3.4 The Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ of Λ is, by definition, a valued translation quiver with the vertex set $\operatorname{Ind}\Lambda$; the valued arrow $L \xrightarrow{(a,a')} M$ with $a := \langle L, \theta^+ M\rangle$, $a' := \langle \theta^- L, M\rangle$ (provided $a \neq 0$); the translation $\tau := \tau^+$ i.e. $\tau L := \tau^+ L$ (provided

$L \notin \operatorname{proj} \Lambda$). Whenever we regard $\mathcal{S}$ as a subset of the vertex set of $\mathfrak{A}(\Lambda)$, we consider $\mathcal{S}$ to be a full subquiver of $\mathfrak{A}(\Lambda)$.

Note that $\phi_{\mathcal{S}}^{+}$ and $\phi_{\mathcal{S}}^{-}$ *can be read from the information (I1) and (I2) of* $\mathcal{S}$.

2 Main theorems

2.1 Definition Let $\mathcal{S}$ be a subset of $\operatorname{Ind} \Lambda$ and $V \in \mathbb{Z}(\operatorname{Ind} \Lambda)$.

(1) A (finite or infinite) sequence $(T_j) = (T_0, T_1, \ldots)$ will be called an $\mathcal{S}^{-}$*-sequence for* V if the following two conditions are satisfied for any j.

(s1) $\quad 0 \neq T_j \in \mathbb{N}\mathcal{S}$

(s2) $\quad \sup\{0, \langle V - \sum_{i=0}^{j-1} \phi^{-} T_i, L\rangle\} \geq \langle T_j, L\rangle$ for any $L \in \operatorname{Ind} \Lambda$.

(2) Let $\mathcal{T}$ be a subset of $\operatorname{Ind} \Lambda$ containing $\mathcal{S}$ and $V \in \mathbb{Z}\mathcal{T}$.

$$[\mathcal{T}, \mathcal{S}]^{-} V := \{\, U \in \mathbb{Z}\mathcal{T} \mid \text{There exists an } \mathcal{S}^{-}\text{-sequence } (T_0, \ldots, T_m) \text{ for } V \text{ such that } V - \textstyle\sum_{i=0}^{m} \phi_{\mathcal{T}}^{-} T_i = U \,\}$$

$$\{\mathcal{T}, \mathcal{S}\}^{-} V := \{\, U \in [\mathcal{T}, \mathcal{S}]^{-} V \mid U|_{\mathcal{S}} \leq 0\}$$

Note that we can determine only from the information (I1) and (I2) of $\mathcal{S}$ whether (T_j) is an $\mathcal{S}^{-}$-sequence for V or not. Similarly, $[\mathcal{T}, \mathcal{S}]^{-} V$ and $\{\mathcal{T}, \mathcal{S}\}^{-} V$ is computable only from the information (I1) and (I2) of $\mathcal{T}$.

Put $\mathcal{S}_p := \mathcal{S} \cap \operatorname{proj} \Lambda, \mathcal{S}_i := \mathcal{S} \cap \operatorname{inj} \Lambda, P_{\mathcal{S}} := \bigoplus_{L \in \mathcal{S}_p} L$ and $I_{\mathcal{S}} := \bigoplus_{L \in \mathcal{S}_i} L$.

2.2 Theorem A *Let* $\mathcal{S}$ *be a non-empty bounded subset of* $\operatorname{Ind} \Lambda$. *Then the following conditions for* $\mathcal{S}$ *are equivalent.*

(c) $\mathcal{S}$ *is cofaithful.*

(c1) For any $V \in {}_{\Lambda}\mathfrak{M}$, *every* $\mathcal{S}^{-}$*-sequence for* V *is finite.*

(c2) $\{\mathcal{S}, \mathcal{S}\}^{-} P_{\mathcal{S}}$ *is not empty.*

2.3 Theorem B *Let* $\mathcal{S}$ *be a bounded subset of* $\operatorname{Ind} \Lambda$. *Then the following conditions for* $\mathcal{S}$ *are equivalent.*

(t) $\mathcal{S}$ *is trivial.*

(t1) $U|_{\mathcal{S}_i} \leq 0$ *for any* $U \in [\mathcal{S}, \mathcal{S} - \mathcal{S}_i]^{-} P_{\mathcal{S}}$.

(t2) $U|_{\mathcal{S}_i} \leq 0$ *for any* $U \in \{\mathcal{S}, \mathcal{S} - \mathcal{S}_i\}^{-} P_{\mathcal{S}}$.

(t3) There exists $U \in \{\mathcal{S}, \mathcal{S} - \mathcal{S}_i\}^{-} P_{\mathcal{S}}$ *such that* $U|_{\mathcal{S}_i} \leq 0$.

(t4) There exists $T \in \mathbb{N}(\mathcal{S} - \mathcal{S}_i)$ *such that* $\phi_{\mathcal{S}}^{-} T \geq P_{\mathcal{S}}$.

2.4 Recovering $\mathfrak{A}(\Gamma)$ from $\mathfrak{A}(\Lambda)$

2.4.1 Definition For an overorder Γ (resp. quotient algebra $\Gamma = \Lambda/I$) of Λ,

$$\dot{(\ )} : {}_{\Lambda}\mathfrak{M} \to {}_{\Gamma}\mathfrak{M} \quad (X \mapsto \dot{X} := \Gamma X \text{ (resp. } X/IX \text{))}$$

$$\underset{\cdot}{(\ )} : {}_{\Lambda}\mathfrak{M} \to {}_{\Gamma}\mathfrak{M} \quad (X \mapsto \underset{\cdot}{X} := \{x \in X \mid \Gamma x \subseteq X \text{ (resp. } Ix = 0 \text{) }\})$$

2.4.2 Proposition *Write* $(\theta^{-} M)^{\cdot} - (\tau^{-} M)^{\cdot}$ *as*

$$(\theta^{-} M)^{\cdot} - (\tau^{-} M)^{\cdot} = U - V \ (U, V \in {}_{\Gamma}\mathfrak{M}, \langle U, V\rangle = 0)$$

Then we have one of the following three results.

(1) If $V \neq 0$, then $M \notin \operatorname{inj}\Gamma$, $V \in \operatorname{Ind}\Gamma$ and the Γ-almost split sequence from M is given by

$$0 \longrightarrow M \longrightarrow U \longrightarrow V \longrightarrow 0$$

(2) If $V = 0$ and $M \in \operatorname{inj}\Gamma$, the complex of the Γ-source map from M is given by

$$0 \longrightarrow M \longrightarrow U \longrightarrow 0 \longrightarrow 0$$

(3) If $V = 0$ and $M \notin \operatorname{inj}\Gamma$, the Γ-almost split sequence from M is given by

$$0 \longrightarrow M \longrightarrow U \oplus M \longrightarrow M \longrightarrow 0$$

Remark that if the case (3) happens, then Λ is an order and $\mathfrak{A}(\Gamma)$ has a connected component of the form of 4.2.4.

2.4.3 Theorem C *Let $\mathcal{T}$ be a subset of* $\operatorname{Ind}\Lambda$ *containing a finite rejectable subset $\mathcal{S}$ and put $\Gamma := \Lambda(\mathcal{S})$. For any $V \in {}_\Lambda\mathfrak{M}$, $\{\mathcal{T},\mathcal{S}\}^- V$ is a singleton set consisting of $\dot{V}|_{\mathcal{T}}$.*

Remark that using the dual version of the Theorem C, we can determine whether M is injective or not. Hence, by 2.4.2 and the Theorem C and the duality 2.5, for any $\mathcal{T}$ containing a finite rejectable subset $\mathcal{S}$, whenever we are given the information (I1) and (I2) of $\mathcal{T}$ in $\mathfrak{A}(\Lambda)$, we can obtain the information (I1) and (I2) of $\mathcal{T} - \mathcal{S}$ in $\mathfrak{A}(\Lambda(\mathcal{S}))$.

2.5 Duality We explain the dual version of the above theorems. Namely, it is obviously valid for $\mathfrak{M}_\Lambda$ (i.e. for ${}_{\Lambda^{op}}\mathfrak{M}$) by the same proof. Taking the duality $(\ )^* : \mathfrak{M}_\Lambda \to {}_\Lambda\mathfrak{M}$, we get the dual result. We give a dictionary of duals here. The map

$$(\ )^* : \mathbb{Z}(\operatorname{Ind}\Lambda^{op}) \to \mathbb{Z}(\operatorname{Ind}\Lambda), X \mapsto X^*$$

is a $\mathbb{Z}$-isomorphism compatible with the inner product $\langle\ ,\ \rangle$, which induces a bijection $\operatorname{proj}\Lambda^{op} \to \operatorname{inj}\Lambda$, $\operatorname{inj}\Lambda^{op} \to \operatorname{proj}\Lambda$. Endomorphism θ^-, τ^-, ϕ^- corresponds to θ^+, τ^+, ϕ^+: For an overorder (resp. quotient algebra) Γ of Λ, $(\dot{\ })$ corresponds to $(\,\underset{\cdot}{\ }\,)$ in the obvious sence, for examples

$$(\theta^- L)^* = \theta^+(L^*), (\dot{L})^* = (\underset{\cdot}{L^*}) \text{ etc.}$$

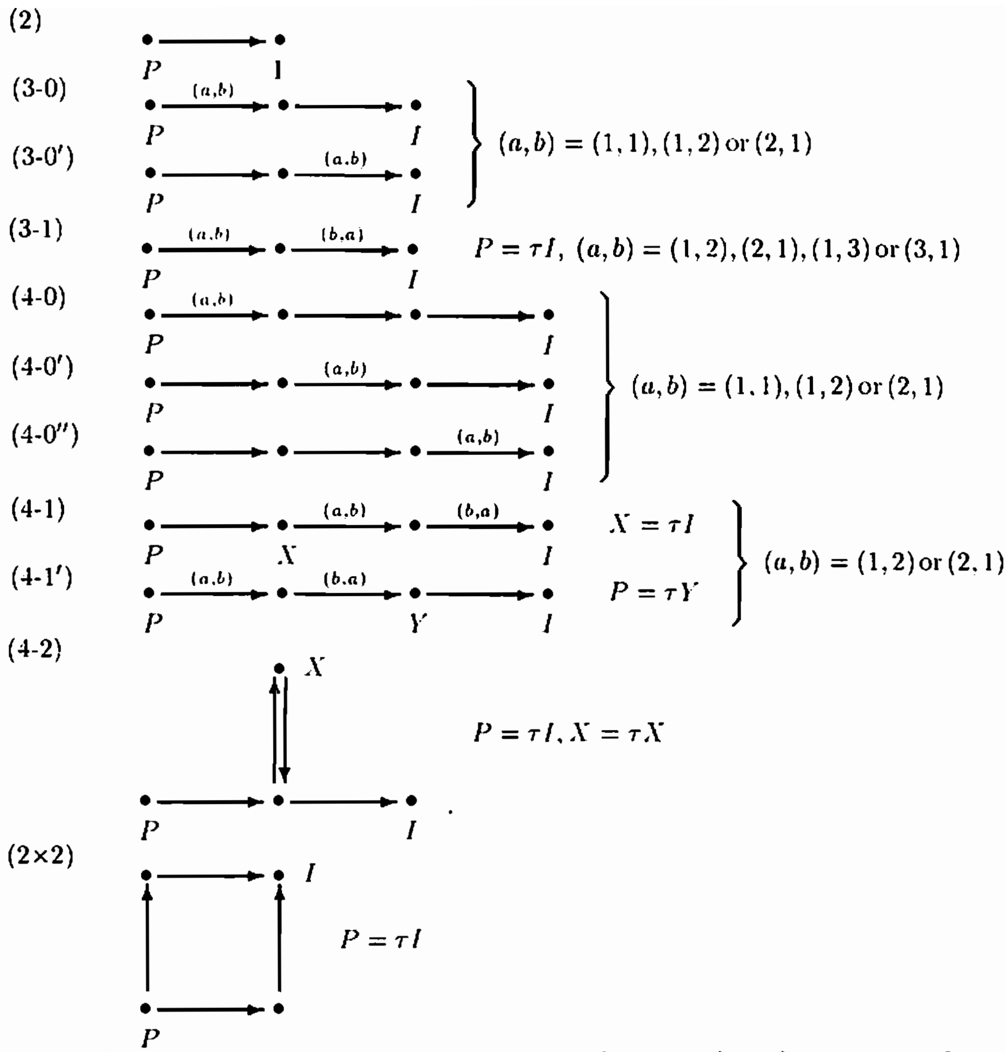
3 Examples

If $\mathcal{S}$ is minimal finite rejectable, then each of $\mathcal{S}_p$ and $\mathcal{S}_i$ is a singleton set (4.2.2), so that we write as $\mathcal{S}_p = \{P\}$, $\mathcal{S}_i = \{I\}$. In the diagrams below, unspecified arrow $\longrightarrow$ has the valuation (1,1).

3.1 Assume that $\mathcal{S}$ has at most four points, $\#\mathcal{S} \leq 4$. If $\mathcal{S}$ is minimal rejectable, then it should have one of the following forms.

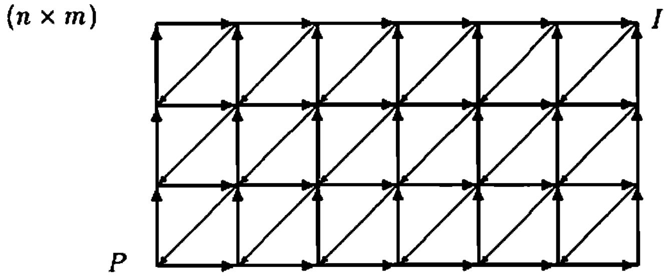
(1) $\bullet$

$P = I$



Here, (1) is the case of D-K Rejection Lemma and appeared in the sequence of Bass orders and almost Bass orders [HN-3] [HN-1][H]. Also (2×2) appeared in the sequence of local orders of finite representation type [HN-2].

3.2 For any $n \geq 1, m \geq 1$, the following square (with $n \times m$ vertices) is minimal rejectable.



where diagonal arrows indicate τ.

There is an infinite sequence of orders of finite representation type $\Lambda_1 \supset \Lambda_2 \supset \dots$ such that $\mathcal{S}_i = \mathrm{Ind}\,\Lambda_i - \mathrm{Ind}\,\Lambda_{i+1}$ has the above form for any i for one fixed (n, m).

4 Applications

4.1 The following 4.1.1 is one of key facts to prove lemma 5.4. As an easy corollary of 4.1.1, we obtain a generalization 4.1.2 of Roiter's Theorem.

4.1.1 Proposition *For* $f : X \to Y, g : X \to Z$, *put* $\mathcal{S} := \{ L \in \mathrm{Ind}\,\Lambda \mid f\,\mathrm{Hom}_\Lambda(Y, L) \neq g\,\mathrm{Hom}_\Lambda(Z, L) \}$. *If* $\mathcal{S}$ *is bounded, then* $\mathcal{S}$ *is finite.*

4.1.2 Corollary *A bounded rejectable subset* $\mathcal{S}$ *of* Ind Λ *is necessarily finite.*

4.2 Our rejection theory has wide applications for the problem characterizing a subquiver of some special type of $\mathfrak{A}(\Lambda)$. For example, as an application of the Theorem C, we obtain a generalization 4.2.1 of Bautista-Brenner's Theorem [**BB**]. Similar arguments give the classificaion in 3.1.

4.2.2 is a direct consequence of the Theorem B, but we can also give more elementary proof. On the other hand, we can apply the Theorem A to give a purely combinatorial proof of well known facts 4.2.3 and 4.2.4.

4.2.1 Proposition *If* $\mathcal{S}$ *is a finite rejectable subset of* Ind Λ, *then any arrow in* $\mathcal{S}$ *has the valuation* $(1, d)$ *or* $(d, 1)$ *where* $d = 1, 2$ *or* 3.

4.2.2 Proposition *Any non-trivial subset* $\mathcal{S}$ *of* Ind Λ *contains at least one projective and at least one injective. Moreover, any minimal non-trivial subset* $\mathcal{S}$ *of* Ind Λ *contains exactly one projective and exactly one injective.*

4.2.3 Proposition [**BS**] *Let* Λ *be an algebra and assume that there are* $L_1, L_2, \dots, L_n \in \mathrm{Ind}\,\Lambda$, $(L_{n+1} := L_1, L_{n+2} := L_2)$ *with irreducible maps* $L_i \to L_{i+1}$ $(1 \leq i \leq n)$. *Then there is some* $3 \leq i \leq n+2$ *such that* $\tau L_i = L_{i-2}$.

4.2.4 Proposition [**W**] *Let* Λ *be a ring indecomposable R-order and assume that there is* $L_0 \in \mathrm{Ind}\,\Lambda$ *with an irreducible map to itself. Then* $\mathfrak{A}(\Lambda)$ *has the following form by some* $n \geq 0$; $L_n \in \mathrm{proj}\,\Lambda \cap \mathrm{inj}\,\Lambda, \tau L_i = L_i (0 \leq i < n)$.

$$\circlearrowleft L_0 \rightleftarrows L_1 \rightleftarrows \dots\dots \rightleftarrows L_{n-1} \rightleftarrows L_n$$

5 Idea of the proof of the main theorems

Let $\mathbb{A} = (A_l, a_l)$ denote a complex of Λ-lattices

$$\mathbb{A}: \longrightarrow A_{l-1} \xrightarrow{a_{l-1}} A_l \xrightarrow{a_l} A_{l+1} \longrightarrow$$

We write the action of Λ-morphism *from right.* Let $T \in {}_\Lambda\mathfrak{M}$ be non-zero and assume that there is a split monomorphism $i : T \to A_n$ satisfying the following property:

$$ia_n \in \mathrm{rad}(T, A_{n+1})$$

where $\mathrm{rad}(T, A_{n+1})$ is defined by the same way as [R] 2.5.

Then it is easily seen that there is a chain homomorphism given by the following commutative diagram. Here, $0 \to T \xrightarrow{\nu} \theta^- T \xrightarrow{\mu} \tau^- T \to 0$ is the complex of Λ-source map from T.

$$\begin{array}{ccccccccccccccc}
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & T & \xrightarrow{\nu} & \theta^- T & \xrightarrow{\mu} & \tau^- T & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathbb{A}: \longrightarrow & A_{n-2} & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & A_{n+2} & \longrightarrow & A_{n+3} & \longrightarrow & 0 & \longrightarrow
\end{array}$$

By a mapping cone, we obtain a new complex

$$\mathbb{A}_1: \longrightarrow A_{n-2} \longrightarrow A_{n-1} \oplus T \longrightarrow A_n \oplus \theta^- T \longrightarrow A_{n+1} \oplus \tau^- T \longrightarrow A_{n+2} \longrightarrow$$

It is easy to show that $\mathbb{A}_1$ have a direct summand of the form

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow \overset{n-1}{T} \xrightarrow{1} \overset{n}{T} \longrightarrow 0 \longrightarrow 0 \longrightarrow$$

By canceling this, we obtain a new complex

$$\mathbb{A}_2: \longrightarrow A_{n-2} \longrightarrow A_{n-1} \longrightarrow (A_n/T) \oplus \theta^- T \longrightarrow A_{n+1} \oplus \tau^- T \longrightarrow A_{n+2} \longrightarrow$$

5.1 Definition (1) Thus obtained $\mathbb{A}_2$ will be called the complex obtained from $\mathbb{A}$ by *rejecting T from A_n*, and will be denoted as

$$\mathbb{A}_2 = \mathbb{A} - \overset{n}{T}.$$

(2) Let $\mathcal{S}$ be a subset of $\mathrm{Ind}\,\Lambda$ and n be an arbitrarily fixed integer. A (finite or infinite) sequence $(\mathbb{A}^{(0)}, \mathbb{A}^{(1)}, \mathbb{A}^{(2)}, \ldots)$ of complexes of Λ-lattices will be called a *successive $\mathcal{S}^-$- rejection sequence with the initial complex* $\mathbb{A}$, if there is a sequence (T_j) with $0 \neq T_j \in \mathbb{N}\mathcal{S}$, by which $\mathbb{A}^{(j)}$ is defined inductively as $\mathbb{A}^{(0)} := \mathbb{A}, \mathbb{A}^{(j+1)} := \mathbb{A}^{(j)} - \overset{n}{T_j}$.

5.2 Lemma *Put* $\mathbb{A} - \overset{n}{T} = (A'_l, a'_l)$.
(r1) $A'_n - A'_{n+1} = A_n - A_{n+1} - \phi^- T$.

(r2) If $X \in {}_\Lambda\mathfrak{M}$ *and* $\langle X, T\rangle = 0$, *then* $a_{n-1}\operatorname{Hom}_\Lambda(A_n, X) = a'_{n-1}\operatorname{Hom}_\Lambda(A'_n, X)$.
(r3) There are exact sequences of homology:

$$0 \longrightarrow H^{n-1}(\mathbb{A}) \longrightarrow H^{n-1}(\mathbb{A}') \longrightarrow \ker\nu \longrightarrow H^n(\mathbb{A}) \longrightarrow H^n(\mathbb{A}')$$
$$\longrightarrow \ker\mu/\operatorname{im}\nu \longrightarrow H^{n+1}(\mathbb{A}) \longrightarrow H^{n+1}(\mathbb{A}') \longrightarrow 0$$
$$0 \longrightarrow H^l(\mathbb{A}) \longrightarrow H^l(\mathbb{A}') \longrightarrow 0 \quad (\ l \neq n-1, n, n+1\ )$$

5.3 Proposition *Let* (T_j) *be an* $\mathcal{S}^-$*-sequence for* V *and* $\mathbb{A}$ *a complex with* $A_n - A_{n+1} = V$. *Then there exists a successive* $\mathcal{S}^-$*-rejection sequence of complexes* $(\mathbb{A}^{(0)}, \mathbb{A}^{(1)}, \ldots)$, $\mathbb{A}^{(0)} := \mathbb{A}, \mathbb{A}^{(j)} := \mathbb{A}^{(j-1)} \overset{n}{-} T_{j-1}$, *with* $\mathbb{A}^{(j)}_n - \mathbb{A}^{(j)}_{n+1} = V_j$.

By 5.3, whenever we are given an $\mathcal{S}^-$-sequence, we can obtain a successive $\mathcal{S}^-$-rejection sequence of complexes. For example, main part of the Theorem A is proven from the following lemma.

5.4 Lemma *Let* $\mathcal{S}$ *be a bounded cofaithful subset of* Ind Λ. *Then any successive* $\mathcal{S}^-$*-rejection sequence with the initial complex* $\mathbb{A}$ *terminates at a finite number of steps.*

Moreover, 5.2 gives many information about these complexes. However they suffice to prove the Theorem B and the Theorem C, our method is rather complicated and we shall omit.

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COHEN-MACAULAY APPROXIMATIONS FROM THE VIEWPOINT OF TRIANGULATED CATEGORIES

KIRIKO KATO

§1 Introduction.

Let $(R, \mathfrak{m}, k)$ be a complete Gorenstein local ring, and let M be a finitely generated R-module. Auslander and Buchweitz introduced the notion of Cohen-Macaulay approximation (1) and a finite projective hull (2), of M, which are the exact sequences dual to each other [1], [6] :

$$0 \to Y_M^R \to X_M^R \overset{\rho_M}{\to} M \to 0, \tag{1}$$

$$\eta_M : 0 \to M \to Y_R^M \to X_R^M \to 0, \tag{2}$$

where X_M^R, X_R^M are maximal Cohen-Macaulay modules and Y_M^R, Y_R^M are modules of finite projective dimension.

If X_M^R and Y_M^R (resp. X_R^M and Y_R^M) have no direct summand in common, according to the inclusion map appeared in the sequence (1) (resp. the projection map in the sequence (2)), it is called the minimal Cohen-Macaulay approximation (resp. the minimal finite projective hull), which exists uniquely up to isomorphisms. We may assume henceforth the minimality of (1) and (2), omitting common summands if necessary.

The above exact sequences suggest an idea to treat a finite module as a kernel or a cokernel of a homomorphism from a finite projective dimensional module to a Cohen-Macaulay module. Indeed, on researching Cohen-Macaulay approximations, there arises a natural question: *If $X_M \cong X_N$, $Y_M \cong Y_N$, do two modules M and N share any common property?* As the simplest case, we obtained the following:

Theorem 12 Suppose $M^* := \mathrm{Hom}_R(M, R) = 0$. If for an R-module N has the property that $X^N \cong X^M$ and $Y^N \cong Y^M$, then $N \cong M$.

We discuss the problem within a framework of the theory of triangulated categories; in other words, we focus on the fact that the category of Cohen-Macaulay modules over a Gorenstein ring is Frobenius. In addition to above two exact sequences (Cohen-Macaulay approximation and finite projective hull), in the section 2 we construct another exact sequence "origin extension" which is the dual of the other two. The notion of origin extensions enables us to consider two R-modules M and N with $X_M \cong X_N$, $Y^M \cong Y^N$ as two elements of an R-module $\mathrm{Ext}_R^1(Y^M, \Omega_1^R(X^M))$.

Unlike a Cohen-Macaulay approximation and a finite projective hull, a non-minimal origin extension does not always includes the minimal origin extension. The existence of a non-trivial non-minimal origin extension obstructs the

The detailed version of this paper will be submitted forpublication elsewhere.

uniqueness of the correspondence between finite modules and elements of the module of the form $\mathrm{Ext}^1_R(Y,X)$.

§2 Origin Extensions.

Thoughout the paper, we fix a complete Gorenstein local ring $(R,\mathfrak{m},k)$, and a "module" always means a finitely generated module over R. We call a module without a free summand a "stable module".

Definition 1 *For a finite R-module M, an origin extension of M is the exact sequence*

$$\zeta : 0 \to X \xrightarrow{\varepsilon_\zeta} M \oplus P \xrightarrow{\rho_\zeta} Y \to 0 \tag{3}$$

with a Cohen-Macaulay module X, a free module P, and a finite projective dimensional module Y.

Definition 2 *An origin extension (3) is called minimal if it satisfies the following conditions:*

MN1) *The Cohen-Macaulay module X is stable.*

MN2) *There exists no common summand with P and Y through ρ_ζ.*

MN3) *For any origin extension $\zeta' : 0 \to X' \to M \oplus P' \to Y' \to 0$ of M, there exist linear maps $a : P \to P'$, $b : Y \to Y'$, and $c : X \to X'$ that make the following diagram commutative.*

$$\begin{array}{ccccccccc} \zeta: & 0 & \to & X & \to & M \oplus P & \to & Y & \to & 0 \\ & & & \downarrow c & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} & & \downarrow b & & \\ \zeta': & 0 & \to & X' & \to & M \oplus P' & \to & Y' & \to & 0. \end{array} \tag{4}$$

Theorem 3 *For an R-module M, there exists a minimal origin extension of M.*

pproof) The minimal finite projective hull $\eta_M : 0 \to M \to Y^M_R \to X^M_R \to 0$ and the syzygy of X^M_R yield a pull-back diagram;

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Omega^1_R(X^M) & = & \Omega^1_R(X^M) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & M \oplus G & \to & G & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & M & \to & Y^M_R & \to & X^M_R & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \\ & & & & \vdots & & & & \\ & & & & \zeta_M & & & & \end{array}$$

where G is a free module and both X_R^M and $\Omega_R^1(X_R^M)$ are stable Cohen-Macaulay modules. Since the middle row splits, the middle column ζ_M is an origin extension of M.

We claim that the exact sequence $\zeta_M \in \operatorname{Ext}_R^1(Y_R^M, \Omega_R^1(X_R^M))$ on the middle column, satisfies the condition MN3). Let ζ' be an arbitral origin extension of M:

$$\zeta' : 0 \to X' \to M \oplus P' \to Y' \to 0.$$

We shall show the existence of maps that make the diagram (4) commutative.

We may assume that X' is stable. It follows from the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & X' & \overset{\varepsilon'}{\to} & M \oplus P' & \to & Y' & \to & 0 \\ & & \uparrow & & \| & & \uparrow & & \\ 0 & \to & C' & \overset{\varepsilon''}{\to} & M \oplus P' & \to & Z' & \to & 0, \end{array}$$

where $X' = C' \oplus V$ with a stable Cohen-Macaulay module C' and a free module V, and Z' is of finite projective dimension because of the induced exact sequence $0 \to V \to Z' \to Y' \to 0$.

Together with the cosyzygy sequence of X', ζ' gives the following push-out diagram:

$$\begin{array}{cccccccccc} & & & 0 & & 0 & & & & \\ & & & \downarrow & & \downarrow & & & & \\ \zeta' : & 0 & \to & X' & \to & M \oplus P' & \to & Y' & \to & 0 \\ & & & \downarrow & & \downarrow & & \| & & \\ \theta' = \theta_{\zeta'} : & 0 & \to & G' & \to & E_{\theta'} & \overset{\rho_{\theta'}}{\to} & Y' & \to & 0 \\ & & & \downarrow & & \downarrow & & & & \\ & & & \Omega_R^{-1}(X') & = & \Omega_R^{-1}(X') & & & & \\ & & & \downarrow & & \downarrow & & & & \\ & & & 0 & & 0 & & & & \end{array}$$

where G' is a free module. We may take $\Omega_R^{-1}(X')$ stable to make the second column the minimal finite projective hull of $M \oplus P'$ hence $E_{\theta'} \cong Y_R^M \oplus P'$, $X' \cong \Omega_R^1(X_R^M)$ and $G = G'$.

Now we shall show that an extension ζ' is carried to ζ_M via the homomorphism $\operatorname{Ext}_R^1(\rho_{\theta'}, \Omega_R^1(X_R^M)) : \operatorname{Ext}_R^1(Y', \Omega_R^1(X_R^M)) \to \operatorname{Ext}_R^1(E_\theta, \Omega_R^1(X_R^M)) \cong \operatorname{Ext}_R^1(Y_R^M, \Omega_R^1(X_R^M))$, which implies the commutative diagram:

$$\begin{array}{cccccccccc} \zeta_M : & 0 & \to & \Omega_R^1(X_R^M) & \to & M \oplus G \oplus P' & \to & Y_R^M \oplus P' & \to & 0 \\ & & & \| & & \downarrow & & \downarrow \rho_{\theta'} & & \\ \zeta' : & 0 & \to & X' & \to & M \oplus P' & \to & Y' & \to & 0 \end{array}$$

Two sequences ζ' and θ' make the pull-back diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
\zeta: & 0 & \to & X' & \to \quad E_\zeta & \to & Y_R^M \oplus P' & \to & 0 \\
 & & & \| & \downarrow \rho_{\sigma'} & & \downarrow \rho_{\theta'} & & \\
\zeta': & 0 & \to & X' & \to \quad M \oplus P' & \to & Y' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & \\
 & & & & \vdots & & \vdots & & \\
 & & & & \sigma' & & \theta' & &
\end{array}$$

Calling the middle row ζ, we easily see $\mathrm{Ext}_R^1(\rho_{\theta'}, X')(\zeta) = \zeta'$. Provided that the sequence σ' splits. Then $E_\zeta \cong M \oplus P' \oplus G$, which implies $\zeta = \zeta_M$ by virtue of Lemma 5. And moreover, we obtain the required diagram on the bottommost two rows in the diagram above, since $\rho_{\sigma'}$ is an splitting epimorphism.

We are now to state that the sequence σ' splits. On the long exact sequence induced by ζ', σ' is the image of θ', while θ' is also the image of the inclusion map in the cosygyzy of X': $0 \to X' \overset{\epsilon_{X'}}{\to} G \to \Omega_R^{-1}(X') \to 0$. So σ' is trivial as an element of $\mathrm{Ext}_R^1(M \oplus P', G)$. (See the diagram below whose columns are induced by the cosyzygy sequence X' and rows by ζ'.)

$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & & \\
\to & \mathrm{End}_R X' & \to & \mathrm{Ext}_R^1(Y', X') & & & \\
 & \Psi & & \Psi & & & \\
 & 1 & & \zeta' & & & \\
 & \downarrow & & \downarrow & & & \\
\to & \mathrm{Hom}_R(X', G) & \to & \mathrm{Ext}_R^1(Y', G) & \to & \mathrm{Ext}_R^1(M \oplus P', G) & \\
 & \Psi & & \Psi & & \Psi & \\
 & \epsilon_{X'} & & \theta' & & \sigma' & \\
 & \downarrow & & & & & \\
 & \mathrm{Hom}_R(X', \Omega_R^{-1}(X')) & & & & &
\end{array}$$

Our final step is splitting off the maximal common free summand Q between G and Y_R^M from ζ_M to obtain the exact sequence $\overline{\zeta}_M$:

$$\overline{\zeta}_M : 0 \to \Omega_R^1(X_R^M) \to M \oplus G/Q \to Y_R^M/Q \to 0$$

The sequence $\overline{\zeta}_M$ satisfies MN1) and MN3) as well as ζ_M, and also MN2) by difinition. Thus $\overline{\zeta}_M$ is a minimal origin extension of M. (q.e.d. for Theorem 3.)

Lemma 4 *Let M be a finite R-module, P a free module. The exact sequence $\theta : 0 \to P \to E_\theta \to M \to 0$ splits if $E_\theta \cong M \oplus P$.*

proof) The corresponding chain map $\theta_\bullet : F_{M\bullet} \to P(-1)$ may be taken as $\theta_i = 0$ $(i \neq 1)$. The mapping cone $Cone\ (\theta_\bullet)_\bullet$ is the minimal free resolution of $M \oplus P$ considered ranks of each free modules. So there exists a chain isomorphism $\alpha_\bullet : F_{M\bullet} \to Cone\ (\theta_\bullet)_\bullet$:

$$\begin{array}{ccc} \downarrow & & \downarrow \\ F_{M2} & \xrightarrow{\ \alpha_2\ } & F_{M2} \\ d_{F_{M2}}\downarrow & & \downarrow d_{F_{M2}} \\ F_{M1} & \xrightarrow{\ \alpha_1\ } & F_{M1} \\ \begin{pmatrix} d_{F_{M1}} \\ 0 \end{pmatrix}\downarrow & & \downarrow \begin{pmatrix} d_{F_{M1}} \\ \theta \end{pmatrix} \\ F_{M0}\oplus P & \xrightarrow{\alpha_0 = \begin{pmatrix} a & b \\ c & d\end{pmatrix}} & F_{M0}\oplus P \\ \downarrow & & \downarrow \\ M\oplus P & & E_\theta \cong M \oplus P \end{array}$$

We have $\theta\alpha_1 = cd_{F_{M1}}$, which means $\theta_\bullet = 0$ as an element of $\mathrm{Ext}^1_R(M, P)$ since α_1 is an isomorphism. (q.e.d. for Lemma 4.)

Lemma 5 *Let Y be a module of finite projective dimension, and X be a stable Cohen-Macaulay module. Let $\zeta \in \mathrm{Ext}^1_R(Y, X)$ be an extension*

$$\zeta : 0 \to X \to M \oplus G \xrightarrow{\rho_\zeta} Y \to 0$$

with a stable module M and a free module G. Suppose ζ has the following properties;

- $Y \cong Y^M_R$,
- $X \cong \Omega^1_R(X^M_R)$,
- $\mathrm{rk}(G) = \mu(X^M_R)$,

then ζ coincides with ζ_M in Theorem 3. In other words, an extension with the three conditions above, is unique in $\mathrm{Ext}^1_R(Y, X)$.

proof) There exists an epimorphism $\pi \in \mathrm{Hom}_R(Y^M \oplus G, Y^M_R)$ such that $\rho_\zeta = \pi\varepsilon_{\eta_{M\oplus G}}$ where $\varepsilon_{\eta_{M\oplus G}}$ is the inclusion map appeared in the minimal finite projective hull of $M \oplus G$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ \zeta: & 0 \to & \Omega^1_R(X^M_R) & \to & M \oplus G & \xrightarrow{\rho_\zeta} & Y^M_R & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \varepsilon_{\eta_{M\oplus G}}} & & \| & & \\ & 0 \to & Z & \to & Y^M_R \oplus G & \xrightarrow{\pi} & Y^M_R & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & X^M_R & = & X^M_R & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Since $Z := \mathrm{Ker}\,\pi$ is Cohen-Macaulay and finite projective dimensional at the same time, Z is free. In view of the leftmost column, we have $\mathrm{rk}(Z) = \mu(X^M_R)$ hence $Z \cong G$. So the middle row splits because of Lemma 4. Consequently, for any ζ with three properties of this Lemma, $\rho_\zeta = \pi\varepsilon_{\eta_{M\oplus G}}$ where π is the split epimorphism $\pi : Y^M_R \oplus G \; Y^M_R$. (q.e.d. for Lemma 5.)

Theorem 6 *The minimal origin extension of an R-module M is unique up to isomorphism. In other word, if two origin extensions of M; $\zeta : 0 \to X \to M \oplus P \to Y \to 0$ and $\zeta' : 0 \to X' \to M \oplus P' \to Y' \to 0$ are both minimal, linear maps a, b and c in the diagram (4) are isomorphisms. The minimal origin extension of M is, after adding some free summand, of the form*

$$\zeta_M : 0 \to \Omega^1_R(X^M) \to M \oplus G_{-1} \to Y^M \to 0$$

where $G_{-1} \; X^M$ is the minimal projective cover.

proof) We may assume that $\zeta' = \overline{\zeta_M}$. The proof of Theorem 3 tells us $X \cong X' = \Omega^1_R(X^M)$. The diagram (4) is decomposed into two parts;

$$\begin{array}{cccccccccc} \zeta: & 0 & \to & X & \to & M \oplus P & \to & Y & \to & 0 \\ & & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \alpha = (\alpha_1 \alpha_2)} & & \| & & \\ \zeta'': & 0 & \to & X' & \to & E & \to & Y & \to & 0 \\ & & & \| & & \downarrow{\scriptstyle \beta = \binom{\beta_1}{\beta_2}} & & \downarrow{\scriptstyle b} & & \\ \overline{\zeta_M} = \zeta': & 0 & \to & X & \to & M \oplus P' & \to & Y' & \to & 0 \end{array} \qquad (5)$$

with

$$\beta\alpha = \begin{pmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{pmatrix} = \begin{pmatrix} \mathrm{id}_M & 0 \\ 0 & a \end{pmatrix}. \qquad (6)$$

Since $\beta_1\alpha_1 = \mathrm{id}_M$, M is a direct summand of E: $E \cong M \oplus Q \oplus N$ (Q:free , N:stable). The sequence ζ'' is an origin extension of $M \oplus N$, which implies

$\Omega^1_R(X^M) \cong \Omega^1_R(X^M \oplus X^N)$ hence N is of finite projective dimension. After partitionwise isomorphic transformations,

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_1 = (1 \quad 0 \quad 0),$$

together with (6), α, β is of the form

$$\alpha = \begin{array}{c} \\ M \\ Q \\ N \end{array} \begin{array}{c} \begin{array}{cc} M & P \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & a_1 \\ 0 & a_2 \end{pmatrix} \end{array}, \quad \beta = \begin{array}{c} \\ M \\ P' \end{array} \begin{array}{c} \begin{array}{ccc} M & Q & N \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & a_4 \end{pmatrix} \end{array}.$$

We shall show that $\zeta'' \cong \overline{\zeta}_{M\oplus N}$. There exist homomorphisms a', b', c' that make the upper half of the following diagram.

$$\begin{array}{ccccccccc}
\overline{\zeta}_{M\oplus N}: & 0 & \to & X & \xrightarrow{\begin{pmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ 0 \end{pmatrix}} & M \oplus P' \oplus N & \to & Y' \oplus N & \to & 0 \\
 & & & \downarrow c' & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \downarrow b' & & \\
\zeta'': & 0 & \to & X' & \to & M \oplus Q \oplus N & \to & Y & \to & 0 \\
 & & & \downarrow \| & & \downarrow \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & a_4 \end{pmatrix} & & \downarrow b & & \\
\zeta': & 0 & \to & X & \xrightarrow{\begin{pmatrix} \varepsilon'_1 \\ \varepsilon'_2 \end{pmatrix}} & M \oplus P' & \xrightarrow{p_{\zeta'}} & Y' & \to & 0
\end{array}$$

Reviewing the proof of Theorem 3, we may take the identity map of X as c'. The equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & a_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & a' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon'_1 \\ \varepsilon'_2 \end{pmatrix}$$

is reduced to $\varepsilon'_2 = a_3 a' \varepsilon'_2$. We claim that $a_3 a'$ is an isomorphism.

The minimal cover $G_0 \xrightarrow{d_{G_0}} X$ induces a homomorphism $x^{P'}$ as

$$\begin{array}{ccc}
G_0 & \xrightarrow{x^{P'}} & P' \\
\downarrow d_{G_0} & & \| \\
X & \xrightarrow{\varepsilon'_2} & P'.
\end{array}$$

This $x^{P'}$ has the same property $x^{P'} = a_3 a' x^{P'}$, since $x^{P'} = \varepsilon'_2 d_{G_0}$.

With respect to matrix representation $a_3 a' = (a_{ij})_{1 \le i,j \le \mathrm{rk}(P')}$, and $x^{P'} = (x_{kl})_{1 \le k \le \mathrm{rk}(P'),\ 1 \le l \le \mathrm{rk}(G_0)}$, the above equation means $(a_3 a' x^{P'})_{ij} = x^{P'}{}_{ij}$, that is,

$$\sum_{k=1}^{\mathrm{rk}(P')} a_{ik}x_{kj} = x_{ij}$$

for $1 \le i \le \mathrm{rk}(P')$, $1 \le j \le \mathrm{rk}(G_0)$. Now suppose that a_3a' is not an isomorphism hence is not an epimorphism. Then a_3a' has at least one row, say, the first row, whose all entries belong to the maximal ideal $\mathfrak{m}$. We have

$$(1-a_{11})x_{1j} = \sum_{k=2}^{\mathrm{rk}(P')} a_{1k}x_{kj}$$

for $1 \le j \le \mathrm{rk}(P')$ with $(1-a_{11})$ a unit, which implies that $x^{P'}$ has a zero row after some row-transformations.

On the other hand, it is easy to see the equivalence of the following conditions:

1) A common summand splits off through $\rho_{\zeta'}$ from P' and Y'.

2) There exists a split epimorphism $s : P'\ R$ such that $s\varepsilon'_2 = 0$.

3) There exists a split epimorphism $s : P'\ R$ such that $sx^{P'} = 0$.

4) After some row-transformations, $x^{P'}$ contains a zero-row.

$$\begin{array}{ccccccccc}
0 & \to & X & \to & M \oplus P' & \xrightarrow{\rho_{\zeta'}} & Y' & \to & 0 \\
 & & & & {\scriptstyle\varepsilon'_2}\searrow & & & & \\
 & & & & & P' & & & \\
 & & & & & \downarrow{\scriptstyle s} & & & \\
 & & & & & R & \cong & R &
\end{array}$$

So we get a contradiction to the condition MN2) of ζ'.

Now that a' is a split monomorphism, the exact sequence $0 \to Y' \oplus N \xrightarrow{b'} Y \to \mathrm{Coker}\, a' \to 0$ splits since $\mathrm{Coker}\, a' \cong P'/Q$ is a free module. Therefore ζ'' is a direct summand of $\overline{\zeta}_{M\oplus N}$, which means $\zeta'' \cong \overline{\zeta}_{M\oplus N}$ due to the minimality of $\overline{\zeta}_{M\oplus N}$.

Next, looking back the upper half of the diagram (5),

$$\begin{array}{lccccccccc}
\zeta : & 0 & \to & X & \to & M \oplus P & \xrightarrow{(\rho_1\,\rho_2)} & Y & \to & 0 \\
 & & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \alpha = \left(\begin{smallmatrix} 1 & 1 \\ 0 & a_1 \\ 0 & a_2 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle \|} & & \\
\zeta'' : & 0 & \to & X & \to & M \oplus P' \oplus N & \xrightarrow{\left(\begin{smallmatrix} \rho'_1 & \rho'_2 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & Y' \oplus N & \to & 0,
\end{array}$$

we get

$$\begin{array}{cc} & \begin{array}{cc} M & P \end{array} \\ (\rho_1 \quad \rho_2) = \begin{array}{c} Y' \\ N \end{array} & \begin{pmatrix} \rho_1' & \rho_2' a_1 \\ 0 & a_2 \end{pmatrix}. \end{array}$$

To see the surjectivity of α, first a_2 is clearly an epimorphism from the above equation. If a_1 is not surjective, neither the homomorphism $M \oplus P \xrightarrow{(\rho_1' \rho_2' a_1)} Y'$ is; since on ζ', ρ_2' is induced from the minimal projective cover P' $\operatorname{Coker} \rho_1'$. It follows that c is surjective, equivalently is isomorphic, because $\operatorname{Coker} c \cong \operatorname{Coker} \alpha$. As a conclusion, $\zeta \cong \zeta''$, which implies $N = 0$ hence $\zeta'' \cong \zeta'$. (q.e.d. for Theorem 6.)

Remark 7 *The minimal origin extension of the direct sum $M \oplus N$ of modules is the direct sum of the minimal origin extension of M and that of N.*

§3 Non-trivially non-minimal extensions.

Any non-minimal Cohen-Macaulay approximation or finite projective hull is the direct sum of the minimal one and some trivial complex. Although it is not the case for non-minimal origin extension as seen in the following example.

Example 8 *Let $R := k[[x,y]]/(xy)$, and $M := k$. We have*

$$\cdots \to R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \xrightarrow{(x\,y)} R \to k \to 0,$$

$$\cdots \to R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} R^2 \to X^M \to 0,$$

$$0 \to R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^2 \to Y^M \to 0.$$

Taking a finite projective dimensional module Y' as

$$\begin{array}{ccccccccc} 0 & \to & R & \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} & R^2 & \to & Y^M & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}} & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \to & R & \xrightarrow{\begin{pmatrix} y^2 \\ x^2 \end{pmatrix}} & R^2 & \to & Y' & \to & 0, \end{array}$$

we get

$$\begin{array}{ccccccccc} 0 & \to & \Omega_1^R(X^M) & \to & M \oplus R^2 & \to & Y^M & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\begin{pmatrix} id_M & 0 \\ 0 & \begin{smallmatrix} y & 0 \\ 0 & x \end{smallmatrix} \end{pmatrix}} & & \downarrow{\scriptstyle\lambda} & & \\ 0 & \to & \Omega_1^R(X^M) & \to & M \oplus R^2 & \to & Y' & \to & 0. \end{array}$$

The the second row is a non-minimal origin extension of M that is not a direct summand of the first row.

Definition 9 *A non-trivially non-minimal origin extension of a module M is a non-minimal origin extension that does not include the minimal origin extension of M as a direct summand.*

Let $0 \to X \to M \ominus P \to Y \to 0$ be an origin extension of a stable R-module M that is not necessarily minimal. We observe that

$$X \cong \Omega_1^R(X^M) \text{ up to free summands,} \tag{7}$$

and

$$0 \to G_{M-1} \to Y^M \oplus P \to Y \to 0 \tag{8}$$

where G_{M-1} X^M is the minimal projective cover. from the argument in the proof of Theorem 3. Through this observasion , non-trivially non-minimal origin extensions exist with respect to the homological property of Y.

Lemma 10 *For a module Y with finite projective dimension, if $\mathrm{Ext}_R^1(Y,R) = 0$, then for any stable Cohen-Macaulay module X, each non-zero element of $\mathrm{Ext}_R^1(Y,X)$ is the minimal origin extension of some stable module. In other words, $\mathrm{Ext}_R^1(Y,X)$ contains no nontrivially-nonminimal origin extension.*

proof) It suffices to prove for a stable Y.

Suppose the contrary; let $0 \to X \to M \oplus P \to Y \to 0$ be a non-minimal origin extension of a stable module M. Then we have a non-split exact sequence (8) $0 \to G_{-1} \to Y^M \oplus P \to Y \to 0$, which contradicts to the assumption. (q.e.d.)

Remark 11 *The converse of the above Lemma is not true. (see [5].)*

Theorem 12 *Suppose $M^* := \mathrm{Hom}_R(M,R) = 0$. If for an R-module N has the property that $X^N \cong X^M$ and $Y^N \cong Y^M$, then $N \cong M$. Moreover, if $\mathrm{Ext}_R^1(M,R) = 0$, then $\mathrm{Ext}_R^1(Y^M, \Omega_1^R(X_M))$ is principally generated by the minimal origin extention ζ_M of M as an an $\mathrm{End}_R\,(Y^M)$-$\mathrm{End}_R\,(\Omega_R^1(X^M))$ bimodule.*

proof) The sequence $0 \to M \to Y_R^M \to X_R^M \overset{\rho_{\eta_M}}{\to} 0$ induces

$$\mathrm{Hom}_R(Y_R^M, R) \cong \mathrm{Hom}_R(X_R^M, R), \tag{9}$$

$$\mathrm{Ext}_R^i(Y_R^M, R) \cong \mathrm{Ext}_R^i(M, R). \quad (i \neq r, 0) \tag{10}$$

From (9) we have $(Y^M)^{**} \cong X^M$. Since X^M is reflexive and ρ_{η_M} is an epimorphism, ρ_{η_M} is nothing but the natural homomorphism $Y^M \to (Y^M)^{**}$. (See the diagram below.)

$$\begin{array}{ccc} Y^M & \overset{\rho_{\eta_M}}{\to} & X^M \\ \downarrow & & \wr\| \\ (Y^M)^{**} & \overset{\rho_{\eta_M}^{**}}{\to} & (X^M)^{**} \end{array}$$

On the othe hand, $\rho_{\eta_N} : Y^M \to X^M \cong (Y^M)^{**}$ is also an epimorphism , hence also isomorphic to the natural map $Y^M \to (Y^M)^{**}$.

To prove the latter part, let $\zeta : 0 \to \Omega_1^R(X_R^M) \to L \oplus P \to Y_R^M \to 0$ an arbitrary element of $\mathrm{Ext}_R^1(Y^M, \Omega_1^R(X_M))$

with a free module P and a stable module L. We have $\mathrm{Ext}_R^1(Y_R^M, R) = 0$ from (10), which implies ζ is isomorphic to the minimal origin extention of L by virtue of Lemma 10. We have $Y^L \cong Y^M \oplus F$ for some free module F and $X^L \cong X^M$ since Y^M is stable because $(Y^M)^{**} \cong X^M$. Moreover, on the minimal projective hull of L $0 \to L \to Y^L \oplus F \stackrel{(\rho_1 \rho_2)}{\to} X^M \to 0$, $\rho_1 = (\rho_1) ** \circ \rho_{\eta_M}$. As for the syzygy sequence of X^M $0 \to \Omega_R^1(X^M) \to G \to X^M \to 0$ with G a free module, induced map $\mathrm{Hom}_R(Y^M, X^M) \to \mathrm{Ext}_R^1(Y^M, \Omega_R^1(X^M))$ is surjective because $\mathrm{Ext}_R^1(Y^M, R) = 0$. Here $\zeta \in \mathrm{Ext}_R^1(Y^M, \Omega_R^1(X^M))$ is the image of $\rho_1 = (\rho_1) ** \circ \rho_{\eta_M}$ while $\overline{\zeta}_M$ is that of ρ_{η_M}. (q.e.d. for Lemma 12.)

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NON-COMMUTATIVE VALUATION RINGS AND THEIR GLOBAL THEORY

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Abstract

This is a survey of last decade on non-commutative valuation rings, and of semi-hereditary and Prüfer orders in a simple Artinian ring which are considered, in a sense, as global theory of non-commutative valuation rings. The article also contains a several open problems.

1.Non-commutative Valuation Rings

Histrorically non-commutative valuation rings of division rings were first treated systematically in Schilling's book[S], which are nowadays called invariant valuation rings, though invariant valuation rings can be traced back to Hasse's work in [H]. Since then various attemptions have made to study the ideal theory of orders in finite dimensional algebras over fields and to describe the Brauer groups of fields by usage of "valuations", "places", "preplaces", ,"value functions" and "psuedoplaces".

In 1984, N. I. Dubrovin defined non-commutative valuation rings of simple Artinian rings with notion of points in the category of simple Artinian rings and obtained significant results on non-commutative valuation rings(namend Dubrovin valuation rings after him)which sygnify that these rings may be the correct definition of valuation rings of simple Artinian rings.

The aim of this section is to give a surmmary of his results. We begin with the definition of invariant valuation rings and their generalization. Let D be a division ring with finite dimension over its center F and let R be a subring of D. Consider the following two conditions:

(T) For every $d \in U(D)$, either $d \in R$ or $d^{-1} \in R$.

(I) For every $d \in U(D)$, $dRd^{-1}=R$.

Here we denote the unit group of a ring S by $U(S)$ and the center of S by $Z(S)$. If R satisfies the condition(T), then it is called a *total* valuation ring of D. If R satifies the conditions (T) and (I), then it is called an *invariant* valuation ring of D. If R is

This is in a final form and no version of it will be submitted for publication elsewhere

invariant, then any one sided ideal of R is two-sided and we can define the *value group* of R as follows: $\Gamma_R = U(D) \setminus U(R)$, which is made into a totally ordered group by $d_1U(R) \leq d_2U(R)$ if and only if $d_1R \supseteq d_2R$ for any $d_1, d_2 \in U(D)$. Then the natural mapping $v : U(D) \longmapsto \Gamma_R$ given by $d \longmapsto dUR)$ satisfies the following two conditions:
(V1) $v(ab) = v(a) + v(b)$ (we use an additive notation for Γ_R).
(V2)$v(a+b) \geq \min\{v(a), v(b)\}$ if $b \neq -a$.
In general, a surjective mapping v: $U(D) \rightarrow G$ which satisfies the conditions (V1) and (V2) is called a *valuation* on D,where G is a totally ordered group. This was an original definition given by Schilling. If v is a valuation on D, then it is easily checked that $R = \{d \in U(D) \mid v(d) \geq 0\} \cup \{0\}$ is an invariant valuation ring of D and $\Gamma_R \cong G$ naturally.

Let Q be a simple Artinian ring with $F = Z(R)$ and let V be a valuation ring of F. An order R in Q is called an *extension* of V to Q provided $R \cap F = V$. The following is the classical result which is concerned with extension.

Theorem 1.1[S]. Let D be a division ring with finite dimension over its center$F$ and let V be a valuation of F. Suppose that V is either complete or Henselian. Then there exists an invariant valuation ring of D extending V to D.

However any valuation V of F is not necessarily to be extended to an invariant valuation ring of D(even to a total valuation ring of D). If D is the quaternion algebra over the rational field, then only 2-adic valuation can be extended to an invariant valuation ring of D(see[GB]). Of course, The class of total valuation rings is much bigger than the class of invariant valuation rings(see [GB1], [GB2] and [M]).

As we have already seen, the invariant and total valuation rings have the following two problems.

(1) Any valuation ring of F is not necessarily to be extended to an invariant(a total) valuation ring of D.
(2) They are not defined in a simple Artinain ring which is not a division ring.

In 1984, N.I. Dubrovin defined non-commutative valuation ring by using the concept of points as follows:

Let Q be a simple Artinian ring. We adjoint a new symbol ∞ to Q and define

$q + \infty = \infty + q = \infty$ for any $q \in Q$,

and

$c \cdot \infty = \infty \cdot c = \infty$ for any $c \in U(Q)$.

Note that we do not define $\infty + \infty$ and $\infty \cdot q$ if $q \notin U(Q)$. We denote this set by (Q, ∞). Now let D be another simple Artinain ring. A mapping f of (Q, ∞) onto (D, ∞) is called a *point* of Q onto D if
(i) $f(1) = 1, f(qr) = f(q)f(r)$ and $f(q + r) = f(q) + f(r)$
for any $q, r \in (Q, \infty)$ whenever the right hand sides of (i) are defined, and for any $q \in Q$ with $f(q) = \infty$, there exist $r, s \in Q$ such that
(ii) $f(qr) \neq \infty, 0$ *and* $f(sq) \neq \infty, 0$ *with* $f(r) \neq \infty$ and $f(s) \neq \infty$.

The points are just a generalization of places in commutative rings(see [B] and [ZS]). However, he has not only solved the problems (1) and (2) above but also obtained significant results which look like genuine non-commutative valuation rings as it will be enumerated in the following.

Proposition 1.2[D1]. Let Q and D be simple Artinian rings and let f be a point of Q onto D. Then $R = \{q \in Q \mid f(q) \neq \infty\}$ is a subring of Q and $M = \{q \in R \mid f(q) = 0\}$ is an ideal of R such that
(1) R/M is a simple Artinain ring, and
(2) for any $q \in Q \backslash R$ there exist $r, s \in R$ with $qr, sq \in R \backslash M$.

Conversely, if R is a subring of Q with ideal M satisfying (1) and (2), then the mapping defined by

$$f(a) = \begin{cases} [a+M], & \text{for } a \in R \\ \infty, & \text{for } a \in Q\backslash R. \end{cases}$$

is a point of R onto R/M.

A subring R of Q satisfying the conditions in Proposition 1.2 is called a *Dubrobin valuation ring* of Q. Let R be a Dubrovin valuation ring of a simple Artinain ring Q. Then he first points out the following:
(a) $M = J(R)$, *the Jacobson radical* of R, and
(b) $qR \cap R \subseteq J(R)$ $(q \in Q) \Longrightarrow q \in J(R)$.

Next, he proves the following by skillfully combining the facts (a) and (b) with the properties of the Jacobson radical, Nakayama's Lemma and etc..

Theorem 1.3[D1]. Let R be a subring of a simple Artinain ring Q. The the following are equivalent:
(1) R is a Dubrobin valuation ring of Q.
(2) R is a local and semi-hereditary order in Q.
(3) R is a local and Bezout order in Q.

Here we give some defintions on terminology described in Theorem 1.3. A ring S is called *local* if $S/J(S)$ is a simple Artinian ring. If any finitely generated one sided ideal of S is projective, then S is called *semi-hereditary*. An order S in a simple Artinian ring is called *Bezout* provided any finitely generated one sided ideal of S is principal.

Furthermore, he proves the following by usage of Theorem 1.3 (3) and the fact that any overring of a Dubrovin valuation ring is again a Dubrovin valuation ring([D1, Theorem 4]).

Proposition 1.4[D1]. Let R be a Dubrovin valuation ring of a simple Artinain ring Q. Then the set of all R-ideals is linearly ordered by inclusion.

In his another paper, he gives an example of ring R in which the set of all R-ideals is linearly ordered but not a Dubrovin valuation ring. To obtain the further detailed results on Dubrovin valuation rings, we assume, in the rest of this section,that *Q is a simple Artinian ring with finite dimension over its center F.* Then any order in Q is a PI-ring. Combining these facts with the results stated before, he proves the following:

Theorem 1.5[D2]. Let R be a Dubrovin valuation ring of Q. The following hold:
(1) $V = F \cap R$ is a valuation ring of F.
(2)$P \in Spec(R)$. Then $C(P) = \{c \in R \mid c:$ *regular mod P*$\}$ is a regular Ore set of R and $R_P = \{ac^{-1} \mid a \in R,\ c \in C(P)\}$ is a Dubrovin valuation ring of R.
(3) The set of all R-ideals is a *commutative* semi-group.
(4) there is a bijective mapping between $Spec(R)$ and $\mathfrak{R}$, the set of all overrings of R given by $P \longmapsto R_P$ and $S \longmapsto J(S)$, where $P \in Spec(R)$ and $S \in \mathfrak{R}$.

The existence theorem has also been proved by Dubrovin ,and later Brung and Gräter gave an elementary proof of it. The conjugacy theorem is due to Wadsworth by using Henselization.

Theorem 1.6. Let V be a valuation ring of F.
(1)(The existence theorem)[D2] and [BG2]. There is a Dubrovin valuation ring R of Q with $V = R \cap F$.
(2)(The conjugacy theorem)[W]. Any Dubrovin valuation ring of Q whose centers are V is conjugate.

We have mainly introduced some results due to Dubrovin. After Dubrovin, some researchers who had been studying invariant and total valuation rings started studying Dubrovin valuation rings and related topics by using Dubrovin's results. For example, Proposition 1.4 and Theorem 1.5 (3) enable us to define the value group as follows:
Let $st(R) = \{q \in Q \mid qR = Rq\}$ is a commutative group and $U(R)$, the set of all units in R, is a subgroup of $st(R)$. The factor group $\Gamma_R = st(R)/U(R)$ is a total abelian group in the following definition;

$$qU(R) \geq sU(R) \Longleftrightarrow qR \subseteq sR$$

,where $q, s \in st(R)$. Γ_R is called a *value group* of R. It is easy to see that Γ_V is naturally embedded in Γ_R, where $V = R \cap F$. However, we can not freely handle Γ_R in order to get some good informations on R as in commutative case. In the case R is integral over V, Wadsworth obtains the following.

Theorem 1.7[W]. Let R be a Dubrovin valuation ring of Q with $V = F \cap R$. Then the following are equivalent:
(1) Any element in R is integral over V.

(2) For any $q \in Q$, there exists $s \in st(R)$ with $RqR = sR = Rs$.

This theorem was extended by Grater[G2] to the case R is a Bezout order in Q. By using the property (2) in Theorem 1.7, we can define a mapping $\omega : Q \longrightarrow \Gamma_R \cup \{\infty\}$, given by $\omega(q) = sU(R)$ and $\omega(0) = \infty$, where $q \in Q$ with $RqR = sR = Rs$ and the mapping ω satisfies the following which look like valuations in commutative rings.
(V1) $\omega(q) = \infty \Longleftrightarrow q = 0, \omega(-1) = 0$.
(V2) $\omega(q + s) \geq min\{\omega(q), \omega(s)\}$.
(V3) $\omega(qs) \geq \omega(q) + \omega(s)$.
(V4) $Im\ \omega = \{\omega(q) \mid q \in Q\} = \omega(st(\omega))$, where $st(\omega) = \{q \in U(Q) \mid \omega(q^{-1}) = -\omega(q)\}$.

As we can guess from the properties (V1)~(V4), in the case R is integral over V, we can use the value group of R. In fact, Wadsworth, Haile and Morandi have used the value group for further development of Dubrovin valuation rings and for characterizing Dubrovin valuation rings in crossed product algebras and tensor product algebras(see [HM], [HMW] and [MW]).

2. Semi-local Bezout Orders

Throughout this section, let Q be a simple Artinian ring with finite dimension over its center F and let $R_1, ..., R_n$ be Dubrovin valuation rings of Q. We say that $R_1, ..., R_n$ have the *intersection property* if

$\psi : \Re(R_1) \cup ... \cup \Re(R_n) \longrightarrow \mathrm{Spec}(R)$

defined by $\psi(S) = J(S) \cap R$ is well-defined and an anti-inclusion-preserving isomorphism, where $R = R_1 \cap ... \cap R_n$, $\Re(R_i)$ is the set of all overrings of R_i and $S \in \Re(R_i)$ for some i. As you can see from the following theorem, this condition is not queer.

Theorem 2.1. Let R be an order in Q. The following are equivalent:
(1) There are a finite number of Dubrovin valuation rings $R_1, ..., R_n$ of Q having the intersection property such that $R = R_1 \cap ... \cap R_n$.
(2) R is a semi-local Bezout order in Q.
(3) R is semi-local and for any prime ideal P of R, $C(P)$ is a regular Ore set of R and R_P is a Dubrovin valuation ring of Q.

The proofs of (1)$\Longrightarrow$ (2) and (2)$\Longrightarrow$ (3) are due to Gräter([G1] and [G2]). The proof of (3)$\Longrightarrow$ (1) is found in [MMU2].

In[G1], Gräter gets the following existence , uniqueness and conjugacy theorems by usage of the intersection property.

Theorem 2.2. Let V be a valuation ring of F.

(1) (The existence theorem). There are a finite number of Dubrovin valuation rings $R_1, ..., R_n$ having the intersection property such that $R = R_1 \cap ... \cap R_n$ is a semi-local Bezout V-order in Q with $V = Z(R)$.
(2) (The conjugacy theorem). Let R and S be a semi-local Bezout V-orders in Q, where $Z(R) = V = Z(S)$. Then R is conjugate to S.
(3)(The uniqueness theorem). Let $R = R_1 \cap ... \cap R_n = S_1 \cap ... \cap S_m$ be a semi-local Bezout V-order in Q with $V = Z(R)$, where $R_1, ..., R_n$ and $S_1, ..., S_m$ are both incomparable Dubrovin valuation rings of Q. Then n = m.

This theorem shows that semi-local Bezout orders are very important class of rings from the viewpoint of orders. The another important result on semi-local Bezout orders is the approximation theorem which is one of the main tools to study semi-local Bezout orders. The approximation theorem was obtained by Morandi[M1]. In addition to [G1] and [G2], we refer the reader to [PR] , [MMU2] for further development of semi-local Bezout orders.

3. Global Theory and Some Open Problems

The following are well-known in commutative rings.

Theorem 3.1[G]. Let D be a domain with its quotient field F.
(1) The following are equivalent:
(i) D is a Prüfer domain, that is, any finitely generated ideal of D is invertible.
(ii) D_P is a valuation ring of F for any prime ideal P of D.
(2) Let K be an algebraic field extension of F and let $R = \{k \in K \mid k \textit{ is integral over } D\}$. If D is Prüfer, then so is R.

Remark.(1) Suppose that D is a Dedekind domain and K is a finite dimension over F, then R is also Dedekind.
(2) Let $F = \mathbf{Q}$, the field of rationals, $D = \mathbf{Z}$,the ring of integers and let K be the algebraic closure of $\mathbf{Q}$. Then, of course, R is a Prüfer domain by Theorem 3.1 and it is known that any prime ideal P of R is idempotent, that is $P = P^2$([G]). This is one of the big differences between Dedekind domains and Prüfer domains. Another difference we want to point out is ; the Krull dimension of any Dedekind domain is one, and it is known that there exists a Prüfer domain which has infinite Krull dimension.
(3)As you can see from Theorem 3.1 that the theory of Prüfer domains is a global theory of valuation theory and contains the theory of infinite number theory.

In commutaive domains, an ideal is invertible if and only if it is projective. However, in non-commutative rings , an invertible ideal is of course projective and the converse is not generally held. This leads us, at least two, the following orders which are considered as global theory of Dubrovin valuation rings; semi-hereditary orders and Prüfer orders. An order in Q is said to be *Prüfer* if any finitely generated one-sided ideal is progenerator([AD]). Semi-hereditary rings have studyed from time to time. However, it seems to me that the study of semi-hereditary and Prüfer orders has just started from the viewpoint of orders ([AD], [MMU1], [MU1] and [MU2]). There are so many unsolved problems in the subjects we have just obserbed. Finally we want to introduce several tpyical ones.

Problem 1. Let R be an order in a simple Artinian ring Q with finite dimension over its center F and let $D = Z(R)$. Suppose that D is a Prüfer domain.
(1) Does there always exists a semi-hereditary D-order in Q?
(2) Moreover, does there always exist a Prüfer D-order?
The answer is affirmative if D is either Noetherian or valuation.
(3) By Zorn's lemma, there exists a maximal D-order. Does any maximal D-order in Q is semi-hereditary?

Problem 2. Let R be a Bezout V-order in a simple Artinian ring with finite dimension over its center F, where $V = Z(R)$, a valuation ring. Is the set of all R-ideals $G(R)$ a commutative semi-group? if the answer is affirmative, then we can get some informations on R by $G(R)$([G2]).

Problem 3. Let Q be a simple Artinian ring with $F = Z(Q)$. Suppose that Q is infinite dimensional over F. Characterize the structure of Dubrovin valuation rings, Prüfer orders and semi-hereditary orders in Q. This is not only replaced *finite dimension* by *infinite dimension* but also contans the theory of quantum type algebras. See [MV] and [VW] for valuation rings in quotient rings of quantum type algebras.

We have roughly given a survey of non-commutative valuation rings and their global theories. We refer the reader to the book [MMU2] for more detailed results and would like to end with mentioning that Alajbegovic and Dubrovin will also publish books on Prüfer rings from Marcel Dekker.

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ON THE RINGS OF THE MORITA CONTEXT WHICH ARE SOME WELL-KNOWN ORDERS

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Abstract

Some necessary and sufficient conditions are given for the ring of a Morita context to be a maximal order (resp. an Asano order, a Dedekind order, a Krull order).

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0. Introduction

A *Morita context* will be a set $M = (R, V, W, S)$ and two maps θ and ψ, where R and S are associative rings with identities, $V = {}_RV_S$ is an R-S bimodule and $W = {}_SW_R$ is an S-R bimodule. The map θ: $V \otimes_S W \to R$ is an R-R bilinear map and ψ: $W \otimes_R V \to S$ is an S-S bilinear map, furthermore these two maps satisfy the associativity conditions that are required to make $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ be a ring. T is called *the ring of the Morita context.*

For any $v \in V$ and $w \in W$, we shall write vw for $\theta(v \otimes w)$, and wv for $\psi(w \otimes v)$. Similarly, $Im\ \theta$ will be denoted by VW, and $Im\ \psi$ by WV. We refer the reader to [1] and [7] for Morita context theory.

Throughout this paper, we assume that $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$ *is the ring of the Morita context.*

1. Maximal orders, Asano orders and Dedekind orders

For any ring R, we denote by $\mathcal{C}_R(0)$ the set of all regular elements of R. Let M be a right R-module. Then M_R is said to be *torsion-free* (with respect to $\mathcal{C}_R(0)$) if $mc \neq 0$ for every $0 \neq m \in M$ and every $c \in \mathcal{C}_R(0)$.

Lemma 1.1.
T is an order in a simple Artinian ring $Q(T)$ if and only if

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[*] The detailed version of this paper has been submitted for publication elsewhere.

(1) R and S are orders in the simple Artinian rings $Q(R)$ and $Q(S)$ respectively,

(2) $vW = 0$ implies $v = 0$, $Vw = 0$ implies $w = 0$ and

(3) $VsW = 0$ implies $s = 0$.

Furthermore, the quotient ring of T has the form:

$$Q(T) \cong \begin{pmatrix} Q(R) & Q(R)\otimes V\otimes Q(S) \\ Q(S)\otimes W\otimes Q(R) & Q(S) \end{pmatrix}.$$

Remark: For simplicity, we denote $Q(R)\otimes V\otimes Q(S)$ by $Q(V)$ and $Q(S)\otimes W\otimes Q(R)$ by $Q(W)$.

We know from [1, Theorem 2 and Corollary 3] that $dimV_S = dimR_R = dim_R R = dim_S W$ and $dimW_R = dimS_S = dim_S S = dim_R V$, where $dimV_S$ denotes the uniform dimension of V as a right S-module. Since V is a torsion-free right S-module with finite uniform dimension, $V\otimes Q(S)$ is isomorphic to a finite direct sum of simple right ideals of $Q(S)$. For any $c \in C_R(0)$, $c(V\otimes Q(S)) \cong V\otimes Q(S)$ and $c(V\otimes Q(S)) \subseteq V\otimes Q(S)$. Hence $c(V\otimes Q(S)) = V\otimes Q(S)$ and so $V\otimes Q(S) = c^{-1}(V\otimes Q(S))$. This implies that $Q(V) = V\otimes Q(S)$. Similarly $Q(V) = Q(R)\otimes V$ and $Q(S)\otimes W = Q(W) = W\otimes Q(R)$.

When T is a prime ring, if one of R, S, V and W is zero, then it is easy to check that T becomes the trivial form. Therefore we all assume that

none of R, S, V and W are zero throughout this paper and T is an order in a simple Artinian ring $Q(T)$ throughout Sections 1 and 2.

Lemma 1.2. *Let A be a nonzero (T,T)-submodule of $Q(T)$. Then A has the form: $\begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$, where I is an (R,R)-submodule of $Q(R)$, J is an (S,S)-submodule of $Q(S)$, V_1 is an (R,S)-submodule of $Q(V)$ and W_1 is an (S,R)-submodule of $Q(W)$, which satisfy: $V_1W \subseteq I$, $W_1V \subseteq J$, $IV \subseteq V_1$, $JW \subseteq W_1$, $WV_1 \subseteq J$, $VW_1 \subseteq I$, $VJ \subseteq V_1$ and $WI \subseteq W_1$.* □

Let A be a (T,T)-submodule of $Q(T)$. Then A is called a *fractional T-ideal* of $Q(T)$ (for short, a T-ideal of $Q(T)$) if :

(1) there exist α, $\beta \in C_T(0)$ such that $\alpha A \subseteq T$ and $A\beta \subseteq T$.

(2) $A \cap C_T(0) \neq \phi$.

Note that the second condition is equivalent to $AQ(T) = Q(T) = Q(T)A$.

A T-ideal A of $Q(T)$ is said to be *integral* if $A \subseteq T$. This is equivalent to say that A is essential as a right ideal of T as well as a left ideal of T.

Following the definition of fractional T-ideals, we now introduce the concept of fractional modules of $Q(V)$ and $Q(W)$ as follows :

Let V_1 be an (R,S)-submodule of $Q(V)$. Then V_1 is called a *fractional (R, S)-module of $Q(V)$* if

(1) $V_1Q(S) = Q(V) = Q(R)V_1$ (note that $V_1Q(S) = Q(V)$ if and only if $Q(R)V_1 = Q(V)$) and

(2) there exist $c \in \mathcal{C}_R(0)$ and $d \in \mathcal{C}_S(0)$ such that $cV_1 \subseteq V$ and $V_1 d \subseteq V$.

If V_1 is contained in V, then it is called *integral*, equivalently $V_1 \subseteq V$ and $V_1 Q(S) = Q(V) = Q(R)V_1$. Similarly we can define *fractional (S, R)-modules of* $Q(W)$ and *integral (S, R)-modules of* $Q(W)$.

To characterize the maximality of T, we need to pick up some integral T-ideals.

Lemma 1.3. *Let T be an order in a simple Artinian ring $Q(T)$. Then:*

(1) If I is an integral R-ideal, then $\begin{pmatrix} I & IV \\ WI & WIV \end{pmatrix}$ *is an integral T-ideal.*

(2) If J is an integral S-ideal, then $\begin{pmatrix} VJW & VJ \\ JW & J \end{pmatrix}$ *is an integral T-ideal.*

In particular, $\begin{pmatrix} R & V \\ W & WV \end{pmatrix}$ *and* $\begin{pmatrix} VW & V \\ W & S \end{pmatrix}$ *are integral T-ideals.*

Proposition 1.4. *Let T be an order in a simple Artinian ring $Q(T)$ and let* $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ *be a (T,T)-submodule of $Q(T)$. Then A is a T-ideal of $Q(T)$ if and only if*

(1) I and J are an R-ideal of $Q(R)$ and an S-ideal of $Q(S)$, respectively.

(2) V_1 and W_1 are a fractional (R,S)-module of $Q(V)$ and a fractional (S,R)-module of $Q(W)$, respectively.

Corollary 1.5. *Let T be an order in a simple Artinian ring $Q(T)$ and let* $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ *be a (T,T)-submodule of $Q(T)$. Then A is an integral T-ideal of $Q(T)$ if and only if :*

(1) I and J are an integral R-ideal and an integral S-ideal, respectively.

(2) V_1 and W_1 are an integral (R,S)-module and an integral (S,R)-module, respectively. □

Let R be an order in a simple Artinian ring Q. Recall that R is said to be a *maximal order* if $O_r(I) = O_l(I) = R$ for all integral R-ideals of Q (see [11]).

For any (R,S)-submodule V_1 of $Q(V)$, we use the following notation : $(S : V_1)_l = \{\tilde{w} \in Q(W) | \tilde{w}V_1 \subseteq S\}$ and $(R : V_1)_r = \{\tilde{w} \in Q(W) | V_1\tilde{w} \subseteq R\}$. In a similar way, we can define $(R : W_1)_l = \{\tilde{v} \in Q(V) | \tilde{v}W_1 \subseteq R\}$ and $(S : W_1)_r = \{\tilde{v} \in Q(V) | W_1\tilde{v} \subseteq S\}$. If T is an order in a simple Artinian ring $Q(T)$, then, by comparing the Goldie dimension (see Remark to Lemma 1.1), we can obtain that $Hom_{Q(R)}(WQ(R), Q(R)) = (Q(R) : WQ(R))_l = Q(R)V$. Thus we easily obtain that $(R : W)_l = Hom(W_R, R_R)$. Similarly, $(R : V)_r = Hom({}_RV, {}_RR)$, $(S : V)_l = Hom(V_S, S_S)$, $(S : W)_r = Hom({}_SW, {}_SS)$. Obviously we have that $(R : V)_r$, $(S : V)_l$ contain W and that $(R : W)_l$, $(S : W)_r$ contain V. However, if T is a maximal order in $Q(T)$, then we have

Theorem 1.6. *Let T be an order in a simple Artinian ring $Q(T)$. Then T is a maximal order in $Q(T)$ if and only if the following are satisfied:*
(1) R and S are maximal orders in $Q(R)$ and $Q(S)$, respectively, and
(2) $(R:W)_l = V = (S:W)_r$ and $(R:V)_r = W = (S:V)_l$.

Remark: (1) If R and S are maximal orders, then $(R:W)_l = (S:W)_r$, i.e., $Hom(W_R, R_R) = Hom({}_SW, {}_SS)$. Since for any $\tilde{v} \in (R:W)_l$, $\tilde{v}W \subseteq R$ implies $W\tilde{v}WV \subseteq WRV \subseteq WV$, so $W\tilde{v} \subseteq O_l(WV) = S$ i.e., $\tilde{v} \in (S:W)_r$. Thus $(R:W)_l \subseteq (S:W)_r$ and the converse inclusion is proved similarly. Hence $(R:W)_l = (S:W)_r$. Similarly, $(R:V)_r = (S:V)_l$. Therefore the sufficient condition of Theorem 1.6 only need that "$V = (R:W)_l$ and $W = (S:V)_l$" or "$V = (S:W)_r$ and $W = (R:V)_r$".

(2) Let Z be the ring of integers. Then it is well-known that the matrix ring $\begin{pmatrix} Z & nZ \\ Z & Z \end{pmatrix}$, where $0, 1 \neq n \in Z$, is not a maximal order. So the condition (2) of Theorem 1.6 is necessary.

Corollary 1.7. *Let R be a maximal order in a simple Artinian ring $Q(R)$ and $e \in R$, $e^2 = e$, $e^* = 1 - e$. Then eRe is a maximal order in $eQ(R)e$. In particular, maximal order is a Morita invariant property .*

In [8] Martin discussed when a skew group ring $R * G$ is a maximal order in a simple Artinian ring. From above corollary we get the fixed ring case.

Corollary 1.8. *Let G be a finite group, $|G|^{-1} \in R$, $e = |G|^{-1}\sum_{g \in G} g$. If $R * G$ is a maximal order in a simple Artinian ring, then $R^G \cong e(R*G)e$ is a maximal order in a simple Artinian ring .* □

Next we consider Asano orders and Dedekind orders.

For any T-ideal A of $Q(T)$, we use the following notation:

$(T:A)_l = \{q \in Q(T) \mid qA \subseteq T\}$ and $(T:A)_r = \{q \in Q(T) \mid Aq \subseteq T\}$.

It is clear that $(T:A)_l$ is a left T-submodule of $Q(T)$, a right $O_l(A)$-submodule of $Q(T)$ and $(T:A)_l \cap C_T(0) \neq \phi$. Furthermore $(T:A)_l A \subseteq T$ and $A(T:A)_l A \subseteq A$. Thus if T is a maximal order, then $(T:A)_l$ is a T-ideal of $Q(T)$, because $O_l(A) = T$.

We define $A_v = (T:(T:A)_l)_r$, a T-ideal containing A. Similarly we can define ${}_vA = (T:(T:A)_r)_l$. We say that A is a *v-ideal* of $Q(T)$ if $A_v = A = {}_vA$.

Following order's case, for any fractional (R,S)-module V_1 of $Q(V)$, we define $V_{1v} = (S:(S:V_1)_l)_r$ and ${}_vV_1 = (R:(R:V_1)_r)_l$. We say that V_1 is a *v-(R, S)-module* of $Q(V)$ if $V_{1v} = V_1 = {}_vV_1$. Similarly we can define $W_{1v}, {}_vW_1$ and "*v-(S, R)-modules*" for any fractional (S,R)-module W_1 of $Q(W)$.

Proposition 1.9. *Let T be a maximal order in a simple Artinian ring $Q(T)$ and let $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ be a T-ideal of $Q(T)$. Then*

(1) $(T:A)_l = \begin{pmatrix} (R:I)_l & (R:W_1)_l \\ (S:V_1)_l & (S:J)_l \end{pmatrix}$ *and*

(1) $(T:A)_r = \begin{pmatrix} (R:I)_r & (S:W_1)_r \\ (R:V_1)_r & (S:J)_r \end{pmatrix}$.

(2) $A_v = \begin{pmatrix} I_v & V_{1v} \\ W_{1v} & J_v \end{pmatrix}$ *and* ${}_vA = \begin{pmatrix} {}_vI & {}_vV_1 \\ {}_vW_1 & {}_vJ \end{pmatrix}$.

Lemma 1.10. *Suppose that T is a maximal order in a simple Artinian ring $Q(T)$. Let V_1 and W_1 be a fractional (R,S)-module of $Q(V)$ and a fractional (S,R)-module of $Q(W)$, respectively. Then $\begin{pmatrix} V_1W & V_1 \\ WV_1W & WV_1 \end{pmatrix}$ and $\begin{pmatrix} VW_1 & VW_1V \\ W_1 & W_1V \end{pmatrix}$ are both T-ideals of $Q(T)$.*

An order R in the quotient ring $Q(R)$ is called *Asano* if any integral R-ideal is invertible. As in the case of an order in a simple Artinian ring (see [7, Proposition 5.2.6]), R is an Asano order in $Q(R)$ if and only if it is a maximal order in $Q(R)$ and any integral R-ideal of $Q(R)$ is a v-ideal. We use this characterization for Asano orders to prove the following:

Theorem 1.11. *Let T be an order in a simple Artinian ring $Q(T)$. Then T is an Asano order in $Q(T)$ if and only if the following are satisfied :*

(1) R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively.

(2) $(R:W)_l = V = (S:W)_r$ *and* $(R:V)_r = W = (S:V)_l$.

(3) any integral (R,S)-module of $Q(V)$ is a v-(R,S)-module.

(4) any integral (S,R)-module of $Q(W)$ is a v-(S,R)-module.

Corollary 1.12. *Let $e \in R, e^2 = e$. If R is an Asano order in a simple Artinian ring $Q(R)$, then eRe is an Asano order in a simple Artinian ring $eQ(R)e$. In particular, Asano order is a Morita invariant property.* □ (see [14, Lemma 4.2]).

According to [7] , we say that an order in a simple Artinian ring is *Dedekind* if it is a maximal order and hereditary.

It is well-known that T is right Noetherian if and only if R_R, S_S, V_S and W_R are right Noetherian (see [7, Proposition 1.1.7]).

Proposition 1.13. *Let T be an order in a simple Artinian ring $Q(T)$. Then T is a Dedekind order if and only if*

(1) R and S are Dedekind orders and

(2) $VW = R$ and $WV = S$.

2. Krull orders

In this section we consider when T is a Krull order by using some methods of divisorially graded rings. For terminology on divisorially graded rings we refer to [9], [10] and [12].

Let $K = \sum \oplus K_g$ $(g \in G)$ be a graded ring by any group G with unit element e and K_e be the part of degree e. Define:

$\mathcal{F}(\sigma_{K_e}) = \{H : \text{right ideal of } K_e \mid Hom_{K_e}(K_e/H : E(Q_e/K_e)) = 0\}$;

$\mathcal{F}(\sigma'_{K_e}) = \{H' : \text{left ideal of } K_e \mid Hom_{K_e}(K_e/H' : E'(Q_e/K_e)) = 0\}$,

where $E(Q_e/K_e)$ (resp. $E'(Q_e/K_e)$) is a right (resp. left) injective envelop of Q_e/K_e as a right K_e-module (resp. as a left K_e-module). So σ_{K_e} (resp. σ'_{K_e}) stands for the idempotent kernel functor cogenerated by right (resp. left) K_e-module Q_e/K_e.

K is said to be *divisorially graded* if the following properties hold:

(1) K is (σ', σ)-torsion-free;

(2) $Q_{\sigma'_{K_e}}(K_gK_h) = K_{gh} = Q_{\sigma_{K_e}}(K_gK_h)$ for any $g, h \in G$,

where $Q_{\sigma_{K_e}}(N)$ (resp. $Q_{\sigma'_{K_e}}(N)$) is the module of quotient of a right (resp. left) module N with respect to σ_{K_e} (resp. σ'_{K_e}).

Note that by Proposition 5.5 of [15, p.147], the right Gabriel topology $\mathcal{F}(\sigma_{K_e}) = \{H : \text{right ideal of } K_e | (K_e : k^{-1}H)_l = K_e \text{ for any } k \in K_e\}$, where $k^{-1}H = \{x \in K_e \mid kx \in H\}$. For the left Gabriel topology $\mathcal{F}(\sigma'_{K_e})$, we have a similar form. For terminology on localization at kernel functors we refer to [15].

Since $T = \begin{pmatrix} R & V \\ W & S \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \oplus \begin{pmatrix} 0 & V \\ W & 0 \end{pmatrix} = T_0 \oplus T_1$ (say), then T is graded by $G = Z/2Z$.

Next we shall prove that T is divisorially graded by $Z/2Z$.

Proposition 2.1. *Let T be a maximal order in a simple Artinian ring $Q(T)$. Then T is divisorially graded by $Z/2Z$.*

Let R be any ring, τ be any idempotent kernel functor on R and $\mathcal{F}(\tau)$ be the right Gabriel topology corresponding to τ. For a submodule N of a right R-module M, we define the τ-closure of N as $cl_\tau(N) = \{m \in M | m\mathcal{H} \subseteq N$ for some $\mathcal{H} \in \mathcal{F}(\tau)\}$. We say that N is *τ-closed* if $N = cl_\tau(N)$ and M is said to be *τ-Noetherian* if M satisfies the ascending chain condition on τ-closed submodules of M. An order in a simple Artinian ring is called *a Krull order* (in the sense of Chamarie [2] and [3]) if it is a maximal order and satisfies the ascending chain condition on τ-closed one-sided ideals. Now we are in a position to prove the main result of this paper which is

Theorem 2.2. *Let T be an order in a simple Artinian ring $Q(T)$. Then T is a Krull order if and only if*

(1) R and S are Krull orders in the simple Artinian rings $Q(R)$ and $Q(S)$, respectively;

(2) $(R:W)_l = V = (S:W)_r$ and $(R:V)_r = W = (S:V)_l$.

Next we consider localizations of T which is a Krull order in a simple Artinian ring. From Proposition 1.9, we know that if $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ is a v-ideal of T, then I and J are v-ideals of R and S, respectively. Furthermore, in the case of prime v-ideals, we have the following more detailed result.

Corollary 2.3. *If R is a Krull order in a simple Artinian ring Q and $0 \neq e^2 = e \in R$, then eRe is a Krull order in a simple Artinian ring eQe. In particular, Krull order is a Morita invariant property.*

Lemma 2.4. *Suppose that T is a maximal order in a simple Artinian ring $Q(T)$. If I is a prime v-ideal of R, then $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ is a prime v-ideal of T, where $V_1 = \{v \in V \mid vW \subseteq I\}$, $W_1 = \{w \in W \mid Vw \subseteq I\}$, $J = \{s \in S \mid VsW \subseteq I\}$.*

Let R be a Krull order in a simple Artinian ring Q. Then note that any prime v-ideal is a maximal v-ideal, i.e., maximal among v-ideals of R. Thus the following theorem follows from Lemmas 1.2, 2.4 and Proposition 1.9.

Theorem 2.5. *Let T be a Krull order in a simple Artinian ring $Q(T)$. Then there exist inclusion preserving (1,1) correspondences among the prime v-ideals of R, S and T. For any prime v-ideal I_1 of R and any prime v-ideal J_2 of S, these correspondences are given by*

$$I_1 \longleftrightarrow \begin{pmatrix} I_1 & V_1 \\ W_1 & J_1 \end{pmatrix}, J_2 \longleftrightarrow \begin{pmatrix} I_2 & V_2 \\ W_2 & J_2 \end{pmatrix}$$

, where $V_1 = \{v \in V \mid vW \subseteq I_1\}$, $W_1 = \{w \in W \mid Vw \subseteq I_1\}$, $J_1 = \{s \in S \mid VsW \subseteq I_1\}$, $I_2 = \{r \in R \mid WrV \subseteq J_2\}$, $V_2 = \{v \in V \mid Wv \subseteq J_2\}$, $W_2 = \{w \in W \mid wV \subseteq J_2\}$. □

Let $\mathcal{D}(R)$ denote the set of all v-ideals of R. If R is a Krull order, then $\mathcal{D}(R)$ forms a free abelian group generated by the maximal v-ideals of R (see [2] and [3]). From Theorem 2.5, we have:

Corollary 2.6. *Let T be a Krull order in a simple Artinian ring $Q(T)$. Then $\mathcal{D}(R) \cong \mathcal{D}(S) \cong \mathcal{D}(T)$ (as group isomorphisms).* □

We closed this section with localizations of a Krull order T at prime v-ideals. Let $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ be a prime v-ideal of T. Then from [2] and [3], we know that the localization T_A of T at A exists and it is a local Dedekind order in $Q(T)$. Here for any ring K, we denote the Jacobson radical of K by $J(K)$ and K is called *local* if $K/J(K)$ is a simple Artinian ring.

From [1], we know that Jacobson radical $J(T)$ is of the form $\begin{pmatrix} J(R) & V_1 \\ W_1 & J(S) \end{pmatrix}$, where $V_1 = \{v \in V \mid vW \subseteq J(R)\} = \{v \in V \mid Wv \subseteq J(S)\}$ and $W_1 = \{w \in W \mid Vw \subseteq J(R)\} = \{w \in W \mid wV \subseteq J(S)\}$.

Hence if T is local, then so are R and S. Furthermore if T is a Dedekind order, then so are R and S by Proposition 1.13. Thus $J(R)$ and $J(S)$ are invertible.

Combining these facts with the methods in Lemma 1.1 and its remark, we have:

Proposition 2.7. *Let* $A = \begin{pmatrix} I & V_1 \\ W_1 & J \end{pmatrix}$ *be a prime v-ideal of a Krull order* T. *Then the localization* T_A *of* T *at* A *is of the form* $\begin{pmatrix} R_I & R_I \otimes V \otimes S_J \\ S_J \otimes W \otimes R_I & S_J \end{pmatrix}$ *with*

$$J(T_A) = \begin{pmatrix} J(R_I) & VJ(S_J) \\ WJ(R_I) & J(S_J) \end{pmatrix} = \begin{pmatrix} J(R_I) & J(R_I)V \\ J(S_J)W & J(S_J) \end{pmatrix}. \square$$

3. Remark

Following [13], we say that T is (R,S)*-faithful* if T satisfies the following conditions:

"$WrV = 0$, where $r \in R$, implies $r = 0$" and

"$VsW = 0$, where $s \in S$, implies $s = 0$".

Note that if T is a prime ring, then T is (R,S)-faithful (see [13, Proposition 3]) and when T is a semiprime ring it is usual condition (see [1]).

Now the (R,S)-faithfulness can be explicitly characterized as follows :

Proposition 3.1. *Let T be an order in a semi-simple Artinian ring $Q(T)$. Then the following are equivalent :*

(1) T is (R,S)-faithful.

(2) VW is an integral R-ideal and WV is an integral S-ideal.

By [4, Theorem 2], "T is an order in a semisimple Artinian ring if and only if (1) R and S are orders in semisimple Artinian rings (2) $vW = 0$ implies $v = 0$ and $Vw = 0$ implies $w = 0$". If we assume that "T is (R,S)-faithful", we can find that all the results in Sections 1 and 2 hold by using Proposition 3.1.

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WEAK GLOBAL DIMENSION AND DUBROVIN VALUATION PROPERTIES OF CROSSED PRODUCTS

HIDETOSHI MARUBAYASHI AND ZHONG YI

Abstract

In this paper some conditions for a skew group ring or a crossed product to have finite weak global dimension are given. Using these results we obtain some necessary conditions and some sufficient conditions for a skew group ring or a crossed product to be a Dubrovin valuation ring.

1. Introduction

In this paper, all rings are associative and have identities, and modules are unitary. For simplicity, we use M_R and ${}_RN$ to denote that M is a right R-module and N is a left R-module respectively. If R is a ring, $J(R)$ is used to denot e its Jacobson radical. For the basic properties and some well-known results of skew group rings and crossed products, see [Mo] and [Pa] for details. For the definition and basic properties of Dubrovin valuation rings, see [D1], [D2] and [Wa] for details. For homological notation and terminology we use [Ro] as a reference.

In section 2, for the use in later sections, we first study the weak global dimension of skew group rings and crossed products. Some necessary conditions and some sufficient conditions for a skew group ring or crossed product to have finite weak global dimension are given. These results are the analogues of the results about the global dimension of skew group rings and crossed products given in [Yi2]. Section 3 is the main part of this paper. Using the results in the previous sections, we obtain some nece ssary conditions and some sufficient conditions for a skew group ring or a crossed product to be a Dubrovin valuation ring. In the skew group ring and commutative coefficient ring case we prove that $R*G$ is a Dubrovin valuation ring if and only if R is semihereditary, G-local and $G(M) = 1$, for some maximal ideal M of R, where $G(M)$ is the inertial group of M. (See section 2 for the definition of inertial groups.) As a special case of this result, we obtain that ! if R is a commutative valuation ring, G is a finite group and $R*G$ is a skew group ring, then $R*G$ is a Dubrovin valuation ring if and only if $G^T =< 1 >$, where G^T is the inertial group of R.

The detailed version of this paper has been submitted for publication elsewhere.

In section 4, some examples are given to show that the results we obtained are the best in their nature.

2. Weak Global Dimension of Skew Group Rings and Crossed Products

Let R be a ring and let M be a (right or left) R-module. We use $f.dh._R(M)$ to denote the flat dimension of M. The weak global dimension of R, denoted by $w.gl.dim.(R)$, is definied by

$w.gl.dim.(R) = sup\{f.dh._R(M) |$ for all right R-modules $M\}$,

see [Ro, p.239] for details. In [Yi2], some necessary and sufficient conditions for a crossed product and a skew group ring to have finite global dimension are given. In this section we prove some results for a crossed product and a skew group ring to have finite weak global dimension. These results are analogues of the results about global dimension and they are essential in the later sections.

2.1 LEMMA. [Yi1, 2.8.2 Lemma] *Let G be a finite group and let $R = \oplus_{g\in G} R_g$ be a strongly G-graded ring. Then*

(i) if M_R is a right R-module, and ${}_RN = \oplus_{g\in G} N_g$ is a graded left R-module, then

$$M \otimes_R N \cong M \otimes_{R_e} N_g$$

for all $g \in G$;

(ii) if $N_R = \oplus_{g\in G} N_g$ is a graded right R-module and ${}_RM$ is a left R-module, then

$$N \otimes_R M \cong N_g \otimes_{R_e} M$$

for all $g \in G$.

Using 2.1 Lemma, we obtain the following proposition. Its parts (i) and (ii) appeared in [Yi1, 2.8.3 Proposition]. Parts (iii) and (iv) can be easily proved by using parts (i) and (ii).

2.2 PROPOSITION. *Let G be a finite group and let $R = \oplus_{g\in G} R_g$ be a strongly G-graded ring. Suppose that H is a subgroup of G, M_R is a right R-module and ${}_RN = \oplus_{g\in G} N_g$ is a graded left R-module. Then*

(i) $Tor_n^R(M, N) \cong Tor_n^{R_e}(M, N_g) \cong Tor_n^{R_H}(M, N_H)$, for all $g \in G$, where $N_H = \oplus_{h\in H} N_h$;

(ii) $f.dh._{R_e}(M) \le f.dh._{R_H}(M) \le f.dh._R(M)$, and equalities hold if

$$f.dh._R(M) < \infty;$$

(iii) $w.gl.dim.(R_e) \le w.gl.dim.(R_H) \le w.gl.dim.(R)$, and equalities hold if $w.gl.dim.(R) < \infty$;

(iv) if R is semihereditary, then R_H is semihereditary for each subgroup H of G, so in particular R_e is semihereditary.

A ring R is called *semilocal* if $R/J(R)$ is a semisimple Artinian ring, and R is called *local* if $R/J(R)$ is a simple Artinian ring. Let R be a ring, G be a group and $R*G$ be a crossed product. If $R/J(R)$ is a G-simple Artinian rin g, that is, the G-invariant ideals of $R/J(R)$ are only itself and 0, then R is called a *G-local ring* . Suppose that M is an ideal of R, we call $G_M = \{g \in G | M^g = M\}$ the *decomposition group* of M, and call $G(M) = \{g \in G | g$ acts on R/M trivially $\}$ the *inertial group* of M. Obviously $G(M) \subseteq G_M$. If R is a local ring with unique maximal ideal M, then we denote $G(M)$ by G^T, and call it the *inertial group of R*.

2.3 LEMMA. *Let R be a ring, G be a finite group and $R*G$ be a crossed product.*

(i) $R*G$ is semilocal if and only if R is semilocal.

(ii) $R*G$ is local if and only if R is G-local and $(R/M)*G_M$ is a local ring for some maximal ideal M of R.

2.4 REMARK. In [Ma, 4.2 Theorem and 4.3 Lemma] some conditions concerning the local properties of group rings and skew group rings are given.

2.5 LEMMA. *Let R be a ring, G be a finite group and $R*G$ be a crossed product. Then $R*G$ is left coherent if and only if R is left coherent.*

The following result must be well-known. (Also see [Ro, Theorem 9.25 and Exer. 9.26].)

2.6 LEMMA. *Let R be a ring. Then R is left semihereditary if and only if $w.gl.dim.(R) \leq 1$ and R is left coherent.*

The following is the weak global dimension version of [Yi1, 2.5.10 Proposition], and the proof is roughly the same as that of [Yi1, 2.5.10 Proposition], by noting that finitely presented flat modules are projective.

2.7 PROPOSITION. *Let R be a ring, G be a finite group and $S = R*G$ be a skew group ring. If there exists a proper ideal M of R such that $char(R/M) = m > 0$ and $G(M)$ has an element $g \neq 1$ of order dividing m, then $w.gl.dim.(R*G) = \infty$.*

Now we are prepared to obtain a sufficient condition for a crossed product to have finite weak global dimension.

2.8 THEOREM. *Let R be a left coherent semilocal ring with $w.gl.dim.(R) < \infty$. Let G be a finite group and let $S = R * G$ be a crossed product. Suppose that for each maximal ideal M of R with $char(R/M) = p > 0$, $(R/M) * G_M$ is s emisimple Artinian, where G_M is the decomposition group of M. Then $w.gl.dim.(R * G) = w.gl.dim.(R) < \infty$. Furthermore, if R is semihereditary, then $R * G$ is also semihereditary.*

2.9 REMARK. The above theorem is an analogue of [Yi2, Theorem 3.2].

2.10 THEOREM. *Let R be a commutative coherent semilocal ring. Let G be a finite group and let $R * G$ be a skew group ring. Then the following are equivalent:*

(i) $w.gl.dim.(R * G) < \infty$;

(ii) (a) $w.gl.dim.(R) < \infty$;

(b) $(R/J(R)) * G$ is semisimple Artinian; (From the proof of 2.3 Lemma (ii), we see that this condition is equivalent to the condition that $(R/M) * G_M$ is a semisimple Artinian ring for each maximal ideal M of R.)

(iii) (a) $w.gl.dim.(R) < \infty$;

(b) for each maximal ideal M of R with $char(R/M) = p > 0$, $G(M)$ contains no elements of order p, where $G(M)$ is the inertial group of M.

2.11 REMARK. 2.10 Theorem is an analogue of [Yi2, Theorem 5.2].

3. Conditions for $R * G$ being a Dubrovin Valuation Ring

Let R be an arbitrary ring, G be a finite group and $R * G$ be a crossed product or a skew group ring. It is easy to see that R being a commutative valuation ring can not imply that $R * G$ is a Dubrovin valuation ring. We can also easily find examples to show that $R * G$ is a Dubrovin valuation ring (even a simple Artinian ring), but R is not even a local ring. In this section we study the conditions under which $R * G$ being a Dubrovin valuation ring. Some necessary conditions and some sufficient condi tions for $R * G$ being a Dubrovin valuation ring are given in terms of the coefficient rings and the acting groups.

Let R be a ring, G be a group and $R * G$ be a crossed product. Suppose that R is an order in a ring Q. Then it is easy to see that the crossed product $R * G$ can be uniquely extended to a crossed product $Q * G$.

At first, in the most general situations, we give some sufficient conditions for $R * G$ being a Dubrovin valuation ring.

3.1 THEOREM. *Let R be an order in an Artinian ring Q, G be a finite group and $R * G$ be a crossed product. If the following conditions are satisfied, then $R * G$ is a Dubrovin valuation ring.*

(i) R is G-local;

(ii) R is semihereditary;

(iii)' G_M is outer on R/M for some maximal ideal M of R.

3.2 REMARKS. (a) If R is G-local, and G_M is outer on R/M for some maximal ideal M of R, then G acts transitively on the set of maximal ideals of R and so for all maximal ideals M of R, G_M is outer on R/M.

(b) From 2.2 Proposition (iv) and 2.3 Lemma (ii) we know that (i) and (ii) of 3.1 Theorem are always necessary conditions for $R * G$ being a Dubrovin valuation ring. But if $R * G$ is a crossed product, even R is commutative, condition (iii)' of the above Theorem is not necessary for $R*G$ being a Dubrovin valuation ring, see 4.2 Example. If R is non-commutative, even $R * G$ is a skew group ring condition (iii)' of 3.1 Theorem is neither necessary for $R * G$ being a Dubrovin valuation ring, see 4.1 Example.

(c) It is obvious that condition (iii)' of 3.1 Theorem is equivalent to $(G_M)_{inn} =< 1 >$. Condition (iii)' can also implies that $J(R * G) = J(R) * G$ and $G(M) =< 1 >$. If R is commutative, $G(M) =< 1 >$ can also implies (iii)' of 3.1 Theorem. Therefore when R is commutative, condition (iii)' of 3.1 Theorem is equivalent to $G(M) =< 1 >$, for some maximal ideal M of R.

Although in the above we remarked that the conditions given in 3.1 Theorem are not necessary for $R * G$ being a Dubrovin valuation ring. In the skew group ring case, we can weaken the condition (iii)' a little bit to obtain some necessary conditions for $R * G$ being a Dubrovin valuation ring.

3.3 THEOREM. *Let R be an arbitrary ring, G be a finite group and $R * G$ be a skew group ring. If $R * G$ is a Dubrovin valuation ring, then we have:*

(i) R is G-local;

(ii) R is semihereditary;

(iii) $G(M) =< 1 >$ for some maximal ideal M of R. (Then $G(M) =< 1 >$ for all maximal ideals M of R.)

3.4 REMARKS. (a) If R is non-commutative, even in the case $|G|^{-1} \in R$ the conditions (i), (ii) and (iii) of 3.3 Theorem are not sufficient conditions for $R * G$ being a Dubrovin valuation ring; see 4.3 Example. In skew group rings and crosse d products theory, it is well-known that if the order of the acting group G is invertible in the coefficient ring R, then many classical properties can be transfered from the coefficient ring to the skew group ring and the crossed product, such as, fi niteness of global dimension, semiprimeness etc. But as indicated by 4.3 Example, there is not much relations between the Dubrovin valuation property and the invertiblity of the order of the acting group in its coefficient ring, mainly because of the loca l property.

(b) Combine the above argument and 3.2 Remarks (b), we see that the conditions (i), (ii) and (iii) of 3.3 Theorem are too weak to be sufficient conditions for $R * G$ being a Dubrovin valuation ring (mainly because of (iii)); but the conditions (i), (ii) a nd (iii)' of 3.1 Theorem are too strong to be necessary (mainly because of (iii)').

(c) If $R*G$ is not a skew group ring, even it is a twisted group ring and R is commutative, (iii) of 3.3 Theorem is not a necessary condition for $R*G$ being a Dubrovin valuation ring, see 4.2 Example.

In 3.3 Theorem, if the coefficient ring R is commutative, then the conditions (i), (ii) and (iii) becomes necessary and sufficient conditions for $R*G$ being a Dubrovin valuation ring. This is one of our main results.

3.5 THEOREM. *Let R be a commutative ring which is an order in an Artinian ring Q. Let G be a finite group and let $R*G$ be a skew group ring. Then $R*G$ is a Dubrovin valuation ring if and only if the following conditions are satisfied:*

(i) R is G-local;

(ii) R is semihereditary;

(iii) $G(M) = <1>$ for some maximal ideal M of R.

3.6 REMARK. In the above Theorem, R is not necessary a domain. 4.4 Example shows that $R*G$ can be a simple Artinian ring (thus is trivially a Dubrovin valuation ring), but R is not even a prime ring.

3.7 PROPOSITION. *Let R be a commutative ring which is an order in an Artinian ring Q, let G be a finite group and let $R*G$ be a skew group ring. Then $R*G$ is a Dubrovin valuation ring if and only if the following conditions are satisf ied:*

(i) The G-invariant ideals of R are linearly ordered;

(ii) R is semihereditary;

(iii) $G(M) = <1>$ for some maximal ideal M of R.

As we remarked before when $R*G$ is a Dubrovin valuation ring, R may not be a valuation ring (even not prime, not local), but if we suppose that R is a commutative domain we can easily see that, in this case, R is always a semi-local Bezout domain, and R^G, the fixed subring of G acting on R, is always a commutative valuation ring.

3.8 COROLLARY. *Let R be a commutative domain, let G be a finite group and let $R*G$ be a skew group ring. If $R*G$ is a Dubrovin valuation ring, then:*

(i) R^G is a valuation ring;

(ii) R is a semi-local Bezout domain.

If the coefficient ring is a commutative valuation ring, we immediately obtain a simple and nice condition for $R*G$ to be a Dubrovin valuation ring.

3.9 COROLLARY. *Let R be a commutative valuation ring, let G be a finite group and let $R*G$ be a skew group ring. Then $R*G$ is a Dubrovin valuation ring if and only if $G^T =<1>$.*

4. Examples

In this section, we collect some examples to show that the results we obtained in the previous sections are the best in their nature. Most of the examples presented here are well-known in skew group rings and crossed products.

4.1 Example. [Yi2, Example 2.6] Let K be a field with $char(K) = p > 0$, let $S = K[x][y; d/dx]$ be the first Weyl Algebra over K and let $R = Q(S)$ be the quotient division ring of S. Define an automorphism g of R by $g: R \to R; r \to xrx^{-1}$. Let $G =<g>$ and let $R*G$ be the skew group ring of G over R. Then $R*G$ is a simple Artinian ring, see [Yi2, Example 2.6] for details. So it is trivially a Dubrovin valuation ring; but $G_{(0)} = G$ is inner on $R/(0) = R$. Therefore the condition (iii)' of 3.1 Theorem is not necessary for $R*G$ being a Dubrovin valuation ring. (Confer 3.2 Remarks (b).)

4.2 Example. [Yi2, Example 6.4] Let K be a field of $char(K) = p > 0$, let $S = K[x, x^{-1}]$, and let $R = K[x^p, x^{-p}] \subset K[x, x^{-1}]$. Then $S = K[x, x^{-1}] = \bigoplus_{i=0}^{p-1} K[x^p, x^{-p}]x^i = R*G$ is a twisted group ring, where $G =<g>$ is a cyclic group of order p. Let $M = (x^p - 1)R$. M is a maximal ideal of $R = K[x^p, x^{-p}]$. Denote $\Omega = R\backslash M$. Since Ω is G-invariant, we have $S_\Omega = R_\Omega * G = R_M * G$. Because S is hereditary, S_Ω is also hereditary. We can directly check that $S_\Omega = S_{(x-1)S}$, so S_Ω is a local ring because $(x-1)S$ is a maximal ideal of S. Thus S_Ω is a Dubrovin valuation ring. But $G = G_{M_M}$ acts on $(R_M)/(M_M)$ trivially. So G acts on $(R_M)/(M_M)$ innerly. This example shows that (iii)' of 3.1 Theorem is not necessary for $R*G$ to be a Dubrovin valuation ring and 3.3 Theorem (iii) is not a necessary condition for! $R*G$ to be a Dubrovin valuation ring if $R*G$ is not a skew group ring.

4.3 Example. Let K be a field of characteristic 0 and let V be a valuation ring in K with unique maximal ideal N. let $Q = M_2(K)$ and let $R = M_2(V)$ be the 2×2 matrix ring over K and V respectively. Then R is a Dubrovin valuation ring in Q with unique maximal ideal $M = \begin{pmatrix} N & N \\ N & N \end{pmatrix}$. Let $u = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \in R$. Then $u^2 = 1$. Define an automorphism g of R by: $g: R \to R; r \to uru^{-1}$. It is clear that g can be uniquely extended to an automorphism of Q, which will still be denoted as g. Let $G =<g>$. Then we can form the skew gro up ring $R*G$ and $Q*G$. Obviously $G(M) =<1>$. By [HLS, Example 1] $Q*G$ is a semisimple Artinian ring, but it is not simple. So the order $R*G$ in $Q*G$ is not prime and thus $R*G$ is not a Dubrovin valuation ring. In this example

$|G|^{-1} \in R$. Thi s example shows that the conditions (i), (ii) and (iii) of 3.3 Theorem are not sufficient conditions for $R*G$ to be a Dubrovin valuation ring. But note that in this example, G_M is inner on R/M. In the situations which G_M is outer on R/M, we have 3.1 Theorem.

The example giving above is motivated by [HLS, Example 1]. If we take $V = K$ to be the trivial valuation ring, then we obtain a trivial example having the same properties. It is just [HLS, Example 1].

4.4 Example. Let K be an arbitrary field and let V be a valuation ring in K. Let $R = V \times V$ and $Q = K \times K$. Define an automorphism g of R by: $g : R \to R; (a, b) \to (b, a)$. It is clear that g can be uniquely extended to an automorphism of Q, which is still denoted as g. Let $G = < g >$. Let $R * G$ and $Q * G$ be the skew group ring of G over R and G over Q respectively. Then it is easy to see that $Q * G$ is a simple Artinian ring and $R * G$ is a Dubrovi n valuation ring in $Q * G$; but R is neither prime nor local. If we take $V = K$, we obtain a trivial example having the same properties.

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GENERARIZATIONS OF THEOREMS OF FULLER

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This is a summary of the authers' paper [5].

Let R be a right artinian ring and e and f a primitive idempotents of R. In [3, Corollary 3.2 and Theorem 3.4], K. R. Fuller showed that the following conditions are equivalent.

(1) eR is an injective right R-module with $S(eR) \cong T(fR)$, where $S(M)$ and $T(M)$ denote the socle and the top of M, respectively.

(2) $S(eR) \cong T(fR)$ and $S(Rf) \cong T(Re)$.

(3) (3ℓ) $\ell_{eR}(r_{Rf}(eI)) = eI$ for each left ideal I of R, and

(3r) $r_{Rf}(\ell_{eR}(Kf)) = Kf$ for each right ideal K of R, where $r_{Rf}(I) = \{a \in Rf | Ia = 0\}$ and $\ell_{eR}(K) = \{b \in eR | bK = 0\}$.

Let R be a semiprimary ring. Then for primitive idempotents e and f of R, (eR, Rf) is called an i-pair in [2] if the above condition (2) is satisfied. In [2, Theorem 1, Proposition 4 and Corollary 1], Y. Baba and K. Oshiro extended t hese results to semiprimary rings to show the following statements.

(a) If R is a semiprimary ring, then the condition (1) is satisfied if and only if both (2) and (3r) are satisfied.

(b) If R is a semiprimary ring satisfying the condition (*) below, then the conditions (1) and (2) are equivalent.

(*) The lattice $\{r_{Rf}(X) | X \subseteq eR\}$ satisfies the ascending chain condition.

Moreover, in [2, Theorem 2], they showed the following statement (c).

(c) If R is a semiprimary ring and (eR, Rf) is an i-pair for primitive idempotents e and f of R, then the following are equivalent.

(c1) Rf is artinian as a right fRf-module.

(c2) eR is artinian as a left eRe-module.

(c3) eR is an injective right R-module and Rf is an injective left R-module.

In this note, for a right R-module M with $S(M) \cong T(fR)$ and $P = EndM$, we consider a pair $({}_PM, Rf_{fRf})$ instead of an i-pair $({}_{eRe}eR, Rf_{fRf})$ and give generalizations of the results (a), (b) and (c) above. In particula r, for a module N_Q, we give a property for the pair $({}_PM, N_Q)$, which is related to Theorem 1.1 in Morita -Tachikawa [6].

Throughout this note we always assume that every ring has an identity and every module is unitary. In particular, R always stands for a semiprimary ring. For a ring H, by M_H (${}_HM$) we stress what M is a right (left) H-module. Le t M be a module. Then $L \leq M$ (resp. $L < M$) means that L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By $S(M)$, $T(M)$ and $E(M)$, we denote the socle, the top and an injective

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hull of M, respectively, and by $|M|$ we denote the composition length of M. Assume every homomorphism always operates from opposite side of scalar. "Acc" ("dcc") means the ascending (descending) chain condition. We denote the set of primitive idempotents of R by $Pi(R)$.

1. Colocal pairs of modules

Let P and Q be rings and ${}_PM$, N_Q and ${}_PU_Q$ be a left P-module, a right Q-module and a P-Q-bimodule, respectively. Let $\varphi : M \times N \to U$ be a P-Q-bilinear map, i. e., a map satisfying the following prop erties:

(1) $\varphi(x_1 + x_2, y) = \varphi(x_1, y) + \varphi(x_2, y)$,

(2) $\varphi(x, y_1 + y_2) = \varphi(x, y_1) + \varphi(x, y_2)$,

(3) $\varphi(px, yq) = p\varphi(x, y)q$;

for any $x, x_1, x_2 \in M, y, y_1, y_2 \in N, p \in P$ and $q \in Q$.

Then, we say that $({}_PM, N_Q)$ is a pair with respect to φ or simply a pair.

Let $({}_PM, N_Q)$ be a pair with respect to φ. Then for $x \in M, y \in N$ and for ${}_PX \leq_P M, Y_Q \leq N_Q$, by xy we denote the element $\varphi(x, y)$, and by XY we denote the P-Q-subbimodule of ${}_PU_Q$ generated by $\{xy | x \in X, y \in Y\}$. Moreover, for $A \subseteq M$ and $B \subseteq N$, we define submodules $r(A)$ $(= r_N(A))$ of N_Q and $\ell(B)$ $(= \ell_M(B))$ of ${}_PM$, as follows: $r(A) = \{y \in N | Ay = 0\}$ and $\ell(B) = \{x \in M | xB = 0\}$, and we call $r(A)$ (resp. $\ell(B)$) the right (resp. left) annihilator of A (resp. of B).

For an arbitrary ring H, we call an H-module V colocal if V has the (non-zero) smallest submodule. We call a pair $({}_PM, N_Q)$ colocal if the module ${}_PU_Q (= {}_P MN_Q)$ is colocal both as a left P-module and as a right Q-module. No te, in case $({}_PM, N_Q)$ is a colocal pair with $U = MN$, we have $S({}_PU) = S(U_Q)$. We call a pair (M, N) left faithful (resp. right faithful) if $\ell(N) = 0$ (resp. $r(M) = 0$), and a pair (M, N) faithful if it is left and right faithful . We denote the class of right annihilator submodules in N_Q by $Ar(M, N)$; that is $Ar(M, N) = \{Y \leq N_Q | Y = r\ell(Y)\}$, and similarly $A\ell(M, N) = \{X \leq_P M | X = \ell r(X)\}$, and the lattice of submodules of ${}_PM$ (resp. N_Q) by $Lat({}_PM)$ (resp. $Lat(N_Q)$). We say that a pair $({}_PM, N_Q)$ satisfies r-ann (resp. ℓ-ann) if $Ar(M, N) = Lat(N_Q)$ (resp. $A\ell(M, N) = Lat({}_PM)$).

Let P be a ring, M a P-R-bimodule and $Q = fRf$; $f \in Pi(R)$. In this case, unless otherwise stated, by the notation $({}_PM, Rf_Q)$ we always mean a pair with respect to the bilinear map $\varphi : M \times Rf \to Mf$ defined by $\varphi(m, a) = ma; m \in M, a \in Rf$.

Lemma 1.1. *Let $({}_PM, N_Q)$ be a colocal pair and put $U = {}_P MN_Q$, $M' = \ell_M(N)$ and $N' = r_N(M)$. Then ${}_PU_Q$ - dual takes simple left P-modules and simple right Q-modules to simples or zero.*

Proof. Let $K = xQ$ be a simple right Q-module. If $0 \neq_P Hom_Q(K, U)$, then $\alpha(x)Q = \alpha(K) = S(U_Q) = S({}_PU) = P\alpha(x)$ for any $0 \neq \alpha \in_P Hom_Q(K, U)$. Hence $P\alpha(x) \geq P\beta(x)$ for any $0 \neq \alpha, \beta \in_P Hom_Q(K, U)$ and consequently $P\alpha \geq P\beta$, which implies ${}_PHom_Q(K, U)$ is simple.

For the following lemma, we give a proof which is different from that of Lemma 1.1 in [5]. The proof is essentially owe to [6, Theorem 1.1] by using Lemma 1.1 above.

Lemma 1.2. *Let $({}_PM, N_Q)$ be a colocal pair, and $Y' < Y \leq N_Q$ with $Y' = r\ell(Y')$. If $(Y/Y')_Q$ is simple, then ${}_P(\ell(Y')/\ell(Y))$ is also simple and $Y = r\ell(Y)$.*

Proof. Put $U = {}_P MN_Q$. From $r\ell(Y') = Y' < Y \leq r\ell(Y)$, we obtain $\ell(Y) < \ell(Y')$, i.e., ${}_P(\ell(Y')/\ell(Y)) \neq 0$. Let $\psi :_P \ell(Y') \to_P Hom_Q(Y/Y'_Q, {}_PU_Q)$ be a map such that

$(x)\psi = \hat{x}$ for any $x \in \ell(Y')$, where $\hat{x} : Y/Y'_Q \to U_Q$ is a left multiplication map by x. Then ψ induces a monomorphism ${}_P(\ell(Y')/\ell(Y)) \to {}_P Hom_Q(Y/Y'_Q, {}_P U_Q)$. Hence ${}_P(\ell(Y')/\ell(Y))$ ($\cong_P Hom_Q(Y/Y'_Q, {}_P U_Q)$) is simple by Lemma 1.1. By the same argument, it follows that $(r\ell(Y)/r\ell(Y'))_Q$ is simple. Hence we have $Y = r\ell(Y)$ from $r\ell(Y') = Y' < Y \leq r\ell(Y)$.

Lemma 1.3. *Let $({}_P M, N_Q)$ be a colocal pair, and Y and Z submodules of N_Q with $Z = r\ell(Z) \leq Y_Q$. If $|(Y/Z)_Q| < \infty$, then $Y = r\ell(Y)$.*

In particular, if $({}_P M, N_Q)$ is right faithful and $|Y_Q| < \infty$, then $Y = r\ell(Y)$.

Theorem 1.4. *(See [2, Lemma 3 and Propositon 5].) Let Q be a semiprimary ring. Assume $({}_P M, N_Q)$ is a colocal pair and put $M' = \ell(N) \leq M$ and $N' = r(M) \leq N$. Then the following conditions are equivalent:*

(1) $Ar(M, N)$ satisfies acc, (or equivalently $A\ell(M, N)$ satisfies dcc).

(2) $|(N/N')_Q| < \infty$.

(3) $|{}_P(M/M')| < \infty$.

Moreover, in case the above conditions are satisfied, we have $X = \ell r(X)$ (resp. $Y = r\ell(Y)$) for any X with $M' \leq X \leq_P M$ (resp. for any Y with $N' \leq Y \leq N_Q$), and $|{}_P(M/M')| = |(N/N')_Q|$.

We call a pair $({}_P M, N_Q)$ right (resp. left) finite provided the lattice $Ar({}_P M, N_Q)$ (resp. $A\ell({}_P M, N_Q)$) satisfies acc and $({}_P M, N_Q)$ finite provided $({}_P M, N_Q)$ is left finite and right finite. As a special case of Theorem 1.4 , we have the following corollary.

Corollary 1.5. *Let Q be a semiprimary ring and $({}_P M, N_Q)$ a right finite faithful colocal pair. Then it holds that $|{}_P M| = |N_Q| < \infty$ and $({}_P M, N_Q)$ satisfies r-ann and ℓ-ann.*

2. Indecomposable injective modules

As mentioned in the introduction, we assume that R always stands for a semiprimary ring.

Let M be a right R-module. Then we call M quasi-injective if for any submodule L of M, any homomorphism $\theta : L \to M$ can be extended to some endomorphism of M. By [4, Theorem 1.1],M is quasi-injective if and only if $HM = M$, where $H = EndE(M_R)$.

Lemma 2.1. *Let M be a right R-module. Then the following are equivalent.*

(1) $S(M_R) \cong T(fR_R)$.

(2) Mf_Q is colocal and $\ell_M(Rf) = 0$.

Cororally 2.2. *Let M be a right R-module, and put $P = EndM$ and $Q = fRf$ ($\cong End_R Rf$); $f \in Pi(R)$. Then the following are equivalent.*

(1) $({}_P M, Rf_Q)$ is a left faithful colocal pair.

(2) ${}_P Mf$ is colocal and $S(M_R) \cong T(fR_R)$.

Moreover, in case the conditions are satisfied, any endomorphism α of $S(M_R)$ can be exended to some endomorphism of M.

Corollary 2.3. *Let e and f be primitive idempotents of R. Then* (eR, Rf) *is an i-pair if and only if* $({}_PeR, Rf_Q)$ *is a faithful colocal pair, where* $P = eRe$ *and* $Q = fRf$.

Lemma 2.4. *Let M be an injective (resp. quasi-injective) right R-module with* $S(M_R) \cong T(fR_R); f \in Pi(R)$. *Then* $({}_PM, Rf_Q)$ *is a faithful (resp. left faithful) colocal pair, where* $P = EndM$ *and* $Q = fRf$.

The following theorem is a slight generalization of Baba-Oshiro [2, Theorem 1].

Theorem 2.5. *(See [2, Theorem 1].) Let M be an indecomposable right R-module. Then the following conditions are equivalent.*

(1) *M is injective.*

(2) $({}_PM, Rf_Q)$ *is a faithful colocal pair satisfying r-ann for some* $f \in Pi(R)$, *where* $P = EndM_R$ *and* $Q = fRf$.

The following theorem shows that in case (M, Rf) is finite, the converse of Lemma 2.4 holds.

Theorem 2.6. *(See [2, Theorem 1 and Corollary 1].) Let M be a right R-module. If* $({}_PM, Rf_Q)$ *is a right finite faithful (resp. left faithful) colocal pair for some* $f \in Pi(R)$, *where* $P = EndM_R$ *and* $Q = fRf$, *then* M_R *is injective (resp. quas i-injective) with* $S(M_R) \cong T(fR_R)$.

Lemma 2.7.[7, Lemma 4] *Let* ${}_PM_R$ *be a P-R-bimodule such that* M_R *is injective and* $X = \ell_M(r_R(X))$ *for any submodule X of* ${}_PM$. *Then* ${}_PM$ *is linearly compact.*

By this lemma, we have the following proposition.

Proposition 2.8. *(See [2, Theorems 1 and 2]) Let M be an indecomposable right R-module and* $({}_PM, Rf_Q)$ *a faithful colocal pair, where* $f \in Pi(R)$, $P = EndM_R$ *and* $Q = fRf$. *Then the following are equivalent.*

(1) *The pair* $({}_PM, Rf_Q)$ *is right finite.*

(2) *The pair* $({}_PM, Rf_Q)$ *satisfies r-ann and* ℓ*-ann.*

(3) M_R *is injective and the pair* $({}_PM, Rf_Q)$ *satisfies* ℓ*-ann.*

As application of colocal pairs, by using Lemma 2.4 and Theorem 2.6, we can give elementary proofs of the following theorem which is due to Baba [1]. (See [8] for the definition of a quasi-projective module and its characterization.)

Theorem 2.9. *(Baba [1, Theorem 1]) Let e and f be primitive idempotents of R and put* $E = E(T({}_RRe))$, $P = eRe$ *and* $Q = fRf$. *If* $Ar({}_PeR, Rf_Q)$ *satisfies acc or dcc, then the following conditions are equivalent.*

(1) ${}_RE$ *is quasi-projective with* $T({}_RE) \cong T({}_RRf)$.

(2) eR_R *is quasi-injective with* $S(eR_R) \cong T(fR_R)$.

(3) $({}_PeR, Rf_Q)$ *is a left faithful colocal pair.*

(4) ${}_PeRf$ *is colocal and* $S(eR_R) \cong T(fR_R)$.

Theorem 2.10. *(Baba [1, Theorem 2]) Let $E = E(T(fR_R))$ and let $({}_{P_i}e_iR, Rf_Q)$ be a right (or left) finite colocal pair for any $i = 1, \ldots, n$, where $e_i, f \in Pi(R), P_i = e_iRe_i$ and $Q = fRf$. Put $P = EndE_R$. Then the following conditions are equivalent.*

(1) $S({}_RRf) \cong T({}_RRe_1) \oplus \cdots \oplus T({}_RRe_n)$.

(2) $T(E_R) \cong T(e_1R_R) \oplus \cdots e \oplus T(e_nR_R)$.

Moreover in case the conditions are satisfied, $S({}_RRf)$ (or equivalently $T(E_R)$) is square-free and the pair $({}_PE, Rf_Q)$ is finite.

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SOME EXAMPLES OF THE DECOMPOSITION MATRIX OF MACKEY FUNCTORS

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Abstract

We computed the decomposition matrices of Mackey functors for some small finite group in the same way of Thévenaz and Webb.

1. INTRODUCTION

In recently J. Thévenaz and P.J. Webb studied the structure of Mackey functors for finite groups. They established the modular representation theory for Mackey functors [7] (vertices, sources, Green correspondents of the projective and simple Mackey functors and blocks) and gave some examples of the computations of the decomposition matrices of Mackey functors. The purpose of this paper is to describe the computations of the decomposition matirices of Mackey functors for some small finite groups. We have made essential use of various computer algebra system GAP to obtain our results about the decomposition of $Q \setminus G/NP$.

The results of this paper is the following.

Proposition 1 *The decomposition matrix of* $\mathrm{Mack}_k(A_6, 1)$

Decomposition matrix

A_6 characteristic 2			$S_{P,V}$											
			1					C_2	C_4	C_2^2		C_2^2		D_8
			1	4_a	4_b	8_a	8_b	1	1	1	2	1	2	1
$S_{Q,W}$	1	1	1					1	1	1		1		1
		5_a	1	1				1		1			1	
		5_b	1		1			1			1	1		
		8_a				1								
		8_b					1							
		9	1	1	1			1	1		1		1	
		10	2	1	1									
	C_2	1						1	1	1	1	1	1	1
		1_a						1		1	1			
		1_b						1				1	1	
		1_c						1	1					
	C_4	1							1					1
		-1							1					
	C_2^2	1								1				1
		-1								1				
		2									1			
	C_2^2	1										1		1
		-1										1		
		2											1	
	D_8	1												1

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2. PRELIMINARIES

Throughout, we shall let k denote a splitting field in characteristic 2 for A_6 and all its subgroups, and let (k, R, K) denote a splitting 2-modular system for A_6. We denote each simple module for a group by its dimension, together with a subscript if there is more than one simple module of that dimension. For a subset X of a finite group G and an element g of G, we put ${}^gX := gXg^{-1}$, $X^g := g^{-1}Xg$. We refer to [7] for some other notations.

Lemma 2 *Let $H \leq G$. Let N be a Mackey functor for H. Then*

$$N\uparrow_H^G(L) = \bigoplus_{g\in[L\backslash G/H]} N(L^g \cap H)$$

for every subgroup L of G. In particular the conjugation σ of Mackey functor $N\uparrow_H^G$ for G is given as follows; if

$$x = \sum_{g\in[L\backslash G/H]} x_g \in N\uparrow_H^G(L)$$

where x_g is the component in $N(L^g \cap H)$, for $a \in G$, we have

$$\sigma_a^L : N\uparrow_H^G(L) \to N\uparrow_H^G({}^aL)\,;\, x \longmapsto \sum_{g\in[L\backslash G/H]} x_{a^{-1}g}.$$

Moreover,

$$\begin{aligned}\sum_{g\in[L\backslash G/H]} x_{a^{-1}g} &= \sum_{g'\in[L\backslash G/H]} x_{l_g g' h_g} \quad (a^{-1}g = l_g g' h_g \in Lg'N_G(H)) \\ &= \sum_{g'\in[L\backslash G/H]} c_{h_g^{-1}}^{L^{g'}\cap H}(x_{g'}) \quad (x_{g'} \in N(L^{g'} \cap NH))\end{aligned}$$

where $c_{h_g^{-1}}^{L^{g'}\cap H}$ is the conjugation of N :

$$c_{h_g^{-1}}^{L^{g'}\cap H} : N(L^{g'} \cap H) \to N(L^{g'h_g} \cap H).$$

We refer to the definition 2.8 of [3] or section 11 of [6] for details.

For a subgroup S of G, we use the following symbols for the normalizer and the Weyl group :

$$NS := N_G(S), \quad WS := W_G(S) = NS/S.$$

For a subgroup H of G and an irreducible WH-module V we can define the simple Mackey functor

$$S_{H,V}^G = (\mathrm{Inf}_{WH}^{NH} S_{1,V}^{WH})\uparrow_{NH}^G$$

for G from the theorem of Thévenaz and Webb [5]. By Lemma 2 we have next lemma which is Proposition (8.8) of [5].

Lemma 3 *Let $L \leq G$. Then*

$$S^G_{H,V}(L) = \bigoplus_{g \in [L \backslash T(H,L)/NH]} \mathrm{Tr}_1^{W_{L^g}(H)}(V)$$

where $T(H,L) = \{g \in G \mid {}^gH \leq L\}$.

Next lemma is corollary (9.10) of [5].

Lemma 4 *Let p be a prime and J a p-perfect subgroup of G. Then the square*

$$\begin{array}{ccc} G_0(\mathrm{Mack}_K(G,J)) & \xrightarrow{\psi_0} & \bigoplus_H G_0(KWH) \\ \downarrow \Delta & & \downarrow d \\ G_0(\mathrm{Mack}_k(G,J)) & \xrightarrow{\psi_p} & \bigoplus_H G_0(kWH) \end{array}$$

commute, where H runs up to conjugacy classes of subgroup of G such that $O^p(H) = J$.

Throughout of this paper, G denotes the alternating group A_6,

$$A_6 = < (1,2,3), (1,2)(3,4), (1,2)(4,5), (1,2)(5,6) > .$$

We shall have cause to look at the following 2-subgroups of G and its Weyl groups.

Lemma 5 *The representatives of conjugacy classes of 2-subgroups of G and some of its normalizers and Weyl groups are as follows.*

2-subgroup P	$N_G(P)$	$W_G(P)$
1	G	G
C_2	D_8	$C_2 \times C_2$
C_4	D_8	C_2
$E_a := C_2 \times C_2$	S_4	S_3
$E_b := C_2 \times C_2$	S_4	S_3
$D := D_8$	D_8	1

3. The evaluation $S_{1,V}$ in characteristic 0

In this section, we shall describe the characters of evaluation $S_{1,V}(Q)$ at 2-subgroup Q for simple Mackey functor indexed by the trivial subgroup of G as a KWQ-module. Since $S_{1,V}(1) = V$ it suffices to compute $S_{1,V}(Q)$ at non-trivial 2-subgroup Q of G. But it is easy to compute the character of WQ-module $S_{1,V}(Q) = \mathrm{Tr}_1^Q(V) = (\sum_{x \in Q} x)V$. By the restrictions of the irreducible characters of G to its 2-local subgroups and the character tables of Weyl groups we have next submatrix of Ψ_0 which is the matrix of ψ_0 of Lemma 4.

Proposition 6 *The submatrix of Ψ_0 indexed by the trivial subgroup of A_6 is*

A_6 characteristic 0			$S_{P,V}$						
			1						
			1	5_a	5_b	8_a	8_b	9	10
Q, W	1	1	1						
		5_a		1					
		5_b			1				
		8_a				1			
		8_b					1		
		9						1	
		10							1
	C_2	1	1	1	1	1	1	2	
		1_a		1	1	1	1	1	1
		1_b		1	1	1	1	1	1
		1_c				1	1	1	2
	C_4	1	1	1	1	1	1	2	
		-1				1	1	1	2
	E_a	1	1	1				1	
		-1		1					1
		2			1	1	1	1	
	E_b	1	1		1			1	
		-1			1				1
		2		1		1	1	1	
	D	1	1	1	1	1	1	2	

4. The evaluation of $S_{1,V}$ in characteristic 2

In this section, we shall describe the character of evaluation $S_{1,V}(Q)$ at 2-subgroup Q for simple Mackey functor indexed by the trivial subgroup of G as a kWQ-module. Since $S_{1,V}(1) = V$ it suffices to compute $S_{1,V}(Q)$ at non-trivial 2-subgroup Q of G. But we need some lemmas to compute the character of WQ-module $S_{1,V}(Q) = \mathrm{Tr}_1^Q(V) = (\sum_{x\in Q} x)V$. The next lemma is a part of Lemma 2.4.1 of [1].

Lemma 7 *Let M and N are kG-modules. Then the following expressions are equal;*
(i) The multiplicity of projective cover P_k of trivial module as a summand of $\mathrm{Hom}(M, N) = M^* \otimes N$,
(ii) The rank of $\sum_{g\in G} g$ in the matrix representations of $\mathrm{Hom}(M, N)$.

By the above lemma, we have next useful lemma.

Lemma 8 *Let $H \le G$ and let V be an simple kG-module. Then the dimension of $S_{1,V}(H) = \mathrm{Tr}_1^H(V)$ is as many times as the multiplicity of P_k as a summand of $V\downarrow_H$ where P_k is the projective cover of k as a kH-module.*

The next Erdmann's result (Theorem 1 of [2]) is useful for the simple modules of $A_6 \cong \mathrm{PSL}(2,9)$.

Lemma 9 *Let $G = \mathrm{PSL}(2,q)$, $q \equiv 1 \bmod 4$. Then (i) The vertex V of a nontrivial simple $B_0(G)$-module S is a Klein 4-group.*
(ii)he Green correspondent fS in NV is a module of length 2 without trivial composition factors.
(iii) If G has 2-conjugacy classes of Klein 4-groups, then both occur as vertices of simple simple $B_0(G)$-modules.

The next result is 2.6 of [4].

Lemma 10 *Let Q be a normal subgroup of G whose index is coprime to the characteristic of k and V a kG-module. Then the Loewy series of $V\downarrow_H$ is the restriction of Loewy series of V.*

Now we compute the character of the evaluation $S_{1,P}(Q)$ over k.

Proposition 11 *The evaluations of $S_{1,V}$ at cyclic subgroup C_2 of order 2 as kWC_2-module are*
(i) $S_{1,1}(C_2)=0$,
(ii) $S_{1,4_a}(C_2)\cong S_{1,4_b}(C_2)\cong 2k$,
(iii) $S_{1,8_a}(C_2)\cong S_{1,8_b}(C_2)\cong 4k$.

Proof. By the dimension of the trivial kG-module and Lemma 8, we can see *(i)*. Since 8_a, 8_b are projective $8_a\downarrow_{C_2}$, $8_b\downarrow_{C_2}$ are the direct sums of kC_2. So *(iii)* follows by Lemma 8. ¿From the ordinary character tables of G (pp 205 of [1]) and $3^2:4$ (Appendix 2),

$$4_a\downarrow_{3^2:4}=4_a,\ 4_b\downarrow_{3^2:4}=4_b$$

wherer 4_a and 4_b are projectives of $3^2:4$. Hence, we obtain *(ii)* by Lemma 8. ∎

Proposition 12 *The evaluations of $S_{1,V}$ at cyclic subgroup C_4 of order 4 as kWC_4-module are*
(i) $S_{1,1}(C_4)=0$,
(ii) $S_{1,4_a}(C_4)\cong S_{1,4_b}(C_4)\cong k$,
(iii) $S_{1,8_a}(C_4)\cong S_{1,8_b}(C_4)\cong 2k$,

Proof. It is similar to the proof of the above proposition. ∎

We have that the vertices of simple kG-modules 4_a 4_b are Klein 4-groups and the vertices are not conjugate in G from Lemma 9 *(iii)*. So, we put E_a be a vertex of 4_b and E_b of 4_a.

Proposition 13 *The evaluations of $S_{1,V}$ at elementary abelian 2-subgroup E_a of order 4 as kWE_a-module are*
(i) $S_{1,1}(E_a)=0$,
(ii) $S_{1,4_a}(E_a)=k$, $S_{1,4_b}(E_a)\cong 0$,
(iii) $S_{1,8_a}(E_a)\cong S_{1,8_b}(E_a)\cong 2k$.

Proof. Since $4_b\downarrow_{NE_a}=2+2$ and the vertex of 4_b is E_a

$$4_b\downarrow_{NE_a}\cong \begin{matrix}2\\2\end{matrix}$$

by Lemma 9 *(ii)*. Since the length of Loewy series of kE_a is 3 $4_a\downarrow_{E_a}$ is not projective by Lemma 10. Hence we have $S_{1,4_b}(E_a)\cong 0$ by Lemma 8. Since $4_a\downarrow_{NE_a}=2k+2$

$$4_a\downarrow_{NE_a}\cong \begin{matrix}k\\2\\k\end{matrix}\ \text{or}\ \begin{matrix}k\\k\end{matrix}\oplus 2$$

by the self duality of 4_a. If we assume the latter case then $k \oplus k$ is the direct summand of $4 \downarrow_{C_2}$, contradicting Proposition 11 (*ii*). The former case follows.

$$4_a \downarrow_D \cong \begin{matrix} k \\ 2 \\ k \end{matrix}$$

from Lemma 10. So $4_a \downarrow_{E_a} \cong kE_a$. Thus we have $S_{1,4_a}(E_a) = k$ by Lemma 8. Since

$$8_a \downarrow_{NE_a} \cong 8_b \downarrow_{NE_a} \cong P_2$$

where P_2 is the projective cover of simple kNE_a-module 2, we obtain (*iii*). The proof of (*i*) is similar to Proposition 11. ∎

Proposition 14 *The evaluations of $S_{1,V}$ at elementary abelian 2-subgroup E_b of order 4 as kWE_b-module are*
(*i*) $S_{1,1}(E_b) = 0$,
(*ii*) $S_{1,4_b}(E_b) \cong k$, $S_{1,4_b}(E_b) = 0$,
(*iii*) $S_{1,8_a}(E_b) \cong S_{1,8_b}(E_b) \cong 2$,

Proof. It is similary to the above proposition. ∎

Proposition 15 *The evaluations of $S_{1,V}$ at dihedral subgroup D of order 8 as kWD-module are*
(*i*) $S_{1,1}(D) = 0$,
(*ii*) $S_{1,4_b}(D) = S_{1,4_b}(D) = 0$,
(*iii*) $S_{1,8_a}(D) \cong S_{1,8_b}(D) \cong k$,

Proof. From Lemma 8 we obtain Proposition 15. ∎

Proposition 16 *The submatrix of Ψ_2 indexed by the trivial subgroup of A_6 is*

A_6 characteristic 2			$S_{P,V}$ 1				
			1	4_a	4_b	8_a	8_b
Q, W	1	1	1				
		4_a		1			
		4_b			1		
		8_a				1	
		8_b					1
	C_2	1		2	2	4	4
	C_4	1		1	1	2	2
	E_a	1		1			
		2				1	1
	E_b	1			1		
		2				1	1
	D	1				1	1

5. THE EVALUATION $S_{P,V}$ INDEXED BY NON-TRIVIAL 2-SUBGROUP

In this section, we shall first describe the characters of evaluation $S_{P,V}(Q)$ at 2-subgroup Q for simple Mackey functor indexed by non-trivial 2-subgroup of G over K. In general if P is non-trivial then the carrier P into Q

$$T(P,Q) = \{g \in G | P^g \leq Q\}$$

is not equal to G. Hence we must have the decomposition of double coset $P \backslash T(P,Q)/NP$. The GAP may be applied to the double coset $Q\backslash G/NP$ to determine the representatives in $T(P,Q)$.

Proposition 17 *(i)* $E_a \setminus T(C_2, E_a)/NC_2 = E_a g_1 NC_2 \cup E_a g_2 NC_2 \cup E_a g_3 NC_2$, *where*

$$\begin{aligned} g_1 &= (), \\ g_2 &= (3,5)(4,6), \\ g_3 &= (1,5,3)(2,6,4). \end{aligned}$$

(ii) $E_b \setminus T(C_2, E_b)/NC_2 = E_b g_1 NC_2 \cup E_b g_2 NC_2 \cup E_b g_3 NC_2$,

where

$$\begin{aligned} g_1 &= (), \\ g_2 &= (2,3)(5,6), \\ g_3 &= (2,3,4). \end{aligned}$$

(iii) $C_4 \setminus T(C_2, C_4)/NC_2 = C_4 NC_2$.

(iv) $D \setminus T(C_2, D)/NC_2 = D g_1 NC_2 \cup D g_2 NC_2 \cup D g_3 NC_2 \cup$,

where

$$\begin{aligned} g_1 &= (), \\ g_2 &= (3,5)(4,6), \\ g_3 &= (2,3)(5,6). \end{aligned}$$

(v) $D \setminus T(C_4, D)/NC_4 = DNC_4$,

(vi) $D \setminus T(E_a, D)/NE_a = DNE_a$.

(vii) $D \setminus T(E_b, D)/NE_b = DNE_b$.

Proposition 18 *The permutations of the representatives $\{g_i\}$ of the double coset $Q \setminus T(P,Q)/NP$ by the action of WQ are as follows.*
(i) If $1A$, $2A$, and $3A$ are the representatives of conjugacy classes of WC_2 then the permutation of $E_a \setminus T(C_2, E_a)/NC_2$ are as follows.

$$1A \mapsto \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad 2A \mapsto \begin{pmatrix} \cdot & 1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad 3A \mapsto \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix}.$$

(ii) If $1A$, $2A$, and $3A$ are the representatives of conjugacy classes of WC_2 then the permutation of $E_b \setminus T(C_2, E_b)/NC_2$ are as follows.

$$1A \mapsto \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}, \quad 2A \mapsto \begin{pmatrix} \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix}, \quad 3A \mapsto \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \end{pmatrix}.$$

Proposition 19 *The submatrix of Ψ_0 indexed by non-trivial 2-subgroup of A_6 is*

A_6 characteristic 0			$S_{P,V}$												
			C_2				C_4		E_a			E_b			D
			1	1_a	1_b	1_c	1	-1	1	-1	2	1	-1	2	1
Q, W	C_2	1	1												
		1_a		1											
		1_b			1										
		1_c				1									
	C_4	1	1				1								
		1_a				1		1							
	C_2^2	1	1						1						
		-1		1						1					
		2	1	1							1				
	C_2^2	1	1									1			
		-1			1								1		
		2	1		1									1	
	D_8	1	3	1	1		1		1		1	1		1	1

Proof. If P and Q are not conjugate in G $S_{P,V}(Q) = 0$ by Lemma 3. Hence we have to calculate $S_{C_2,V}(Q)$, $S_{P,V}(D)$ where $C_2 < Q$, and $1 \neq P < D$. Since we can compute by the same manner we only describe the character of $S_{C_2,V}(E_a)$. By Lemma 3 and Lemma 17 (i) we have

$$S^G_{C_2,V}(E_a) = \bigoplus_{i=1}^{3} \mathrm{Tr}_1^{W_{E_a^{g_i}}(C_2)}(V)$$

where $W_{E_a^{g_i}}(C_2)$ isomorphic to C_2 for $1 \leq i \leq 3$. Now, we commute the character of

$$\mathrm{Tr}_1^{W_{E_a^{g_i}}(C_2)}(V) = S_{1,V}^{WC_2}(W_{E_a^{g_i}}(C_2))$$

as $W_{WC_2}(W_{E_a^{g_i}}(C_2))$-module. Since $W_{WC_2}(W_{E_a^{g_i}}(C_2)) \simeq W_{NC_2}(N_{E_a^{g_i}}(C_2)) \cong C_2$

$$\mathrm{Irr}(W_{WC_2}(W_{E_a^{g_i}}(C_2))) = \{1, 1_a\}.$$

From the appendix of [7] we see the characters of $S_{1,V}^{WC_2}(W_{E_a^{g_i}}(C_2))(i = 1, 2, 3)$.

Thus we have determined the characters of $S_{P_{C_2,V(E_a)}}$ as WE_a-module by Proposition 18 (i) and GAP as follows.

	1	2A	2B	3C
$S_{C_2,1}(E_a)$	1	0	0	0
$S_{C_2,-1}(E_a)$	0	1	0	0
$S_{C_2,2}(E_a)$	0	1	0	0

The others follow in the same manner. We have now proved Proposition 19. ∎

Proposition 20 *The submatrix of* Ψ_2 *indexed by non-trivial 2-subgroup of* A_6 *is*

A_6 characteristic 2			$S_{P,V}$ C_2 1	C_4 1	C_2^2 1	C_2^2 2	C_2^2 1	C_2^2 2	D_8 1
$S_{Q,W}$	C_2	1	1						
	C_4	1		1					
	C_2^2	1			1				
		2				1			
	C_2^2	1					1		
		2						1	
	D_8	1			1		1		1

Proof. Immediately from Lemma 8 and the proof of Proposition 19. ∎

Proof of Proposition 1. We obtain the matrix of Ψ_0 by Proposition 6, 18 and the matrix Ψ_2 by Proposition 16, 20. Thus we obtain the matrix of Δ from

$$\Delta = \Psi_0 d \Psi_2^{-1}$$

in Lemma 4. ∎

Appendix

Table 1 The ordinary character table of $3^2 : 4$

1A	2A	3A	3B	4A	4B
1	1	1	1	1	1
1_a	1	1	1	-1	-1
1_b	-1	1	1	A	$-A$
1_c	-1	1	1	$-A$	A
4_a	.	-2	1	.	.
4_b	.	1	-2	.	.

where $A = ER(-2) = i2$.

Table 2 The 2-modular character table of $3^2 : 4$

1A	3A	3B
1	1	1
4_a	-2	1
4_b	1	-2

Table 3

Decomposition matrix of $\mathrm{Mack}_k(D_{12}, 1)$

D_{12} characteristic 2			$S_{P,V}$						
			1		C_2		C_2	C_2	C_2^2
			1	2	1	2	1	1	1
$S_{Q,W}$	1	1	1		1		1	1	1
		1_a	1		1				
		1_b	1				1		
		1_c	1					1	
		2		1		1			
		2_a		1					
	C_2	1			1				1
		-1			1				
		2				1			
	C_2	1					1		1
		-1					1		
	C_2	1						1	1
		-1						1	
	C_2^2	1							1

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TRIPLE REPRESENTATIONS OF KUPISCH SEMIGROUPS

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ABSTRACT. We investigate semigroup rings of Kupisch semigroups over which we can define a notion of piled modules. We study such modules to get a criterion for such a ring to be Gorenstein or Auslander-Gorenstein.

1. Introroduction.

Gorenstein 環とは self-injective dimension が, 左右とも有限のネター環のことである。このとき, 左右の次元は一致するので, それを Gorenstein 環の次元という。次元が 0 の Gorenstein 環とは, 準フロベニウス環に他ならず, ネター的遺伝環は, 次元が 1 の Gorenstein 環である。可換 Gorenstein 環については Bass [2] の結果は可換 Gorenstein 環論の基礎であると同時に, Gorenstein 環論一般の, ある意味での, 目標にもなっている。

また, Weyl Algebra $A_n(\mathbb{C})$ を典型とする, 代数多様体上の Differential Operator Rings や, $\mathbb{C}^n$ に作用する有限群 G による $A_n(\mathbb{C})$ の invariant subring 等は, 適当な条件下では Gorenstein 環になる。楕円曲線に対して定義される Sklyanin Algebras は global dimension が有限の Noetherian domain (したがって Gorenstein 環) であるし, Sklyanin algebra の拡張と見なされるべき Artin-Schelter の意味での Regular Algebras は, 定義から自明であるが, ネター環であれば, global dimension が有限の Gorenstein 環である。これらを代数幾何的 Gorenstein 環と呼ぼう。これらについては岩永の解説 [5,6] 及び, そこにあげられている文献を見られたい。また, 代数閉体上の可解リー環の universal envelopping algebra $U(\mathfrak{G})$ も Gorenstein 環になる ([1])。

代数幾何的 Gorenstein 環 A の特徴は, おおむね Auslander-Gorenstein graded algebra であり, それ自身 domain であるか, そうでなければ $gr(A)$ は可換な domain になりやすい。また, そこでの議論では, (Gabriel-Rentschler の意味での) Krull 次元とか, Gel'fand-Kirillov 次元が有効に働く。実際, 代数幾何的 Gorenstein 環の self-injective dimension は, その環の Krull 次元, または, Gel'fand-Kirillov 次元で与えられることが多い。一方, $U(\mathfrak{G})$ については, その self-injective dimension は, $\mathfrak{G}$ の 係数体上での次元に等しい。Malliavin 等の仕事を見ていくと, Bass の結果のアナロジーを追っているように見える。このような意味では, 代数幾何的 Gorenstein 環も $U(\mathfrak{G})$ も可換 Gorenstein 環に近いとも言える。

一方, 体上有限次元な多元環で, いくらでも次元の高い Gorenstein 環は存在する。また, Tiled Orders についても事情は同じである。これらの環は fully bounded であり, その Krull 次元は, それぞれ 0 であり, 1 である。一般には graded にはならない。

このように質の異なる Gorenstein 環がある以上, それらを一般的に取り扱うことを探求するのは, 自然かつ必然であると言ってもよい。この観点に立ったときの主題は,

This paper is in a final form and no version of it will be submitted for publication elsewhere.

当然ながら，Gorenstein 環の minimal injective resolution の解明である。こうしたことの一般的結果は，実はあまり多くはない。解説 [7,8] と宮地 [10]，およびそこに引用されている文献を見ればほぼ足りる。

筆者の意図は，その minimal injective resolution を具体的に記述できる Gorenstein 環の新しいクラスを作ることであった。その結果は[13] に著したが，その見取り図を与えることが本稿の目的である。なお，後藤氏は筆者の設定よりも一般的な Noetherian Algebras の設定で考察している。その概要は本報告集に記載されるはずである。

2. Basic Notions.

いわゆる Gabriel Quiver が，relations としては，zero relations と commutativity relations しかもたず，かつ，non-zero paths が全体としては有限である場合を考えよう。通常通り，各 vertex を自明な path と見なせば，それは non-zero な idempotent である。これら全体に 0 を付け加えた集合を S で表し，paths の結合を S の積と考えれば，S は次の条件をみたす。

(1) S は 0 を零元として含む有限半群である。
(2) S の nonzero idempotents 全体を $I(S) = \{e_1, e_2, \cdots, e_n\}$ とおけば，$S \neq \{0\}$
(3) $e_i \cdot e_j = 0$ if $i \neq j$
(4) $\forall s \in S$ に対して，$\exists e_i, e_j \in S$ s.t. $e_i s e_j = s$
(5) $T = S \setminus I(S)$ とおけば，$\exists m \geq 1$ s.t. $T^m = \{0\}$

Definition 1. 上記の条件 (1) ~ (5) をみたす半群 Sを *Kupisch semigroup*という。

今後，Kupisch semigroup というときは，上記の記号を固定する。この半群はすでに [12] で扱っているが，その重要性については，たとえば [3] を見られたい。

S を Kupisch semigroup，A を単位元をもつ任意の ring とする。このとき，自然に半群環 $A[S]$ を定義すると，$e_1 + \cdots + e_n$ は $A[S]$ の単位元となる。それで $A \subseteq A[S]$ と考える。また，自然に $S \subseteq A[S]$ である。さらに $A[S]$ は $S \setminus \{0\}$ を basis とする free A-module である。

これ以降，我々は次の設定で考える。

R を (Noetherian) discrete valuation ring，$J = (\pi)$ を R の Jacobson radical，k を R の剰余体，K を R の商体とする。Kupisch semigroup S に対して，３つの半群環；$\Lambda = R[S]$，$\Gamma = K[S]$，$\Delta = k[S]$ を考える。$(\Lambda, \Gamma, \Delta)$ を S の *triple representation* という。この環 Λ はかなり特殊なクラスに属する。

Lemma 2. Λ は *FBN(=fully bounded Noetherian) ring で semiperfect かつ Krull 次元が 1 である。*

（FBN 環と Krull 次元については [4] を，semiperfect ring については[14] を見られたい。）

$N = \sum_{s \in T} Rs \subseteq \Lambda$ は，Λ の nilpotent radical である。上記の補題の実質的な意味は，$\forall e \in I(S)$ に対して，$\Lambda e = Re \oplus \sum_{s \in T_e} \oplus Rse$ の unique maximal submodule が $Je \oplus \sum_{s \in T_e} \oplus Rse$ で与えられることである。すなわち，$Rad(\Lambda e) = Je \oplus \sum_{s \in T_e} \oplus Rse$. また，$X = \Lambda e / Ne$ は R-torsionfree な Λ-module で，$X/JX \cong \Delta e / Rad(\Delta e) \cong \Lambda e / Rad(\Lambda e)$. 以上の記号で，

Lemma 3. *Λ-simple modules 全体の集合と Δ-simple modules 全体の集合は一致し，* $\{\Lambda e / Rad(\Lambda e)$
$| e \in I(S)\}$ *で与えられる。また，Γ-simple modules 全体の集合は，* $\{\Gamma \otimes_\Lambda (\Lambda e / Ne) | e \in I(S)\}$ *で与えられる。*

3. Piled Modules.

${}_{\Lambda}X$ に対して，$X(i)=\{x\in X|\pi^i x=0\}$ とおく。これは X の Λ-submodule で，

$$0=X(0)\subseteq X(1)\subseteq X(2)\subseteq\cdots$$

Definition 4. ${}_{\Lambda}X$ *は下記の条件をみたすとき，* ${}_{\Delta}M$ *を slice とする piled module という。*

(1) $X=\bigcup X(i)$

(2) $JX(i+1)=X(i)$ *for* $\forall i\geq 1$.

(3) $M\cong X(1)$ *(as Δ-modules.)*

明らかに piled module X は R-torsion module で π^i を乗ずることによって得られるΛ-epimorphism で，$X/X(j)\cong X$，$X(j+1)/X(j)\cong X(1)$，$X(j+1)/X(1)\cong X(j)$ なる同型が得られ，$Soc(X)$ は essential in X で $Soc(X)\subsetneq X(1)$。

与えられた Δ-module M に対して，それを slice とする piled module の存在は気にかかる問題であるが，次は自明である。

Lemma 5. *有限生成 R-torsionfree な Λ-module L に対して，$M=L/JL$，$X=KL/JL$ (ただし，$KL=K\otimes_R L$) とおけば，X は M を slice とする piled module である。*

前節で考察したとおり，Δ-simple module は上記の補題の条件をみたす。また，$\forall e\in I(S)$ に対して $\Delta e\cong(\Lambda e/J\Lambda e)$ は明らか。さらに [13, Proposition 2.7] の証明を見れば，任意の indecomposable injective Δ-mopdule も上記の補題の条件をみたす。よって，

Proposition 6. *有限生成 Δ-module M について，M が simple or projective or injective ならば，M を slice としてもつ piled module は存在する。とくに，$e\in I(S)$ について，$\Gamma e/\Lambda e$ は Δe を slice にもつ piled module である。*

この他，有限生成 Δ-module M が Δ-module としての射影次元が≤ 1 ならば M を slice としてもつ piled module の存在が知られるが，その存在のための一般的条件は不明である。

Lemma 2 と [9, Corollary 3.6] によって，次が証明できる。

Proposition 7. *piled module ${}_{\Lambda}X$ について，X : Λ-injective $\Longleftrightarrow$ $X(1)$: Δ-injective.*

なお，[13] では $\Longrightarrow$ の証明に $X(1)$ の有限生成性を仮定したが，この条件は不要である。(佐藤真久氏の注意)。以上から，injective Δ-module Q に対して，それを slice としてもつ piled modue は一意に存在するから，それを $\mathcal{P}(Q)$ と表す。また，以下では A-module Z の minimal injective resolution の i-th term を $E^i_A(Z)$ で，i-th cosyzygy を$\Omega^i_A(Z)$ で表す。さらに，$\mathrm{id}_A(Z)$，$\mathrm{fd}_A(Z)$，$\mathrm{pd}_A(Z)$ で，それぞれ ${}_AZ$ の移入次元，平坦次元，射影次元を表す。

Corollary 8. *injective Δ-module Q に対して，$\mathcal{P}(Q)\cong E^0_{\Lambda}(Q)$.*

Lemma 9. *X と Y は，ともに piled module で，X は Y の submodule とすれば，Y/X も piled module で slice は $Y(1)/X(1)$ である。*

以上から次の結果が得られる。

Theorem 10. *${}_{\Lambda}X$ が slice ${}_{\Delta}M$ をもつ piled module ならば，$E^i_{\Lambda}(X)\cong\mathcal{P}(E^i_{\Delta}(M))$ で，$\Omega^i_{\Lambda}(X)$ は $\Omega^i_{\Delta}(M)$ を slice としてもつ piled module である。*

Corollary 11. $id_\Lambda(\Gamma e/\Lambda e) = id_\Delta(\Delta e)$ *for* $\forall e \in I(S)$.

次に piled moduel の平坦次元を考える。[13, Proposition 4.5] の証明を検討すれば，下記のことがわかる。

Proposition 12. ${}_\Lambda X$ *が有限生成 slice* ${}_\Delta M$ *をもつ piled module で，* $pd_\Delta M < \infty$ *ならば* ,
$fd_\Lambda(X) = pd_\Delta(M) + 1$. *特に，* $fd_\Lambda(E^{\cdot}_\Lambda(\Gamma e/\Lambda e)) = pd_\Delta(E^{\cdot}_\Delta(\Delta e))$ *for all* $e \in I(S)$.

4. Minimal injective resolutions for Λ, Γ and Δ.

$\Gamma = K\Lambda$ で，とくに Γ は Λ の classical quotient ring であることから，次が得られる。

Lemma 13. $e \in I(S)$ *に対して，* $E^0_\Lambda(\Lambda e) \cong E^0_\Gamma(\Gamma e)$ *であり，また次の exact sequence がある。*

$$0 \to \Gamma e/\Lambda e \to \Omega^1_\Lambda(\Lambda e) \to \Omega^1_\Gamma(\Gamma e) \to 0.$$

$\Gamma e/\Lambda e$ は Δe を slice にもつ piled module だからその minimal injective resolution は Theorem 10 からわかる。また，$\Omega^1_\Gamma(\Gamma e)$ の Λ-injective hull はΓ-injective hull と一致する。さらに，$E^0_\Lambda(\Gamma e/\Lambda e)$ と $E^1_\Gamma(\Gamma e)$ の Λ-uniform dimensions がともに有限であることを考えれば，$E^1_\Lambda(\Lambda e) \cong E^0_\Lambda(\Gamma e/\Lambda e) \oplus E^1_\Gamma(\Gamma e)$ であり，かつ，

$$0 \to \Omega^1_\Lambda(\Gamma e/\Lambda e) \to \Omega^2_\Lambda(\Lambda e) \to \Omega^2_\Gamma(\Gamma e) \to 0$$

なる exact sequence が得られる。この論法を帰納的に繰り返すことができる。よって，

Theorem 14. $\forall e \in I(S)$ *に対して，Λ-module* Λe *の minimal injective resolution は次で与えられる。*

$$0 \to \Lambda e \to E^0_\Gamma(\Gamma e) \to E^1_\Gamma(\Gamma e) \oplus E^0_\Lambda(\Gamma e/\Lambda e) \to E^2_\Gamma(\Gamma e) \oplus E^1_\Lambda(\Gamma e/\Lambda e) \to$$

$$\cdots \to E^i_\Gamma(\Gamma e) \oplus E^{i-1}_\Lambda(\Gamma e/\Lambda e) \to \cdots$$

これと Corollary 11 とから直ちに次がわかる。

Corollary 15. $id_\Lambda(\Lambda e) = max\{id_\Delta(\Delta e) + 1\ ,\ id_\Gamma(\Gamma e)\}$.
特に， Λ *: Gorenstein* $\Longleftrightarrow$ Δ *: Gorenstein and* Γ *: Gorenstein.*

実は，われわれの設定の特殊性から，

Lemma 16. Λ *が Gorenstein で，その次元を* n *とすれば，* $E^n_\Lambda(\Lambda)$ *は essential socle をもつ。*

また，Γ−module は Λ−module としての socle をもたないから，$n = \mathrm{id}_\Lambda(\Lambda e) < \infty$ とすると，
$E^n_\Gamma(\Gamma e) = 0$. 言い換えると，$\mathrm{id}_\Gamma(\Gamma e) \leq n - 1$. よって，

Theorem 17. Λ *が Gorenstein ならば，* $id(\Gamma) \leq id(\Delta) = id(\Lambda) - 1$.

ネター環 A 及び整数 $j > 0$ について，$\mathrm{fd}_A E^i({}_A A) \leq i$ が $\forall i < j$ について成立するとき，A は *j-Gorenstein* と言われる。これは左右同値である。(Auslander による。) さらに，A が次元 n の Gorenstein 環のとき，$(n+1)$-Gorenstein ならば，*Auslander-Gorenstein* と言われる。実は，n 次元の Gorenstein 環 A は n-Gorenstein であれ

ば, Auslander-Gorenstein である。Inroduction で述べた Gorenstein 環の実例の多くは Auslander-Gorenstein である。

われわれの環 Λ について調べてみると，まず，Γ-module M については，$\mathrm{pd}_\Gamma(M) = \mathrm{fd}_\Lambda(M)$ となる。よって，Theorem 14 と Proposition 12 から，

$$\begin{aligned} fd_\Lambda(E^i_\Lambda(\Lambda e)) &= pd_\Gamma(E^i_\Gamma(\Gamma e)) && \text{for } i = 0, \\ &= max\{fd_\Delta(E^{i-1}_\Delta(\Delta e)) + 1, pd_\Gamma(E^i_\Gamma(\Gamma e))\} && \text{for } \forall i \geq 1. \end{aligned}$$

したがって，Corollary 15 を考えれば，

Proposition 18. Λ : *Auslander-Gorenstein* $\Longleftrightarrow$ Δ *and* Γ : *Auslander-Gorenstein.*

5. Global dimensions of Λ, Γ and Δ.

M を Δ-module で，有限生成な R-torsionfree Λ-module X があって $M \cong X/JX$ となるものとする。(simple Δ-module はこの性質をみたす)。X の Λ-module としての minimal projective resolution を

$$\cdots \longrightarrow P_m \xrightarrow{\epsilon_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{\epsilon_0} X \longrightarrow 0$$

とする。$i \geq 1$ に対して，$X_i = \mathrm{Im}(\epsilon_i) = \mathrm{Ker}(\epsilon_{i-1})$ とおく。このとき，$Q_0 = P_0/JP_0$ は M の Δ-projective cover で，かつ，$\epsilon_0(JP_0) = JX$ だから，下記の commutative diagram with exact rows and columns を得る。

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & JP_0 & \longrightarrow & JX & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X_1 & \longrightarrow & P_0 & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & 0 \\ & & 0 & & 0 & & 0 & & \end{array}$$

さて，the composite map : $P_0 \to X \to M$ を ρ で表せば，$0 \to JP_0 + X_1 \to P_0 \xrightarrow{\rho} M \to 0$ なる short exact sequence を得る。そこで $i \geq 1$ について，$M_i = JP_{i-1} + X_i$ とおく。これは P_{i-1} の 有限生成な Λ-submodule で特に R-free であるから，$JP_{i-1} \cap X_i = JX_i$ が $\forall i \geq 1$ について成立する。このことを用いて次が証明される。

Lemma 18. $j \geq 1$ について，$X_j \neq 0$ と仮定する。$\rho_j : JP_{j-1} \oplus P_j \to M_j$ を $\rho_j(x, y) = -x + \epsilon_j(y)$ によって定義すれば，ρ_j は M_j の (Λ-*module としての*) *projective cover を与え，かつ，* $\mathrm{Ker}(\rho_j) = \{(\epsilon_j(y), y) | y \in M_{j+1}\} \cong M_{j+1}$ である。

さて，$\mu_0 = \rho$, $\mu_1 = \rho_1$ とおき，$j \geq 2$ については，μ_j = the composite map : $JP_{j-1} \oplus P_j \xrightarrow{\rho_j} M_j \xrightarrow{\sigma_j} \mathrm{Ker}(\rho_{j-1}) \hookrightarrow JP_{j-2} \oplus P_{j-1}$ とする。ここに $\sigma_j : M_j \to \mathrm{Ker}(\rho_{j-1})$ は上記の補題の同型である。よって，

$$\cdots \longrightarrow JP_{j-1} \oplus P_j \xrightarrow{\mu_j} JP_{j-2} \oplus P_{j-1} \to \cdots \to JP_1 \oplus P_2 \xrightarrow{\mu_2} JP_0 \oplus P_1 \xrightarrow{\mu_1} JP_0 \xrightarrow{\mu_0} M \to 0$$

は上記の補題から，M の Λ-module としての minimal projective resolution である。

Δ-module M は R-torsion だから $\mathrm{pd}_\Lambda(M) = m + 1$ for $0 \le m \le \infty$ と書ける。以下，この記号で考える。

Lemma 19. $i < m$ ならば，$JP_i \subsetneq M_{i+1}$. また，$m < \infty$ ならば $JP_m = M_{m+1}$.

そこで $i < m$ に対して $\overline{M_{i+1}} = M_{i+1}/JP_i$，$\overline{P_i} = P_i/JP_i$ とおくと，$0 \neq \overline{M_{i+1}} \subseteq Rad(\overline{P_i})$ となる。このことから，上記の M の minimal Λ-projective resolution は，M の minimal Δ-projective resolution を引き起こすことがわかる。実際，$i < m$ として，$\tau_i : JP_{i-1} \oplus JP_i \to JP_{i-1}$ を ρ_i の restriction とすれば，下記の commutative diagram with exact rows and columns を得る。

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & JP_i & \longrightarrow & JP_{i-1} \oplus JP_i & \xrightarrow{\tau_i} & JP_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & M_{i+1} & \longrightarrow & JP_{i-1} \oplus P_i & \xrightarrow{\rho_i} & M_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M_{i+1}} & \longrightarrow & \overline{P_i} & \longrightarrow & \overline{M_i} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

ここで α，β，γ は inclusion maps である。なお，可換性は $\mathrm{Ker}(\tau_i) = \{(\epsilon_i(y), y) | y \in JP_i\}$ を意味する。ここで最下段の short exact sequence は $\overline{M_i}$ のΔ-module としての projective cover となる。以上の考察は次のことを意味する。

Proposition 20. *有限生成 R-torsionfree な Λ-module X について，$M = X/JX$ となる Δ-module M については，$pd_\Lambda(M) = pd_\Delta(M) + 1$. (有限無限を問わず！)*

さきに注意しておいたように，Δ-simple module M は Proposition 20 の条件をみたす。また，Λ は Krull 次元が 1 であるから，[11, Corollary 4] により，$gl.dim\Lambda = sup\{pd_\Lambda(M) | M : simple\}$ である。よって，Lemma 3 により

Theorem 21. *$gl.dim\Lambda = gl.dim\Delta + 1$ が有限無限に関わらず成立する。*

また，ネター環 A については global dimension 有限ならば，$gl.dimA = id(A)$ であり，有限無限に関わらず $gl.dim\Gamma \le gl.dim\Lambda$ であるから，Theorem 17 と Theorem 21 から

Corollary 22. *$gl.dim\Gamma \le gl.dim\Delta$.*

6. Remarks.

(1) $id(\Gamma) = 0 < id(\Delta) = \infty$ となる例は存在する。
(2) gl.dim $\Gamma <$ gl.dim $\Delta < \infty$ となる例も存在する。
(3) S に対応する Gabriel quiver の relations がすべて zero-relations ならば，gl.dim Γ =gl.dim Δ.(Burgess-Fuller-E. L. Green- Zacharia)

(4) for all $n \geq 2$ について，Λ が n 次元の Auslander-Gorenstein ring となるものが存在する。しかも，global dimension が無限のものも有限のものも存在する。

(5) 体の拡大 $F \subseteq F'$ について $gl.dimF[S] = gl.dimF'[S]$ が成り立つ。したがって，$gl.dimF[S]$ は体の標数にのみ依存する。

(6) 体 F に対して，Kupisch semigroup S の半群環 $F[S]$ の self-injective dimension は，$F[S]$ が finite representation type になるならば，確実に計算できる。この意味で，本稿で述べた結果は単に Λ の self-injective dimension が Δ や Γ のそれによって記述できることを示しているのではない。

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DEGENERATIONS OF FINITE DIMENSIONAL MODULES AND TAME ALGEBRAS

ANDRZEJ SKOWROŃSKI

0. Introduction

The present notes are an extended version of two lectures given during the 29-th Symposium on Ring Theory and Representation Theory held at Kashikojima in November 1996.

The class of finite dimensional algebras (associative, with an identity) over an algebraically closed field may be divided in two disjoint classes. One class consists of tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces with two noncommuting endomorphisms, for which the classification is a well-known unsolved problem. Hence, we can realistically hope to describe indecomposable finite dimensional modules only over tame algebras. Among tame algebras we may distinguish the class of representation-finite algebras, having only finitely many isomorphism classes of indecomposable modules. The representation theory of representation-finite algebras is presently rather well understood. In practice, we have enough methods to decide whether a given algebra is representation-finite and, if it is the case, to describe all its indecomposable modules. A representation theory of arbitrary tame algebras is presently only emerging. We are still looking for methods to describe the indecomposable modules over tame algebras and efficient criteria for the tame representation type. On the other hand, even for representation-finite algebras, the representation theory of arbitrary finite dimensional modules is relatively poor. In this article we are interested in the geometry of modules of a fixed dimension. We shall report on recent advances in the investigation of degenerations of finite dimensional modules and connection with the representation type of algebras.

We divide the notes into the following parts:

1. Preliminaries on module categories.
2. Tame and wild algebras.
3. Affine varieties of modules.

This paper is a final form and no version of it will be submitted for publication elsewhere.

4. Degenerations of modules.
5. Degenerations in Auslander-Reiten components.
6. Degenerations to indecomposable modules.

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1. Preliminaries on module categories

Throughout this article A will denote a fixed finite dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite dimensional (over K) right A-modules and by $\operatorname{ind} A$ its full subcategory consisting of indecomposable modules. The term module is used for an object of $\operatorname{mod} A$ if not specified otherwise. We shall denote by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, that is, the ideal of $\operatorname{mod} A$ generated by all noninvertible morphisms in $\operatorname{ind} A$. The infinite radical $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ of $\operatorname{mod} A$ is the intersection of all finite powers $\operatorname{rad}^{i}(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. We shall denote by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$, where A^{op} is the opposite algebra to A. Without loss of generality we may assume that A is connected and basic.

Let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of A into a sum of primitive orthogonal idempotents. Then it is known that:

- $P_1 = e_1 A, \ldots, P_n = e_n A$ is a complete set of pairwise nonisomorphic indecomposable projective A-modules;
- $S_1 = e_1 A / e_1 \operatorname{rad} A, \ldots, S_n = e_n A / e_n \operatorname{rad} A$ is a complete set of pairwise nonisomorphic simple A-modules;
- $I_1 = D(Ae_1), \ldots, I_n = D(Ae_n)$ is a complete set of pairwise nonisomorphic indecomposable injective A-modules.

It is well-known that any X from $\operatorname{mod} A$ has a finite chain of submodules

$$0 = X_0 \subseteq X_1 \subseteq \ldots \subseteq X_l = X$$

with X_i / X_{i-1} being simple for all $1 \leq i \leq l$, called the composition series of X. A useful point of view concerning the composition series of a module from $\operatorname{mod} A$ is to consider the Grothendieck group $K_0(A)$ of the category $\operatorname{mod} A$. It follows from the Jordan-Hölder theorem that the images of the isomorphism classes $[S_1], \ldots, [S_n]$ of the simple A-modules form a $\mathbb{Z}$-basis of the group $K_0(A)$, and hence we may identify $K_0(A)$ with $\mathbb{Z}^n$. We may then assign to each module M from $\operatorname{mod} A$ its dimension-vector $\underline{\dim} M \in K_0(A) = \mathbb{Z}^n$, being the collection of the multiplicities of $S_1, \ldots, S_n$ in the composition series of M. It is well-known that

$$(\dim_K \operatorname{Hom}_A(P_i, M))_{1 \leq i \leq n} = \underline{\dim} M = (\dim_K \operatorname{Hom}_A(M, I_i))_{1 \leq i \leq n}.$$

One of the main objectives of the representation theory of finite dimensional algebras is to study modules having a fixed dimension-vector.

We denote by Γ_A the Auslander-Reiten quiver of A. We shall agree to identify an indecomposable A-module with the vertex of Γ_A corresponding to it. By a component of Γ_A we mean a connected component of the quiver Γ_A. Recall that a component $\mathcal{C}$ of Γ_A is called preprojective if $\mathcal{C}$ has no oriented cycle and each module in $\mathcal{C}$ belongs to the $\operatorname{Tr} D$-orbit of a projective module. Dually, $\mathcal{C}$ is said to be preinjective if $\mathcal{C}$ contains no oriented cycle and each module in $\mathcal{C}$ belongs to the $D\operatorname{Tr}$-orbit of an injective module. Following [26] a component $\mathcal{T}$ of Γ_A is said to be a tube if it contains an oriented cycle and its geometric realization $|\Gamma| = S^1 \times \mathbb{R}_0^+$, where S^1 is the unit circle and $\mathbb{R}_0^+$ the set of nonnegative real numbers. A tube $\mathcal{T}$ of Γ_A containing no injective modules (respectively, projective modules) is said to be a ray tube (respectively, coray tube). Finally, a component $\mathcal{C}$ of Γ_A is said to be a quasi-tube [27] if its translation subquiver formed by all vertices which are not projective-injective is a tube.

The Auslander-Reiten quiver Γ_A of A describes mainly the quotient category $\operatorname{mod} A/\operatorname{rad}^\infty(\operatorname{mod} A)$. If A is representation-finite then $\operatorname{rad}^\infty(\operatorname{mod} A) = 0$ and we may recover the morphisms in $\operatorname{mod} A$ from the quiver Γ_A. On the other hand, if A is representation-infinite and X, Y are indecomposable A-modules lying in different components of Γ_A, then $\operatorname{Hom}_A(X,Y) = \operatorname{rad}^\infty(X,Y)$. Hence $\operatorname{rad}^\infty(\operatorname{mod} A)$ contains lot of information on the representation theory of A. In order to study the behaviour of components of Γ_A in the category $\operatorname{ind} A$, the author introduced in [30] the component quiver Σ_A of A. The vertices of Σ_A are the components of Γ_A. Two components $\mathcal{C}$ and $\mathcal{D}$ are connected in Σ_A by an arrow $\mathcal{C} \to \mathcal{D}$ if $\operatorname{rad}^\infty(X,Y) \neq 0$ for some (indecomposable) modules X from $\mathcal{C}$ and Y from $\mathcal{D}$. Clearly, if A is representation-finite then Σ_A consists of exactly one vertex, namely the quiver Γ_A. Following [31] a component $\mathcal{C}$ of Γ_A is said to be generalized standard if Σ_A has no loop at the vertex $\mathcal{C}$, or equivalently, $\operatorname{rad}^\infty(X,Y) = 0$ for all modules X, Y from $\mathcal{C}$. Observe that any preprojective component of Γ_A is a source of Σ_A and any preinjective component of Γ_A is a sink of Σ_A, and hence all such components are generalized standard.

For basic background on the representation theory of algebras we refer to [6] and [26].

2. Tame and wild algebras

An intuitive notion of wild algebras was built on investigations of A. L. S. Corner and S. Brenner who showed that there are algebras A such that for any finite dimensional algebra B there is a full exact embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$. Presently, following Y. Drozd [14], we say that an algebra A is wild if there is a $K\langle x,y\rangle$–A-bimodule Q, where $K\langle x,y\rangle$ is the polynomial K-algebra in two noncommuting variables such that ${}_{K\langle x,y\rangle}Q$ is free of finite rank and the functor

$$- \otimes_{K\langle x,y\rangle} Q : \mathrm{mod} K\langle x,y\rangle \to \mathrm{mod}\, A$$

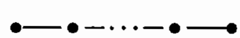
preserves indecomposability and isomorphism classes. It is known that if B is a finite dimensional K-algebra then there is a full exact embedding $\mathrm{mod} B \to \mathrm{mod} K\langle x,y\rangle$. Hence, if A is wild, the classification of indecomposable A-modules is as complicated as that for any finite dimensional K-algebra B, an impossible task! Following [14] an algebra is said to be tame if, for any dimension d, there exists a finite number of $K[x]$-A-bimodules Q_i, $1 \le i \le n_d$, which are free finite rank left modules over the polynomial K-algebra $K[x]$ in one variable, and all but finitely many isomorphism classes of indecomposable A-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} Q_i$ for some $\lambda \in K$ and some i. Denote by $\mu_A(d)$ the least number of $K[x]$-A-bimodules satisfying the above conditions for d. Then, following [27], A is said to be of polynomial growth if there is a positive integer m such that $\mu_A(d) \le d^m$ for all $d \ge 1$. From the validity of the fameous second Brauer-Thrall conjecture we know that an algebra A is representation-finite (ind A has only finitely many pairwise nonisomorphic objects) if and only if $\mu_A(d) = 0$ for all $d \ge 1$.

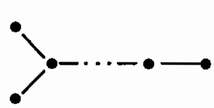
The following Tame and Wild Theorem proved in 1979 by Y. Drozd [14] is remarkable for the modern representation theory of algebras.

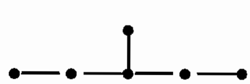
Theorem 2.1. *Every algebra A is either tame or wild, and not both.*

It is known that an algebra A is hereditary if and only if A is isomorphic to the path algebra KQ of a finite quiver Q without oriented cycles. The following classical result due to P. Gabriel [15] describes all representation-finite hereditary algebras.

Theorem 2.2. *Let A be a hereditary algebra. Then A is representation-finite if and only if A is isomorphic to the path algebra of one of the Dynkin quivers*

$\mathbb{A}_n$:  (n-vertices, $n \ge 1$)

$\mathbb{D}_n$:  (n-vertices, $n \ge 4$)

$\mathbb{E}_6$: 

$\mathbb{E}_7$: 

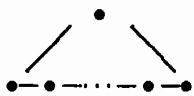
$\mathbb{E}_8$: 

where •—• *means* • ⟶ • *or* • ⟵ •.

The following theorem due to L. Nazarova [21] and P. Donovan–M. R. Freislich [13] (see also V. Dlab–C. M. Ringel [11]) describes all representation-infinite tame hereditary algebras.

Theorem 2.3. *Let A be a hereditary algebra. Then A is representation-infinite tame if and only if A is the path algebra of one of the extended Dynkin quivers*

$\tilde{\mathbb{A}}_n$:

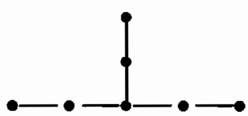


($n+1$-vertices, $n \geq 1$)

$\tilde{\mathbb{D}}_n$:

($n+1$-vertices, $n \geq 4$)

$\tilde{\mathbb{E}}_6$:



$\tilde{\mathbb{E}}_7$:

$\tilde{\mathbb{E}}_8$:

where •—• *means* • ⟶ • *or* • ⟵ •.

It is also known (see [11]) that if A is tame hereditary then the component quiver Σ_A is directed (has no oriented cycles) and $\mu_A(d) \leq 1$ for all $d \geq 1$.

The problem of when a finite dimensional algebra A of finite global dimension (even of global dimension 2) is tame remains still open. Recently the author obtained two general criteria for the tame representation type of triangular algebras, that is, algebras whose ordinary quiver has no oriented cycles. An important role in these investigations is played by two integral quadratic forms on $K_0(A)$: the Euler and the Tits form.

Let A be triangular and $K_0(A) = \mathbb{Z}^n$. Consider the Cartan matrix

$$C_A = (\dim_K \mathrm{Hom}_A(P_i, P_j))_{1 \leq i,j \leq n}$$

of A. Then C_A is invertible over $\mathbb{Z}$, and we get an integral quadratic form $\chi_A : \mathbb{Z}^n \to \mathbb{Z}$ on $K_0(A) = \mathbb{Z}^n$ defined by

$$\chi_A(x) = x C_A^{-t} x^t \qquad \text{for } x \in \mathbb{Z}^n.$$

It has been proved by C. M. Ringel [26] that if X is an A-module then

$$\chi_A(\underline{\dim}\, X) = \sum_{i\geq 0}(-1)^i \dim_K \mathrm{Ext}_A^i(X, X).$$

Hence, χ_A is called the Euler form of A.

The Tits form q_A of A is the integral quadratic form $q_A : \mathbb{Z}^n \to \mathbb{Z}$ defined by

$$q_A(x) = \sum_{i=1}^n x_i^2 - \sum_{i,j=1}^n x_i x_j \dim_K \mathrm{Ext}_A^1(S_i, S_j) + \sum_{i,j=1}^n x_i x_j \dim_K \mathrm{Ext}_A^2(S_i, S_j).$$

In [7] K. Bongartz proved that if gl.dim $A \leq 2$ then $\chi_A = q_A$. The Euler form χ_A (respectively, Tits form q_A) is said to be weakly nonnegative provided $\chi_A(x) \geq 0$ (respectively, $q_A(x) \geq 0$) for all $x \in \mathbb{Z}^n$ with nonnegative coordinates.

Recall that following D. Happel–I. Reiten–S. O. Smalø [16] an algebra A is called quasitilted (almost hereditary) if gl.dim $A \leq 2$ and for each module X from ind A we have $\mathrm{pd}_A X \leq 1$ or $\mathrm{id}_A X \leq 1$. It is known that any quasitilted algebra is triangular. The following characterization of tame quasitilted algebras has been proved by the author in [33].

Theorem 2.4. *Let A be a quasitilted algebra. The following conditions are equivalent:*

(i) *A is tame.*
(ii) *A is of polynomial growth.*
(iii) *q_A $(= \chi_A)$ is weakly nonnegative.*
(iv) $\dim_K \mathrm{Ext}_A^1(X, X) \leq \dim_K \mathrm{End}_A(X)$ *for any module X in* ind A.
(v) *Σ_A is directed.*
(vi) *A is tame tilted or tame of canonical type.*

The last condition means that A is a tame tilting of a hereditary algebra or a canonical algebra (in the sense of [26]).

In the representation theory of finite dimensional algebras an important role is played by the simply connected algebras. The importance of simply connected algebras follows from the fact that often we may reduce, with the help of coverings, the study of modules over an algebra to that for the corresponding simply connected algebras. Recently strongly simply connected algebras introduced by the author in [28] have been investigated extensively. Recall that a triangular algebra A is called strongly simply connected if, for every convex subcategory C of A, the Hochschild cohomology group $H^1(C, C)$ vanishes (that is, any derivation $\delta : C \to C$ is inner). In particular, any algebra whose ordinary quiver is a tree is strongly simply connected. We note also that strongly simply connected algebras may be of arbitrary large finite global dimension. In [36] (respectively, [22]) a class of minimal wild (respectively, tame minimal nonpolynomial growth) strongly simply connected algebras, called hypercritical algebras (respectively, *pg*-critical algebras), has been classified by quivers and relations. We have the following characterization of polynomial growth strongly simply connected algebras established by the author in [32] (the conditions (iii) and (iv) in the joint work with J. A. de la Peña [23]).

Theorem 2.5. *Let A be a strongly simply connected algebra. The following conditions are equivalent:*

(i) *A is of polynomial growth.*
(ii) *A does not contain a convex subcategory which is hypercritical or pg-critical.*
(iii) *The Tits form q_A is weakly nonnegative and $\operatorname{Ext}_A^2(X,X)=0$ for any module X in $\operatorname{ind} A$.*
(iv) *$\dim_K \operatorname{Ext}_A^1(X,X) \le \dim_K \operatorname{End}_A(X)$ and $\operatorname{Ext}_A^r(X,X)=0$ for any module X in $\operatorname{ind} A$ and $r \ge 2$.*
(v) *Σ_A is directed.*

3. Affine varieties of modules

In this section we assign to A a family of affine varieties of modules. This allows to study the finite dimensional A-modules using geometric methods, in particular methods from algebraic transformation groups and invariant theory (see [17], [18], [20]).

Let $a_1 = 1, a_2, \dots, a_m$ be a K-basis of A. Then we have the associated structure constants a_{kji}, $1 \le i,j,k \le m$, defined by

$$a_j a_i = \sum_{k=1}^{m} a_{kji} a_k.$$

Let d be a positive integer. Then the affine variety $\operatorname{mod}_A(d)$ of d-dimensional (right) A-modules consists of m-tuples $M = (M_1, \dots, M_m) \in M_{d\times d}(K)^m$ of $d \times d$ matrices with coefficients in K such that M_1 is the identity matrix and

$$M_j M_i = \sum_{k=1}^{m} M_k a_{kji}$$

for all $1 \le i,j \le m$. Observe that $\operatorname{mod}_A(d)$ is a closed subset of K^{md^2} in the Zariski topology. A d-dimensional A-module M can be regarded as a K-algebra homomorphism

$$M : A \to \operatorname{End}_K(K^d) = M_{d\times d}(K),$$

and hence we may assign to M the m-tuple $(M_1, \dots, M_m) \in \operatorname{mod}_A(d)$ given by $M_i = M(a_i)$ for any $1 \le i \le m$. We shall identify a d-dimensional A-module M with the point $(M_1, \dots, M_m)$ of $\operatorname{mod}_A(d)$ corresponding to it.

Let $G(d) = \mathrm{Gl}_d(K)$. Then $G(d)$ acts on $\mathrm{mod}_A(d)$ by conjugation:

$$g * M = (gM_1g^{-1}, \dots, gM_mg^{-1})$$

for all $g \in G(d)$ and $M = (M_1, \dots, M_m) \in \mathrm{mod}_A(d)$. Hence $\mathrm{mod}_A(d)$ is an affine variety with a natural action of the affine (reductive) algebraic group $G(d)$. For a module $M \in \mathrm{mod}_A(d)$ we denote by $\mathcal{O}(M)$ its $G(d)$-orbit $G(d)M = \{g*M;\ g \in G\}$. Observe that two d-dimensional A-modules M and N are isomorphic if and only if $\mathcal{O}(M) = \mathcal{O}(N)$. Indeed, $M \simeq N$ if and only if there exists a K-linear isomorphism $g : M = K^d \to K^d = N$ such that the diagrams

$$\begin{array}{ccc} K^d & \xrightarrow{M(a_i)} & K^d \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ K^d & \xrightarrow{N(a_i)} & K^d \end{array}$$

$1 \le i \le m$, are commutative, or equivalently $N = g * M$.

We are interested in the orbit closures $\overline{\mathcal{O}(M)}$ of modules $M \in \mathrm{mod}_A(d)$ in the Zariski topology. The following facts are well-known (see [17], [18]).

Lemma 3.1. *Let M be a module in* $\mathrm{mod}_A(d)$. *Then*

(i) *$\overline{\mathcal{O}(M)}$ is a union of $G(d)$-orbits.*
(ii) *$\mathcal{O}(M)$ is open in $\overline{\mathcal{O}(M)}$.*
(iii) *$\overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$ is a union of $G(d)$-orbits of smaller dimension than $\mathcal{O}(M)$.*
(iv) *$\overline{\mathcal{O}(M)}$ contains exactly one orbit of minimal dimension (closed orbit) given by the semisimple A-module with dimension-vector equal $\underline{\dim}\, M$.*

Let $1 = e_1 + \ldots + e_n$ be the decomposition of the identity of A into a sum of primitive orthogonal idempotents, $\underline{d} = (d_1, \dots, d_n) \in \mathbb{N}^n$ be a dimension-vector and $d = d_1 + \ldots + d_n$. For each $1 \le i \le n$, denote by E_i the matrix of the projection

$$K^d = \bigoplus_{i=1}^{n} K^{d_i} \to K^{d_i}.$$

Then we may consider the affine subvariety

$$\mathrm{mod}_A(\underline{d}) = \{M \in \mathrm{mod}_A(d) \mid M(e_i) = E_i \text{ for all } 1 \le i \le n\}$$

of $\mathrm{mod}_A(d)$. Moreover, the affine algebraic subgroup $G(\underline{d}) = \prod_{i=1}^{n} \mathrm{Gl}_{d_i}(K)$ of $G(d) = \mathrm{Gl}_d(K)$ acts on $\mathrm{mod}_A(\underline{d})$ by conjugation. We have the following facts.

Lemma 3.2. *Let M and N be two modules in* $\mathrm{mod}_A(\underline{d})$. *Then*

(i) *$M \simeq N$ if and only if $G(\underline{d})M = G(\underline{d})N$.*
(ii) *$N \in \overline{\mathcal{O}(M)} = \overline{G(d)M}$ in* $\mathrm{mod}_A(d)$ *if and only if $N \in \overline{G(\underline{d})M}$ in* $\mathrm{mod}_A(\underline{d})$.

In this setting a number of interesting questions arises very naturally: what are the irreducible components, orbits closures and singularities in the varieties $\mathrm{mod}_A(d)$ (respectively, $\mathrm{mod}_A(\underline{d})$). It is known that all varieties $\mathrm{mod}_A(d)$, $d \geq 1$, are nonsingular if and only if A a hereditary algebra. Moreover, the following known facts show a connection between homological and geometric properties of modules.

Theorem 3.4. *Let M be a module in $\mathrm{mod}_A(d)$. Then*

(i) *If $\mathrm{Ext}^1_A(M,M) = 0$ then $\mathcal{O}(M)$ is an open subset of $\mathrm{mod}_A(d)$ and its closure $\overline{\mathcal{O}(M)}$ is an irreducible component of $\mathrm{mod}_A(d)$.*

(ii) *If $\mathrm{Ext}^2_A(M,M) = 0$ then M is a nonsingular point of $\mathrm{mod}_A(d)$.*

4. Degenerations of modules

One of the important problems in the geometric classification of finite dimensional modules is to study their degenerations. For $M, N \in \mathrm{mod}_A(d)$, we say that N is a degeneration of M if N belongs to the closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(d)$, and we denote that fact by $M \leq_{\deg} N$, and not by $N \leq_{\deg} M$ as one might expect. Then $\leq_{\deg}$ is a partial order on the set of $G(d)$-orbits in $\mathrm{mod}_A(d)$ (equivalently, the set of isomorphism classes of d-dimensional A-modules). We have the following fact.

Lemma 4.1. *Let $M, N \in \mathrm{mod}_A(d)$. Then $M \leq_{\deg} N$ if and only if there is an affine variety Z and a (regular) morphism $\mu : Z \to \mathrm{mod}_A(d)$ such that $\mu(z) \simeq M$ for some open and dense subset U of Z, and $\mu(z') \simeq N$ for a point $z' \in Z$.*

Proof. Assume $M \leq_{\deg} N$. Then we may take $Z = \overline{\mathcal{O}(M)}$ and $\mu : Z \to \mathrm{mod}_A(d)$ the canonical embedding. We know by Lemma 3.1 that $U = \mathcal{O}(M)$ is an open dense subset of $Z = \overline{\mathcal{O}(M)}$. Clearly, $M \simeq Z$ for any $Z \in \mathcal{O}(M)$ and $N \in \overline{\mathcal{O}(M)}$ by our assumption. Conversely, assume that there are an affine variety Z, a morphism $\mu : Z \to \mathrm{mod}_A(d)$, and an open dense subset U of Z such that $\mu(z) \simeq M$ for any $z \in U$ and $\mu(z') \simeq N$ for some point $z' \in Z$. Then $\mu(U) \subseteq \mathcal{O}(M)$, and $Z = \overline{U} = \mu^{-1}(\overline{\mathcal{O}(M)})$. Hence, $N = \mu(z') \in \overline{\mathcal{O}(M)}$, and therefore $M \leq_{\deg} N$.

A classical example of degenerations is provided by filtrations of modules. For a filtration

$$F : M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_s \supseteq M_{s+1} = 0$$

of an A-module M we denote by $\mathrm{gr}_F(M)$ the associated graded A-module

$$\bigoplus_{i=1}^{s} M_i/M_{i+1}.$$

Moreover, for a one-parameter subgroup $\chi : K^* \to G(d)$ and modules $M, N \in \mathrm{mod}_A(d)$, we write $\lim_{t\to 0} \chi(t) * M = N$ if the regular map $\varphi : K^* \to \mathrm{mod}_A(d)$

which assigns to each $t \in K^* = K \setminus \{0\}$ the point $\chi(t) * M \in \text{mod}_A(d)$ has an extension to a regular map $\overline{\varphi} : K \to \text{mod}_A(d)$ and $N = \overline{\varphi}(0)$. We have then the following classical fact (see [18]).

Theorem 4.2. *Let $M, N \in \text{mod}_A(d)$. Then M has a filtration $F : M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_s \supseteq M_{s+1} = 0$ with $\text{gr}_F(M) \simeq N$ if and only if there exists a one-parameter subgroup $\chi : K^* \to G(d)$ such that $\lim_{t\to 0} \chi(t) * M = N$.*

As a direct consequence of Lemma 4.1 and Theorem 4.2 we get the following

Corollary 4.3. *Let $F : M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_s \supseteq M_{s+1} = 0$ be a filtration of a module $M \in \text{mod}_A(d)$. Then $M \leq_{\text{deg}} \text{gr}_F(M)$.*

In particular, if $0 \to X \to M \to Y \to 0$ is an exact sequence of A-modules, then we conclude that $M \leq_{\text{deg}} X \oplus Y$.

The geometric structure of the modules over the truncated polynomial algebra $A = K[x]/(x^m)$ has been completely analyzed. In this case, $\text{mod}_A(d)$ consists of $d \times d$ matrices M with $M^m = 0$. In particular we know from the results by H. Kraft–C. Procesi [19] (characteristic 0) and S. Donkin [12] (positive characteristic) that the orbit closures are normal varieties. Moreover, we know the following fact (see [18]).

Theorem 4.4. *Let $A = K[x]/(x^m)$ and $M, N \in \text{mod}_A(d)$. Then $M \leq_{\text{deg}} N$ if and only if $\text{rk}(M^i) \geq \text{rk}(N^i)$ for all i, $1 \leq i \leq m$.*

It is not clear how to characterize the partial order $\leq_{\text{deg}}$ on the isomorphism classes of d-dimensional modules in terms of the representation theory. There has been an important work by S. Abeasis and A. del Fra [1], K. Bongartz [8], [10] and C. Riedtmann [25] connecting $\leq_{\text{deg}}$ with other partial orders $\leq_{\text{ext}}$, $\leq_{\text{virt}}$ and $\leq$ on the isomorphism classes of modules in $\text{mod}_A(d)$ which are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N :\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\text{mod}\, A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number s.
- $M \leq_{\text{virt}} N :\Leftrightarrow M \oplus X \leq_{\text{deg}} N \oplus X$ for some A-module X.
- $M \leq N :\Leftrightarrow [X, M] \leq [X, N]$ holds for all modules X in $\text{mod}\, A$.

Here and later on we abbreviate $\dim_K \text{Hom}_A(X, Y)$ by $[X, Y]$. The fact that $\leq$ is a partial order on the isomorphism classes of modules follows from a result of M. Auslander [4]. We have also the following related facts.

Lemma 4.5. *Let $M, N \in \text{mod}_A(d)$. If $M \leq N$ then $\underline{\dim}\, M = \underline{\dim}\, N$.*

Proof. We know that $\underline{\dim}\, M = ([P_i, M])_{1 \leq i \leq n}$ and $\underline{\dim}\, N = ([P_i, N])_{1 \leq i \leq n}$, where $P_1, \ldots, P_n$ is a complete set of pairwise nonisomorphic indecomposable projective A-modules. Then $M \leq N$ implies $\underline{\dim}\, M \leq \underline{\dim}\, N$. But since $\dim_K M = d = \dim_K N$, we get $\underline{\dim}\, M = \underline{\dim}\, N$.

Lemma 4.6. *Let $M, N \in \operatorname{mod}_A(d)$. Then $M \leq N$ if and only if $[M, X] \leq [N, X]$ holds for all modules X in* $\operatorname{mod} A$.

Proof. First observe that if $[M, X] \leq [N, X]$ holds for all modules X in $\operatorname{mod} A$ then

$$\underline{\dim}\, M = ([M, I_i])_{1 \leq i \leq n} \leq ([N, I_i])_{1 \leq i \leq n} = \underline{\dim}\, N$$

where I_i, $1 \leq i \leq n$, is a complete set of pairwise nonisomorphic indecomposable injective A-modules, and consequently $\underline{\dim}\, M = \underline{\dim}\, N$. Hence, we may assume that $\underline{\dim}\, M = \underline{\dim}\, N$. But then, for X an indecomposable nonprojective A-module and Y an indecomposable injective A-module Y, we have the following Auslander–Reiten formulas [5]:

$$\begin{aligned} [X, M] - [M, D\mathrm{Tr}\, X] &= [X, N] - [N, D\mathrm{Tr}\, X], \\ [M, Y] - [\mathrm{Tr}\, DY, M] &= [N, Y] - [\mathrm{Tr}\, DY, N]. \end{aligned}$$

Then the required equivalence follows.

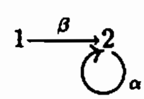
The following result (see [10], [25]) is remarkable for our investigations.

Theorem 4.7. *For $M, N \in \operatorname{mod}_A(d)$ the following implications hold:*

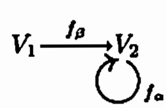
$$M \leq_{\mathrm{ext}} N \Rightarrow M \leq_{\mathrm{deg}} N \Rightarrow M \leq_{\mathrm{virt}} N \Rightarrow M \leq N.$$

Unfortunately, the reverse implications are not true in general, and it is interesting to find out when they are. This is the case for all modules over the truncated polynomial algebra $K[X]/(x^m)$. The following example shows that $\leq_{\mathrm{ext}}$ and $\leq_{\mathrm{deg}}$ do not coincide even for modules over very simple representation-finite algebras.

Example 4.8. Let Q be the quiver

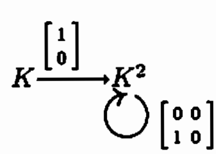


KQ the path algebra of Q and $A = KQ/(\alpha^2)$. Then A is a 5-dimensional algebra and $\operatorname{mod} A$ is equivalent to the category of finite-dimensional K-representations



of Q satisfying the condition $f_\alpha^2 = 0$. It is not difficult to show that $\operatorname{ind} A$ has only 7 pairwise nonisomorphic objects, and hence A is representation-finite. Consider the A-modules

M :



$$N: \quad K \xrightarrow{\begin{bmatrix}0\\1\end{bmatrix}} K^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}$$

Then M and N are nonisomorphic indecomposable A-modules of dimension 3. Let $\mu : K \to \text{mod}_A(3)$ be the regular map given by

$$\mu(\lambda): \quad K \xrightarrow{\begin{bmatrix}\lambda\\1\end{bmatrix}} K^2 \circlearrowleft \begin{bmatrix}0&0\\1&0\end{bmatrix}$$

for all $\lambda \in K$. It is easy to check that $\mu(\lambda) \simeq M$ for all $\lambda \in K^* = K \setminus \{0\}$. Clearly, $\mu(0) = N$. Applying now Lemma 4.1 we get $M <_{\text{deg}} N$. On the other hand, since N is indecomposable, we have $M \not\leq_{\text{ext}} N$.

Example 4.9. Let Q be the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2 \underset{\sigma}{\overset{\gamma}{\rightrightarrows}} 3$$

and A the bound quiver algebra $KQ/(\sigma\alpha, \gamma\beta, \gamma\alpha - \sigma\beta)$. Then A is a tame algebra with $\mu_A(d) \leq 2$ for all $d \geq 1$. Consider the following indecomposable A-modules

$$P : K \underset{\begin{bmatrix}0\\1\end{bmatrix}}{\overset{\begin{bmatrix}1\\0\end{bmatrix}}{\rightrightarrows}} K^2 \underset{[0\ 1]}{\overset{[1\ 0]}{\rightrightarrows}} K$$

$$U_\lambda : 0 \rightrightarrows K \underset{\lambda}{\overset{1}{\rightrightarrows}} K$$

$$V_\lambda : K \underset{1}{\overset{-\lambda}{\rightrightarrows}} K \rightrightarrows 0$$

for $\lambda \in K$. Observe that P is an indecomposable projective module P_1, and U_λ, V_λ are homogeneous modules over the corresponding Kronecker algebras. Then one can prove that for $\lambda, \mu \in K$ the following facts hold:

$$P \leq_{\text{deg}} U_\lambda \oplus V_\mu \Leftrightarrow \lambda = \mu$$

$$P \leq_{\text{virt}} U_\lambda \oplus V_\mu \qquad \text{for all } \lambda, \mu.$$

Hence, for $\lambda = 0$, $\mu = 1$ we have $P \leq_{\text{virt}} U_0 \oplus V_1$ but $P \not\leq_{\text{deg}} U_0 \oplus V_1$.

The property $M \leq_{\text{virt}} N$ is called in [25] a virtual degeneration. Observe that if $M \leq_{\text{virt}} N \Leftrightarrow M \leq_{\text{deg}} N$ then we have a cancelation property of degenerations. The above example shows that in general it is not the case. But we have the following important result due to C. Riedtmann [25].

Proposition 4.10. *Let M, N, Z be modules in* mod A. *Then*

(i) *If there is an exact sequence* $0 \to Z \to Z \oplus M \to N \to 0$ *then* $M \leq_{\deg} N$.

(ii) *If there is an exact sequence* $0 \to N \to M \oplus Z \to Z \to 0$ *then* $M \leq_{\deg} N$.

In [25] C. Riedtmann proved also the following theorem.

Theorem 4.11. *Let A be a representation-finite algebra, $d \geq 1$, and $M, N \in \operatorname{mod}_A(d)$. Then $M \leq N \Rightarrow M \leq_{\mathrm{virt}} N$.*

It is still an open question whether, for a representation-finite algebra A, the partiall orders $\leq_{\deg}$ and $\leq$ (equivalently, $\leq_{\deg}$ and $\leq_{\mathrm{virt}}$) coincide. From Example 4.9 we know that it is not the case for representation-infinite (tame) algebras. But we have the following positive result proved by K. Bongartz [8], [10].

Theorem 4.12. *Let A be the path algebra KQ of a Dynkin quiver Q (of type $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$). Then the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\deg}$, $\leq_{\mathrm{virt}}$, $\leq$ coincide in each variety* $\operatorname{mod}_A(d)$.

In [9] K. Bongartz proved also the following fact.

Theorem 4.13. *Let A be the path algebra KQ of an extended Dynkin quiver Q (of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$). Then the partial orders $\leq_{\deg}$, $\leq_{\mathrm{virt}}$, $\leq$ coincide in each variety* $\operatorname{mod}_A(d)$.

Recently (after Symposium at Kashikojima) my student G. Zwara has proved (see [39]) the following fact which completes the above theorem.

Theorem 4.14. *Let A be the path algebra KQ of an extended Dynkin quiver Q. Then the partial orders $\leq_{\mathrm{ext}}$ and $\leq_{\deg}$ coincide in each variety* $\operatorname{mod}_A(d)$.

An algebra A is called biserial if the radical of any indecomposable nonuniserial projective, left or right, A-module is a sum of two uniserial submodules whose intersection is simple or zero. Note that the algebras condsidered in Examples 4.8 and 4.9 are biserial. We have the following result proved by G. Zwara in [38].

Theorem 4.15. *Let A be a representation-finite biserial algebra. Then the partial orders $\leq_{\deg}$ and $\leq$ coincide in each variety* $\operatorname{mod}_A(d)$.

We refer also to [38] for a criterion when the partial orders $\leq_{\mathrm{ext}}$ and $\leq_{\deg}$ coincide for all modules over representation-finite biserial algebras. In particular, we have from [38] the following interesting fact.

Theorem 4.16. *Let A be a block of the group algebra $K[G]$ of a finite group G. If A is representation-finite, then the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\deg}$, $\leq_{\mathrm{virt}}$, $\leq$ coincide in each variety* $\operatorname{mod}_A(d)$.

5. Degenerations in Auslander-Reiten components

In this section we are interested in the following problem. Let $\mathcal{C}$ be a (connected) component in the Auslander-Reiten quiver Γ_A of A. Denote by $\mathrm{add}(\mathcal{C})$ the additive category of $\mathcal{C}$, that is, the full subcategory of $\mathrm{mod}\, A$ formed by all modules which are isomorphic to finite direct sums of modules from $\mathcal{C}$. We may ask when $M \leq_{\mathrm{deg}} N$ for two modules M and N in $\mathrm{add}(\mathcal{C})$ of the same dimension. By Lemma 4.5 we may assume that $\underline{\dim}\, M = \underline{\dim}\, N$. In order to discuss the above question it is convenient to consider an additional partial order. For $M, N \in \mathrm{add}(\mathcal{C})$ with $\underline{\dim}\, M = \underline{\dim}\, N$ we set

$$M \leq_{\mathcal{C}} N :\Leftrightarrow [X, M] \leq [X, N] \qquad \text{for all (indecomposable) modules } X \text{ in } \mathcal{C}.$$

One can prove that $M \leq_{\mathcal{C}} N$ if and only if $[M, X] \leq [N, X]$ for all modules X in $\mathcal{C}$. Moreover, $\leq_{\mathcal{C}}$ is a partial order on the set of isomorphism classes of modules in $\mathrm{add}(\mathcal{C})$ having the same dimension-vector (see [34]). We note also that by Lemma 3.2 and 4.5 the study of degenerations of modules in $\mathrm{mod}_A(d)$ is equivalent to that in the corresponding subvarieties $\mathrm{mod}_A(\underline{d})$, where $\underline{d}$ ranges all dimension-vectors $\underline{d} = (d_1, \dots, d_n)$ with $d_1 + \dots + d_n = d$. Observe also that

$$M \leq N \Rightarrow M \leq_{\mathcal{C}} N \qquad \text{for } M, N \in \mathrm{add}(\mathcal{C}) \cap \mathrm{mod}_A(\underline{d}).$$

For the preprojective components and preinjective components we have the following theorem proved by K. Bongartz [8].

Theorem 5.1. (i) *Let $\mathcal{P}$ be a preprojective component of Γ_A. Then the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$, $\leq_{\mathrm{virt}}$, $\leq$, $\leq_{\mathcal{P}}$ coincide for all modules in* $\mathrm{add}(\mathcal{P})$.

(ii) *Let $\mathcal{Q}$ be a preinjective component of Γ_A. Then the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$, $\leq_{\mathrm{virt}}$, $\leq$, $\leq_{\mathcal{Q}}$ coincide for all modules in* $\mathrm{add}(\mathcal{Q})$.

Following [26] an algebra A is said to be representation-directed if Γ_A is finite and directed. Clearly, then A is representation-finite and Γ_A is both a preprojective and preinjective component. We then have the following consequence of the above theorem, which generalizes Theorem 4.13.

Theorem 5.2. *Let A be a representation-directed algebra. Then the partial orders $\leq_{\mathrm{ext}}$, $\leq_{\mathrm{deg}}$, $\leq_{\mathrm{virt}}$ and $\leq$ coincide for all A-modules.*

An important future of the preprojective (respectively, preinjective) components is that they consist of indecomposable modules not lying on oriented cycles of nonzero nonisomorphisms between indecomposable modules (directing modules [26]), and hence such modules are uniquely determined (up to isomorphism) by their dimension-vectors. On the other hand, by an independent result due to L. Peng–J. Xiao [24] and the author [29], the Auslander-Reiten quiver Γ_A of any algebra A

has at most finitely many $D\mathrm{Tr}$-orbits containing directing modules. Hence, a real problem is to study the degenerations of modules having nondirecting indecomposable direct summands. Examples of components containing many cycles are tubes (see Section 1). In recent investigations of tame simply connected algebras appeared a natural generalization of the notion of tube called coil, introduced by I. Assem and the author in [2], [3]. Roughly speaking a coil is a translation quiver whose underlying topological space, modulo projective-injective vertices, is homeomorphic to a crawned cylinder. Special types of coils are quasi-tubes [27] whose underlying topological space, modulo projective-injective vertices, is homeomorphic to a tube. It is shown that coils can be obtained from stable tubes by a sequence of admissible operations. Moreover, it was shown by the author in [32] (compare Theorem 2.5) that a strongly simply connected algebra A is of polynomial growth if and only if every nondirecting indecomposable A-module lies in a coil of a standard multicoil of Γ_A. We note also that quasi-tubes frequently appear in the Auslander-Reiten quivers of selfinjective algebras. For such components we have the following fact proved by the author and G. Zwara in [34].

Theorem 5.3. *Let $\mathcal{C}$ be a generalized standard quasi-tube of an Auslander-Reiten quiver Γ_A. Then the partial orders $\leq_{\text{ext}}$, $\leq_{\text{deg}}$, $\leq_{\text{virt}}$, $\leq$ and $\leq_{\mathcal{C}}$ coincide for all modules in* $\operatorname{add}(\mathcal{C})$.

If $\mathcal{C}$ is a generalized standard coil of Γ_A which is not a quasi-tube then there exist in $\mathcal{C}$ (indecomposable) modules M and N such that $\underline{\dim} M = \underline{\dim} N$ and $M <_{\text{deg}} N$, and clearly $M \not\leq_{\text{ext}} N$. But we have still the following fact proved by G. Zwara [37].

Theorem 5.4. *Let $\mathcal{C}$ be a generalized standard coil of an Auslander-Reiten quiver Γ_A. Then the partial orders $\leq_{\text{deg}}$, $\leq_{\text{virt}}$, $\leq$, $\leq_{\mathcal{C}}$ coincide for all modules in* $\operatorname{add}(\mathcal{C})$.

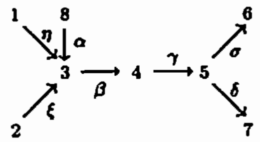
In general we have the following fact proved by G. Zwara in [37], which generalizes Theorem 4.11.

Theorem 5.5. *Let $\mathcal{C}$ be an arbitrary generalized standard component of an Auslander-Reiten quiver. Then the partial orders $\leq_{\text{virt}}$, $\leq$ and $\leq_{\mathcal{C}}$ coincide for all modules in* $\operatorname{add}(\mathcal{C})$.

6. Degenerations to indecomposable modules

In this section we are interested in the problem when there exists in $\operatorname{mod}_A(d)$ a proper degeneration $M <_{\text{deg}} N$ with N indecomposable. Observe that in such a case $M \not\leq_{\text{ext}} N$. Example 4.8 shows that there are proper degenerations to indecomposable modules over very simple representation-finite algebras. Consider an another example.

Example 6.1. Let $A = KQ/I$ where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by $\gamma\beta\alpha$. Observe that A is a one-point extension of the hereditary algebra $H = K\Delta$, where Δ is the extended Dynkin quiver of type $\widetilde{\mathbb{D}}_6$ given by the vertices $1, 2, \dots, 7$, by an indecomposable regular H-modules of regular length 2. Hence A is a tame but nonpolynomial growth algebra. It has been proved in [35] that for any positive integer $r \geq 2$ there exist exact sequences

$$0 \to N_t \to N_t \oplus M_{t+1} \to M_t \to 0$$

$1 \leq t \leq r-1$, in $\operatorname{mod} A$, where $N_1, \dots, N_{r-1}, M_1, \dots, M_r$ are pairwise nonisomorphic indecomposable A-modules (even lying in one component of Γ_A). Then, applying Proposition 4.10, we get a sequence of degenerations

$$M_r <_{\text{deg}} M_{r-1} <_{\text{deg}} \dots <_{\text{deg}} M_2 <_{\text{deg}} M_1$$

in $\operatorname{mod}_A(d)$, where $d = \dim_K M_1$. Therefore, we have arbitrary long sequences of proper degenerations of indecomposable A-modules.

In the remaining part of this article we shall present some results proved in the joint work with G. Zwara [35], which show a strong relationship between the considered problem and the representation type of an algebra.

Applying the above example, Theorem 2.1 and Proposition 4.10, one may prove the following fact.

Theorem 6.2. *Let A be an algebra. Assume there is an integer m such that for any sequence*

$$M_r <_{\text{deg}} M_{r-1} <_{\text{deg}} \dots <_{\text{deg}} M_2 <_{\text{deg}} M_1$$

with $M_1, \dots, M_r \in \operatorname{ind} A$, the inequality $r \leq m$ holds. Then A is tame.

As a direct consequence we get the following

Corollary 6.3. *Let A be an algebra. Assume that, for any proper degeneration $M <_{\text{deg}} N$ of A-modules, the module N is decomposable. Then A is tame.*

The following theorem shows that we may characterize the tame quasitilted algebras completely in terms of degenerations of modules.

Theorem 6.4. *Let A be a quasitilted algebra. Then A is tame if and only if for any proper degeneration $M <_{\deg} N$ of A-modules the module N is decomposable.*

Proof. One implication follows from the above corollary. Assume A is tame. Let $M <_{\deg} N$ be a proper degeneration in a variety $\text{mod}_A(d)$. Suppose N is indecomposable. Let C be a component of Γ_A containing N. Further, take an indecomposable direct summand X of M. Since $M <_{\deg} N$ implies $M < N$, we get $0 \neq [X,M] \leq [X,N]$ and $0 \neq [M,X] \leq [N,X]$. Hence $[X,N] \neq 0 \neq [N,X]$. Moreover, X is not isomorphic to N because $\dim_K M = d = \dim_K N$, X is a direct summand of M and $M <_{\deg} N$. Now it follows from Theorem 2.4 that the component quiver Σ_A of A is directed, and moreover any component of Γ_A containing an oriented cycle is a ray or coray tube. Therefore, we deduce $X \in C$, and hence C is a tube. Clearly, then $M \in \text{add}(C)$. Since C is a generalized standard tube (hence a quasi-tube) we infer by Theorem 5.3 that $M <_{\deg} N$ implies $M \leq_{\text{ext}} N$. But then N is decomposable, a contradiction.

We shall note that for arbitrary tame quasitilted algebras the partial orders $\leq_{\text{ext}}$ and $\leq_{\deg}$ do not coincide (see Example 4.9).

Applying Theorem 2.5 we proved in [35] the following characterization of polynomial growth strongly simply connected algebras.

Theorem 6.5. *Let A be a strongly simply connected algebra. The following conditions are eqiuvalent:*

(i) *A is of polynomial growth.*
(ii) *For A-modules M, M', N such that $M <_{\deg} N$, $M' <_{\deg} N$ and N is indecomposable, $M \simeq M'$ and is indecomposable.*
(iii) *There exists an integer m such that for any sequence*

$$M_r <_{\deg} M_{r-1} <_{\deg} \cdots <_{\deg} M_2 <_{\deg} M_1$$

with $M_1, \ldots, M_r$ indecomposable A-modules, the inequality $r \leq m$ holds.

Observe that the condition (ii) means that for any indecomposable A-module N over a polynomial growth strongly simply connected algebra A we have at most one proper degeneration $M <_{\deg} N$, and, in such a case, the module M is also indecomposable. In fact we proved in [35] that all indecomposable degenerations are given by the Riedtmann's Proposition 4.10.

Theorem 6.6. *Let A be a strongly simply connected algebra of polynomial growth, and M, N be two indecomposable A-modules. The following conditions are equivalent:*

(i) $M <_{\deg} N$.

(ii) *There exists a nonsplittable short exact sequence of A-modules*

$$0 \to Z \to Z \oplus M \to N \to 0$$

with Z indecomposable.

(iii) *There exists a nonsplittable short exact sequence of A-modules*

$$0 \to N \to M \oplus Z \to Z \to 0$$

with Z indecomposable.

We end the article with the following consequence of Theorems 2.5, 6.5 and 6.6.

Corollary 6.7. *Let A be an algebra whose ordinary quiver is a tree. Then A is of polynomial growth if and only if for any proper degeneration $M <_{\deg} N$ of A-modules, the module N is decomposable.*

Finally, Example 6.1 shows that there are tame algebras with ordinary quiver a tree having many proper degenerations of indecomposable modules.

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ON GENERALIZED DIMENSION SUBGROUUPS

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Abstract

This is a summary of the general Fox problem and the generalized dimension subgroup problem centering around our joint paper [10]. Let G be any group, and H be a normal subgroup of G. Then Hartl identified the subgroup $G \cap (1 + \Delta^3(G) + \Delta(G)\Delta(H))$ of G. In [10], we give an independent proof of the result of Hartl, and we identify two subgroups $G \cap (1 + \Delta(H)\Delta(G)\Delta(H) + \Delta([H,G])\Delta(H))$, $G \cap (1 + \Delta^2(G)\Delta(H) + \Delta(K)\Delta(H))$ of G for some subgroup K of G containing $[N,G]$.

Gを群とし, ZGを整数環Z上の群Gの群環とし$\Delta(G) = < g - 1 \mid 1 \neq g \in G >_Z$を$ZG$の添加イデアルとする. このとき, $(\Delta(G))^n = \Delta^n(G)$とおき

$$D_n(G) = G \cap (1 + \Delta^n(G)) = \{ g \in G \mid g - 1 \in \Delta^n(G)\}, n \geq 1$$

をGの第n次元部分群という. このとき, 明らかに, $D_n(G) \supseteq \gamma_n(G)$が成り立つ. 次元部分群問題とは「$D_n(G)$の構造を決定すること」である。この問題に関しては, 各$n \geq 4$に対して$exp(D_n(G)/\gamma_n(G)) = 2$となる群$G$が存在することが証明されており、また$exp(D_n(G)/\gamma_n(G))$は1または2であることが予想されることから, 完全な形で解決が得られそうである.

ここでは, この次元部分群問題を一般化することを考える.

1. 一般 Fox 問題

一般 Fox 問題とは, $G \trianglerighteq H$について$G \cap (1 + \Delta^n(G)\Delta(H))$の構造を決定することである. この問題に対して, 現在までに得られている結果は次の通りである.

1) Bergman–Dicks (1975): 次が成立する.

$$G \cap (1 + \Delta(G)\Delta(H) = \gamma_2(H) = H'.$$

2) K. Khambadkone (1985)[5], Vermani–Razdan–Karan (1985)[11]: Hが少し特殊な場合に, $G \cap (1 + \Delta^2(G)\Delta(H))$, $G \cap (1 + \Delta^3(G)\Delta(H))$を求めた.

3) Narain Gupta (1987)[1]: Fを自由群, RをFの正規部分群とするとき

$$F \cap (1 + \Delta^n(F)\Delta(R)) = \sqrt{G(n,R)},\ n \geq 1$$

である. ただし, $G(n,R) = < \prod_{t(m)}[R_{t_1}, R_{t_2}, \cdots, R_{t_m}] \mid 2 \leq m \leq n-1 >$とし, $R_t = R \cap \gamma_t(F)$, $t(m) = (t_1, t_2, \cdots, t_m)$, $t_i > 0$, $t_1 + t_2 + \cdots + t_i^\vee + \cdots + t_m \geq n \, (1 \leq i \leq m)$とする.

This paper is in final form and no version of it will be submitted for publication elsewhere

4) Curzio–C.K.Gupta (1995) [1]:　次が成立する.

$$G \cap (1 + \Delta^2(G)\Delta(H)) = K_G(H)\gamma_3(H)[H \cap G', H \cap G'].$$

ただし, $K_G(H)$ は, Hのある部分群である.

5) Tahara–Vermani–Razdan [10]: H が少し特殊な場合に, $G \cap (1 + \Delta^2(G)\Delta(H))$ を見やすい形にした.

2. 一般化された次元部分群問題

一般化された次元部分群問題 とは $G \trianglerighteq H$, $n \geq 3$ に対して　$G \cap (1 + \Delta^n(G) + \Delta(G)\Delta(H))$ の構造を決定することである. この問題について Karan–Vermani (1988) は $G = H \cdot K$, $H \cap K \subseteq \zeta(G)$ (G の中心) とするとき $G \cap (1 + \Delta^3(G) + \Delta(G)\Delta(H))$ の構造を求めた [4].

ここでは $n = 3$ のとき, 群 G の一般の正規部分群 H に対して, 一般化された次元部分群の構造を求める. そのために まず自由群 F とその正規部分群 R について定理 1 を証明する.

定理 A (Hartl, Tahara-Vermani-Razdan). G を群, H を任意の正規部分群とするとき

$$\begin{aligned} & G \cap (1 + \Delta^3(G) + \Delta(G)\Delta(H)) \\ = \;& \gamma_3(G) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in HG' \ (x, y \in G) > \end{aligned}$$

である.

定理 1. Fを自由群, Rを Fの正規部分群とするとき

$$\begin{aligned} & F \cap (1 + \Delta^3(F) + \Delta(F)\Delta(R)) \\ = \;& \gamma_3(F) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in RF' \ (x, y \in F) > \end{aligned}$$

である.

証明 ある $m \geq 1$ に対して $x^m, y^m \in RF'$ $(x, y \in F)$ となる元 $[x^m, y]$ 全体を U とおく. また自由加群 F の自由基底を $\{x_1, x_2, \cdots, x_r\}$ とし,

$$R = < x_1^{e_1}\xi_1, x_2^{e_2}\xi_2, \cdots, x_r^{e_r}\xi_r, \xi_{r+1}, \cdots >^F$$

とする. ここで, $e_1 | e_2 | \cdots, e_r > 0, \xi_k \in F'$ とできる.

まず, $\gamma_3(F)U \subseteq F \cap (1 + \Delta^3(F) + \Delta(F)\Delta(R))$ を示す. U の任意の生成元 $[x^m, y]$ について

$$\begin{aligned} [x^m, y] - 1 &\equiv \{(x^m - 1)(y - 1) - (y - 1)(x^m - 1)\} \pmod{\Delta^3(F)} \\ &\equiv \{(x - 1)(y^m - 1) - (y - 1)(x^m - 1)\} \pmod{\Delta^3(F)} \end{aligned}$$

が成り立つ. ここで, $(x-1)(y^m-1) - (y-1)(x^m-1) \in \Delta(F)\Delta(RF') \subseteq \Delta(F)\Delta(R) + \Delta^3(F)$ であるから

$$[x^m, y] \in F \cap (1 + \Delta^3(F) + \Delta(F)\Delta(R))$$

が成立するので，$U \subseteq F \cap (1+\Delta^3(F)+\Delta(F)\Delta(R))$ である．したがって $\gamma_3(F)U \subseteq F \cap (1+\Delta^3(F)+\Delta(F)\Delta(R))$ を得る．

つぎに，$F \cap (1+\Delta^3(F)+\Delta(F)\Delta(R)) \subseteq \gamma_3(F)U$ を示す．$F \cap (1+\Delta^3(F)+\Delta(F)\Delta(R))$ の任意の元を w とすると

$$w \in F \cap (1+\Delta^3(F)+\Delta(F)\Delta(R)) \subseteq [F,R]\gamma_3(F)$$

である．ここで

$$[F,R] =< [x_i^{e_i}\xi_i, x_j], [\xi_q, x_k] \mid 1 \le i,j,k \le r, r+1 \le q >^F$$

であるから次の式を得る．

$$\begin{aligned} w &\equiv \prod_{i<j}[x_i, x_j^{e_j}]^{a_{ij}} \prod_{q<k}[x_k, x_q^{e_q}]^{b_{kq}} \ (mod\ \gamma_3(F)), (a_{ij}, b_{ij} \in \mathbf{Z}) \\ &\equiv \prod_{k=1}^{r-1}\prod_{k<i}[x_k^{e_k}, x_i]^{a_{ki}(e_i/e_k)-b_{ik}} \ (mod\ \gamma_3(F)) \\ &\equiv \prod_{k=1}^{r-1}[x_k^{e_k}, \prod_{k<i} x_i^{a_{ki}(e_i/e_k)-b_{ik}}] \ (mod\ \gamma_3(F)). \end{aligned}$$

ここで $s_k(w) = \prod_{k<i} x_i^{a_{ki}(e_i/e_k)-b_{ik}}$ とおけば $w \equiv \prod_{k=1}^{r-1}[x_k^{e_k}, s_k(w)]\ (mod\ \gamma_3(F))$ となるので，$x_k^{e_k} \in RF'$ を考慮すれば

$$s_k(w)^{e_k} \in RF' \ (1 \le k \le r-1)$$

を示せばよいことがわかる．ところで

$$w \ \equiv \textstyle\prod_{k=1}^{r-1}[x_k^{e_k}, s_k(w)]\ (mod\ \gamma_3(F))$$

より

$$w-1 \ \equiv \textstyle\sum_{k=1}^{r-1}\{(x_k^{e_k}-1)(s_k(w)-1)-(s_k(w)-1)(x_k^{e_k}-1)\}\ (mod\ \Delta^3(F)$$

であり、さらに $(s_k(w)-1)(x_k^{e_k}-1) \in \Delta(F)\Delta(RF') \subseteq \Delta(F)\Delta(R)+\Delta^3(F)$ より

$$\begin{aligned} w-1 &\equiv \sum_{k=1}^{r-1}\{(x_k^{e_k}-1)(s_k(w)-1)\ (mod\ \Delta^3(F)+\Delta(F)\Delta(R)) \\ &\equiv \sum_{k=1}^{r-1}(x_k-1)(s_k(w)^{e_k}-1)\ (mod\ \Delta^3(F)+\Delta(F)\Delta(R)) \end{aligned}$$

を得る．したがって，$\sum_{k=1}^{r-1}(x_k1-1)(s_k(w)^{e_k}-1) \in \Delta^3(F)+\Delta(F)\Delta(R)$ である．このとき，$x_k-1\ (1 \le k \le r)$ は自由基底であるから

$$s_k(w)^{e_k}-1 \in \Delta^2(F)+ZF\Delta(R) = \Delta^2(F)+\Delta(R),\ 1 \le k \le r-1$$

となり，よって $s_k(w)^{e_k} \in F \cap (1+\Delta^2(F)+\Delta(R)) \subseteq RF'$ が得られる．

以上から, U の定義により $[x_k^{e_k}, s_k(w)] \in U$ となり, よって

$$w = \prod_{k=1}^{r-1} [x_k^{e_k}, s_k(w)] \in \gamma_3(F)U$$

が成立するので,

$$F \cap (1 + \Delta^3(F) + \Delta(F)\Delta(R)) \subseteq \gamma_3(F)U$$

が分る. かくして次を得る.

$$F \cap (1 + \Delta^3(F) + \Delta(F)\Delta(R)) = \gamma_3(F)U. \qquad \square$$

定理Aを利用することによって次の定理Bを得る.

定理 B. 群 G の正規部分群 H 対して

$$\begin{aligned} & G \cap (1 + \Delta(H)\Delta(G)\Delta(H) + \Delta([H,G])\Delta(H)) \\ = \ & \gamma_3(H) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in [H,G] \ (x, y \in H) > \end{aligned}$$

である.

$G \trianglerighteq H$ とし自然な完全列

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} G/H \longrightarrow 1$$

に対し, π の切断を $\rho : G/H \to G$ とすれば, G/Hの任意の元α, β について, $H \ni \omega(\alpha, \beta)$ が存在して $\rho(\alpha)\rho(\beta) = \rho(\alpha\beta)\omega(\alpha, \beta)$ が成り立つ. このとき $K =< [H,G], \omega(\alpha,\beta) \mid \alpha, \beta \in G/H >$ とおけば次が得られる.

定理 C. 群 G の正規部分群 H について

$$\begin{aligned} & G \cap (1 + \Delta^2(G)\Delta(H) + \Delta(K)\Delta(H)) \\ = \ & \gamma_3(H) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in K \ (x, y \in H) > \end{aligned}$$

である.

定理Cの系として、部分群 H が特別な場合に次の系D, Eを得る.

系 D. $G \trianglerighteq H, \omega(\alpha,\beta) \in H \cap G' \ (\alpha, \beta \in G/H)$ とすれば

$$\begin{aligned} & G \cap (1 + \Delta^2(G)\Delta(H)) \\ = \ & \gamma_3(H) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in H \cap G' \ (x, y \in H) > \end{aligned}$$

である.

系 E. $G = H \cdot K \subseteq H \cap G'$ とすれば

$$\begin{aligned} & G \cap (1 + \Delta^2(G)\Delta(H)) \\ = \ & \gamma_3(H) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in H \cap G' \ (x, y \in H) > \end{aligned}$$

である.

上の系に関して、自然な写像　$HG'/G' \to G/G' \to G/HG'$ が分解すれば系 E が成立する. このような分解する例として, 次のいずれかが成立すればよい.

1. G/Hが自由アーベルである,
2. HがGにおいて可除である,
3. HがGにおいて分解する.

従って, この 1, 2 または 3 の場合に 系 E が成立することが分る.

系 D, E の結果から, 一般の正規部分群について $n=2$ の場合にある予想が得られる. 最後にそれを紹介しよう.

予想 F 群 G, その正規部分群 Hについて次が成立する.

$$\begin{aligned} & G \cap (1+\Delta^2(G)\Delta(H)) \\ = \; & \gamma_3(H) < [x^m, y] \mid \text{ある } m \geq 1 \text{ が存在して } x^m, y^m \in H \cap G' \; (x, y \in H) > . \end{aligned}$$

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V-RING THEOREM RELATIVE TO HEREDITARY TORSION THEORIES

TSUDA KENTARO

ABSTRACT AND INTRODUCTION

V-rings and their generalizations have been studied by many authors. Recently C. Faith and P. Menal [2] gave a duality theorem for semisimple right R-modules, so called V-Ring Theorem, which characterizes V-rings. In this paper, we will investigate relative V-Ring Theorem to hereditary torsion theories, using known results. Throughout this paper, R denotes a ring with unit, every right R-module is unital, and $Mod - R$ denotes the category of right R-modules. For a right R-module M, $rad(M)$ and $soc(M)$ will denote the Jacobson radical of M and the socle of M, respectively. In particular J denotes $rad(R)$. $E(M)$ and $Z(M)$ will denote the injective hull of M and the singular submodule of M, respectively. G. Michler and O. Villamayor [6] showed that every simple right R-module is injective if and only if every right ideal of R is an intersection of maximal right ideals, or equivalently the Jacobson radical of any right R-module is zero. If a ring R satisfies these equivalent conditions, R is called a right V-ring. B. Johns showed that if a right Noetherian ring R in which every right ideal is a right annihilator ideal, 1then $r_R(J) = soc(M) = l_R(J)$, where $r_R(S)$ (resp. $l_R(S)$) denotes the right annihilator ideal (resp. left annihilator ideal) of a subset S of R in R, (see [4]). C. Faith and P. Menal called such a right Noetherian ring a *right Johns ring*, and showed that $r_R(soc(M)) = J = l_R(soc(M))$, (see [3]). C. Faith and P. Menal proved that a ring R is a right V-ring if and only if there exists some semisimple right R-module M satisfying the double annihilator condition with respect to right ideals, that is $I = r_R l_M(I)$ for any right ideal I of R, where $l_M(S) = \{m \in M \mid mS = 0\}$ and $r_R(S) = \{r \in R \mid Sr = 0\}$. This characterization of a right V-ring is called *V-Ring Theorem*. C. Faith and P. Menal showed that if a ring R is a right Johns ring, then $I/J = r_{R/J} l_{soc(R_R)}(I/J)$ for every right ideal I/J of R/J, i.e. R/J is a right V-ring by V-Ring Theorem. V-rings relative to hereditary torsion theories were introduced and studied by Y. Takehana in [9], and V-rings relative to stable torsion theories were studied by K. Varadarajan in [7]. We will establish V-Ring Theorem relative to hereditary torsion theories.

1. A NOTE ON $V(\mathfrak{F})$-RINGS

For fundamental definitions and results related to torsion theories, we refer to ([5]). A family $\mathfrak{F}$ of right ideals of R is called a *right Gabriel topology* if $\mathfrak{F}$ satisfies following

The final version of this paper will be submitted for publication elsewhere.

axioms:

(A1) If $I \in \mathfrak{F}$ and any $a \in R$, then $(I : a) \in \mathfrak{F}$, where $(I : a) = \{r \in R \mid ra \in I\}$;

(A2) If I is right ideal and there exists $J \in \mathfrak{F}$ such that $(I : a) \in \mathfrak{F}$ for every $a \in J$, then $I \in \mathfrak{F}$.

It is well known that a right Gabriel topology $\mathfrak{F}$ is a filter, i.e. satisfies:

(A3) If $J \in \mathfrak{F}$ and $J \subseteq I$, then $I \in \mathfrak{F}$;

(A4) If I and J in $\mathfrak{F}$, then $I \cap J \in \mathfrak{F}$.

Let $\mathfrak{F}$ be a nonempty right Gabriel topology. The hereditary torsion class of $Mod-R$ associated with $\mathfrak{F}$ is defined by setting $\mathcal{C}(\mathfrak{F}) = \{M \in Mod-R \mid r_R(x) \in \mathfrak{F}$ for all $x \in M\}$. And for a hereditary torsion class $\mathcal{C}$, the right Gabriel topology associated with $\mathcal{C}$ is defined by setting $\mathfrak{F}(\mathcal{C}) = \{I$ *right ideal* $\mid R/I \in \mathcal{C}\}$. It is well known that there is one to one corespondence between hereditary torsion theories (classes) and right Gabriel topologies (i.e. $\mathcal{C}(\mathfrak{F}(\mathcal{C})) = \mathcal{C}$ and $\mathfrak{F}(\mathcal{C}(\mathfrak{F})) = \mathfrak{F}$). Any right R-module in $\mathcal{C}(\mathfrak{F})$ is called a ***torsion right R-module***. A right R-module M is called ***$\mathfrak{F}$-injective***, if $Hom_R(-, M)$ preserves the exact sequence of the following form:

$$0 \to I \to R \to R/I \to 0 \quad with \quad I \in \mathfrak{F}.$$

A submodule L of M is called an ***$\mathfrak{F}$-submodule***, if $(L : x) \in \mathfrak{F}$, for all $x \in M$. We denote by $\mathfrak{F}(M)$ the family of all $\mathfrak{F}$-submodules of M. And we note that L is an $\mathfrak{F}$-submodule if and only if M/L is a torsion right R-module in $\mathcal{C}(\mathfrak{F})$. Let E be a right R-module. If there is a monomorphism $0 \to M \to E$ such that E is $\mathfrak{F}$-injective and $M \in \mathfrak{F}(E)$, then E is called an ***$\mathfrak{F}$-injective hull*** of M, which is denoted by $E_{\mathfrak{F}}(M)$. Furthermore $E_{\mathfrak{F}}(M) = \{x \in E(M) \mid (M : x) \in \mathfrak{F}\}$ is unique up to isomorphism.

We need the next lemma.

Lemma 1.1. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $\mathcal{C} = \mathcal{C}(\mathfrak{F})$ be a hereditary torsion class of $Mod-R$ associated with $\mathfrak{F}$. And let C be a right R-module. Then the following conditions are equivalent.*

(1) *C cogenerates each right R-module in $\mathcal{C}$.*

(2) *For each simple right R-module S in $\mathcal{C}$, C contains a copy of $E_{\mathfrak{F}}(S)$.*

Proof. (1)$\Rightarrow$ (2). Let S be a torsion simple right R-module. Since $E_{\mathfrak{F}}(S) \in \mathcal{C}$, Therefore there is a set Y such that $0 \to E_{\mathfrak{F}}(S) \xrightarrow{f} C^Y$. Let $p_i : C^Y \to C$ be the natural projection for each $i \in Y$. But since S is an essential right $\mathfrak{F}$-submodule of $E_{\mathfrak{F}}(S)$, there exists $i \in Y$ such that $ker(p_i f) = 0$. Therfore C contains a copy of $E_{\mathfrak{F}}(S)$.

(2)$\Rightarrow$(1). Let M be a torsion right R-module and let m be a nonzero element of M. Then we have $mR \cong R/r_R(m)$. Since $r_R(m)$ is a proper right idealof R, there exists a maximal right ideal L of R in $\mathfrak{F}$ such that $r_R(m) \subseteq L$. Hence if we set $S = R/L$, mR has the torsion simple homomorphic image S. Since $E_{\mathfrak{F}}(S)$ is $\mathfrak{F}$-injective and $mR \in \mathcal{F}(M)$, Proposition 6.2.in [S] implies that there exists an $f \in Hom_R(M, E_{\mathfrak{F}}(S))$ with $f(m) \neq 0$. But since $E_{\mathfrak{F}}(S) \subseteq C$ by (ii), C cogenerates M. □

Let $\{S_i\}_{i\in A}$ be a complete isomorphic set of simple right R-modules in $\mathcal{C} = \mathcal{C}(\mathfrak{F})$, $\sum_{i\in A} \oplus E_{\mathfrak{F}}(S_i)$ (resp. $\prod_{i\in A} E_{\mathfrak{F}}(S_i)$) denotes the direct sum (resp. direct product) of each $E_{\mathfrak{F}}(S_i)$ $(i \in A)$. By Lemma 1.1, we see that $\sum_{i\in A} \oplus E_{\mathfrak{F}}(S_i)$ and $\prod_{i\in A} E_{\mathfrak{F}}(S_i)$ cogenerates any torsion right R-module in $\mathcal{C} = \mathcal{C}(\mathfrak{F})$.

Proposition 1.2. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $C = C(\mathfrak{F})$ be a hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then the following are equivalent:*

(1) *Every torsion simple right R-module in C is $\mathfrak{F}$-injective.*
(2) *$rad(M) = 0$ for any torsion right R-module M in C.*
(3) *$rad(E_{\mathfrak{F}}(S)) = 0$ for any torsion simple right R-module S in C.*
(4) *For $M \in Mod - R$ and every submodule N of M with $M/N \in C$, N is an intersection of maximal submodules of M, equivalently, $rad(M/N) = 0$.*
(5) *Every right ideal I in $\mathfrak{F}$ is an intersection of maximal right ideals.*

Proof. (1)⇒(2) and (5)⇒(1) are similarly proved as in the proof of Theorem 2 in ([9]). (2)⇒(3). It is obvious.

(3)⇒(1). We show that $E_{\mathfrak{F}}(S) = S$ for each simple right R-module S in C. Suppose that $E_{\mathfrak{F}}(S)$ is not equal to S. We may assume that each L which is a maximal submodule of $E_{\mathfrak{F}}(S)$ is non-zero. But since S is an essential $\mathfrak{F}$-submodule of $E_{\mathfrak{F}}(S)$, we have $S \subseteq L$. Hence $S \subseteq rad(E_{\mathfrak{F}}(S))$. This is a contradiction. (1)⇒(4). Since $\prod_{i\in A} E_{\mathfrak{F}}(S_i)$ cogenerates any torsion right R-module in C and each $E_{\mathfrak{F}}(S_i)$ is simple, it is clear that $rad(M/N) = 0$. (4)⇒(5). If we set $M = R$ in (4), we have (5). □

Y. Takehana [9], a ring satisfing the equivalence conditions (1),(2), and (5) of the preceding proposition is called a $V(C(\mathfrak{F}))$-ring. A ring R in which any simple right R-module is either injective or projective is called a GV-ring, (see [8]). Let $\mathcal{T}$ be a hereditary torsion class in $Mod - R$. K. Varadarajan [7], a right $V(\mathcal{T})$-rings are studied for stable torsion classes $\mathcal{T}$ (i.e. $\mathcal{T}$ is closed under taking injective hull). As a special case of this setting, K. Varadarajan proved that if $\mathcal{T}$ is a Goldie torsion class (it is well known that Goldie torsion theory is a stable torsion theory.), the $V(\mathcal{T})$-ring is just a GV-ring. In this paper, we start with right Gabriel topologies coresponding to hereditary torsion theories. Therefore we call a right $V(C(\mathfrak{F}))$-ring a right $V(\mathfrak{F})$-ring.

2. V-Ring Theorem relative to hereditary torsion theories

We need the following notations. Let M, N be a right R-modules, and $S = Hom_R(M, N)$. For each subset X of M and each subset Z of S, we shall define the left annihilator of X in S , $l_S(X) = \{f \in S \mid f(x) = 0 \text{ for all } x \in X\}$ and the right annihilator of Z in M, $r_M(Z) = \{x \in M \mid f(x) = 0 \text{ for all } f \in Z\}$. We note that

$$l_S(X) = \{f \in S \mid X \subseteq ker(f)\}, \quad r_M(Z) = \cap_{f\in Z} ker(f)$$

The following proposition including its proof is a slight modification of Proposition 3.5 in ([1]).

Proposition 2.1. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $C = C(\mathfrak{F})$ be the hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then a ring R is a right $V(\mathfrak{F})$-ring if and only if there exists a semisimple right R-module E such that $N = r_M l_{M^*}(N)$ for $M \in Mod - R$ and every submodule N of M with $M/N \in C$, where $(-)^* = Hom(-, E)$.*

Proof. Sufficiency. It suffices to prove that for $M \in Mod - R$ and every submodule N of M with $M/N \in \mathcal{C}$, N is an intersection of maximal submodules of M. The assumption, $N = r_M l_{M^*}(N)$ implies that

$$f : M/N \longrightarrow E^{l_{M^*}(N)} \ , \ m + N \longmapsto (f(m))_{f \in l_{M^*}(N)}.$$

is a well defined monomorphism. Since the M/N embeds the direct product of copies of semisimple right R-module E, the proof of sufficiency is completes.

Necessity. Suppose that R is a right $V(\mathfrak{F})$-ring. We note that $S = \sum_{i \in A} \oplus E_{\mathfrak{F}}(S_i)$ is semisimple and it cogenerates any torsion right R-module in $\mathcal{C}$. We show that S satisfies the double annihilator condition for $M \in Mod - R$ and every submodule N of M with $M/N \in \mathcal{C}$. Hence there are some set Y and a monomorphism $M/N \longrightarrow N^Y$, and so there exists a family $(f_i)_{i \in Y}$ with $f_i \in M^*$ such that $N = \cap_{i \in Y} ker(f_i)$. Then

$$N \subseteq r_M l_{M^*}(N) = \cap_{f \in l_{M^*}(N)} ker(f) \subseteq \cap_{i \in Y} ker(f_i) = N.$$

So $N = r_M l_{M^*}(N)$. □

Corollary 2.2. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $\mathcal{C} = \mathcal{C}(\mathfrak{F})$ be the hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then a ring R is a right $V(\mathfrak{F})$-ring if and only if there exists a semisimple right R-module E such that $N = r_M l_{M^*}(N)$ for any torsion right R-module M in $\mathcal{C}$ and every submodule N of M.*

Corollary 2.3. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $\mathcal{C} = \mathcal{C}(\mathfrak{F})$ be the hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then a ring R is a right $V(\mathfrak{F})$-ring if and only if there exists a semisimple right R-module E such that $N = r_{E_{\mathfrak{F}}(S)} l_{E_{\mathfrak{F}}(S)^*}(N)$ for any torsion simple right R-module S in $\mathcal{C}$ and every submodule N of $E_{\mathfrak{F}}(S)$.*

In Proposition2.1, if we take $M = R$, we have $M^* = Hom_R(R, U) \cong U$. Therefore we obtain the following.

Corollary 2.4. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $\mathcal{C} = \mathcal{C}(\mathfrak{F})$ be the hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then a ring R is a right $V(\mathfrak{F})$-ring if and only if there exists a semisimple right R-module such that $I = r_R l_M(I)$ for every right ideal I in $\mathfrak{F}$.*

Remark 2.5. Let $\mathfrak{F}$ be a nonempty right Gabriel topology, and let $\mathcal{C} = \mathcal{C}(\mathfrak{F})$ be the hereditary tosion class of $Mod - R$ associated with $\mathfrak{F}$. If there exists a semisimple right R-module W satisfies the double annihilator condition $N = r_{E_{\mathfrak{F}}(S)} l_{E_{\mathfrak{F}}(S)^*}(N)$ for any torsion simple right R-module S in $\mathcal{C}$ and N is an every submodule of $E_{\mathfrak{F}}(S)$, where $E_{\mathfrak{F}}(S)^* = Hom_R(E_{\mathfrak{F}}(S), W)$. Then W cogenerates any torsion right R-module in $\mathcal{C}$.

Proof. Since R is a right $V(\mathfrak{F})$-ring by Corollary 2.3, every torsion simple right R-module S in $\mathcal{C}$ is $\mathfrak{F}$-injective. So it suffices show that W contains a copy of each S by Lemma 1.1. If $S^* = 0$, we have $S = r_S l_{S^*}(0)$. This is a contradiction of $0 = r_S l_{S^*}(0)$. Therefore since S is simple, W contains a copy of S. □

Then the following corollary which is almost actually due to C. Faith and P. Menal (Corllary [2]).

Remark 2.6. Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $C = C(\mathfrak{F})$ be the hereditary tosion class of $Mod - R$ associated with $\mathfrak{F}$. If there exists a semisimple right R-module W satisfying the double annihilator condition $I = r_R l_W(I)$ for every $I \in \mathfrak{F}$. Then W cogenerates any torsion right R-module in C.

Proof. Since R is a right $V(\mathfrak{F})$-ring by Corollary 2.4, every simple right R-module S in C is $\mathfrak{F}$-injective. So it sufficies to show that W contains a copy of each S by Lemma 1.1. But, since $S \cong R/M$ where M is a maximal right ideal in $\mathfrak{F}$, we have $M = r_R l_W(M)$. Since M is maximal, there exists $w \in W$ such that $M = ann_R(w)$. Hence $wR \cong V$. Thus we obtain a monomorphism $0 \to S \to W$. Clearly W cogenerates any torsion right R-module in C. □

For a right $V(\mathfrak{F})$-rings, by the preceding corollaries and remarks, we easily obtain a duality theorem for semisimple right R-modules.

Theorem 2.7. *Let $\mathfrak{F}$ be a nonempty right Gabriel topology and let $C = C(\mathfrak{F})$ be the hereditary torsion class of $Mod - R$ associated with $\mathfrak{F}$. Then a ring R is a right $V(\mathfrak{F})$-ring if and only if there exists a semisimple right R-module E such that E satisfies the following equivalent conditions, where $(-)^* = Hom_R(-, E)$*

(1) $N = r_{E_{\mathfrak{F}}(S)} l_{E_{\mathfrak{F}}(S)^*}(N)$, *for any torsion simple right R-module in C and every submodule N of $E_{\mathfrak{F}}(S)$.*
(2) $N = r_M l_{M^*}(N)$, *for any torsion right R-module M in C and every submodule N of M.*
(3) $I = r_R l_E(I)$, *for every right ideal I in $\mathfrak{F}$.*

We call this theorem, $V(\mathfrak{F})$-*Ring Theorem.*

Let $\mathcal{E}$ be the family of essential right ideals of R, and $\mathfrak{F}_G = \{I \mid$ there exists $J \in \mathcal{E}$ such that $I \subseteq J$ and $(I : a) \in \mathcal{E}$ for all $a \in J\}$. $\mathfrak{F}_G$ is called the Goldie topology. And $G = Z_2$ defined $Z_2(M)/Z(M) = Z(M/Z(M))$ for any right R-module M is called the Goldie exact radical coresponding to $\mathfrak{F}_G$. In this situation, it is well known that $C(\mathfrak{F}_G) = \{M \mid r_R(x) \in \mathfrak{F}_G$ for all $x \in M\} = \{M \mid G(M) = M\}$. Furthermore M is $\mathfrak{F}$-injective then M is injective, where $M \in C(\mathfrak{F}_G)$, and that $M \in \mathfrak{F}_G$ if and only if $Z(M)$ is an essential right R-module in M. We apply Proposition 1.2 to the family of right ideals $\mathfrak{F}_G$. Then a $V(\mathfrak{F}_G)$-ring is called a $V(G)$-ring. Recall the $V(\mathfrak{F}_G)$-ring is just GV-ring (see [7]).

Corollary 2.8. *Let $\mathfrak{F}_G$ be the Goldie topology. Then a ring R is a right GV-ring if and only if there exists a semisimple right R-module E such that E satisfies the following equivalent conditions.*

(1) $N = r_{E(S)} l_{E(S)^*}(N)$, *for any singular simple right R-module and every submodule N of $E(S)$.*
(2) $N = r_M l_{M^*}(N)$, *for every submodule N of M with $Z(M)$ essential submodule of M.*
(3) $I = r_R l_E(I)$, *for every right ideal I in $\mathfrak{F}_G$.*

We call this corollary, GV-Ring Theorem.

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FULLY PRIME RINGS AND RELATED RINGS

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Introduction

A ring with the property that every ideal is prime is called a fully prime ring and the structure of such rings and the structures of rings which are related to fully prime rings were studied in Blair-Tsutsui [2] and Tsutsui [10]. It is well known that a commutative fully prime ring is a field. Perhaps the most amusing and useful fact in investigating the structure of fully prime rings is the following necessary and sufficient condition for a ring to be fully prime: Every ideal is idempotent and the set of ideals is linearly ordered. It should also be noted at this point that a fully prime ring is, in general, not even semiprimitive.

Conditions similar to the fully prime condition have received attention in the literature. R.C. Courter [4] and several other authors have studied those rings in which every ideal is semiprime; K. Koh [8] studied those rings in which every right ideal is prime; and Y. Hirano [7] studied rings in which every ideal is completely prime.

My talk at the symposium was solely contrived to introduce some basic structure theory of fully prime rings, its goal being to find fellow ring theorists of similar interest. In this proceeding, I shall summarize the essential contents of the two papers noted above with a few additional examples and corollaries, and provide several open problems. Interested readers' success in solving those problems are sincerely hoped.

Throughout, a ring will mean an associative noncommutative ring. However, there are several occasions where we will find it useful to consider ideals to be subrings and, for this reason, we will not assume that our rings necessarily have a multiplicative identity element.

Definitions. A ring in which every ideal is prime will be called a *fully prime ring*. A ring all of whose nonzero proper ideals are prime will be called an *almost fully prime ring*. A ring with involution $(*)$, satisfying the additional condition that every $(*)$-ideal is $(*)$-prime will be called a *fully* $(*)$*-prime ring*. A ring in which every ideal is idempotent is called a *fully idempotent ring*.

The reader should be warned that some authors use the term fully idempotent to mean that every right ideal is idempotent.

Though this article is in its final form, it is essentially a summary of two papers, Blair-Tsutsui [2] and Tsutsui[10].

Theorem 1. (Theorem 1.2 of Blair-Tsutsui [2] and Theorem 3.1 of Tsutsui [10])
A ring is fully prime if and only if it is fully idempotent and the set of (two sided) ideals is linearly ordered. A ring with an involution ($*$) is fully ($*$)-prime if and only if every ($*$)-ideal is idempotent and the set of ($*$)-ideals is linearly ordered.

An almost fully prime ring fails to be fully prime either because the set of ideals is not linearly ordered or because not every ideal is idempotent. We consider each of these possibilities in the following two theorems. As a consequence of those theorems, we note that an almost fully prime ring which is not prime has minimal nonzero ideals.

Problem. Does a fully prime ring necessarily have a minimal nonzero ideal ?

Theorem 2. (Theorem 2.1 of Tsutsui [10]) Let R be a ring whose set of ideals is not linearly ordered. Then R is almost fully prime if and only if R is a fully idempotent ring with exactly two minimal ideals and the set of ideals is linearly ordered except the minimal ideals.

Theorem 3. (Theorem 2.2 of Tsutsui [10]) Let R be a ring which is not prime and whose set of ideals is linearly ordered. Then R is almost fully prime if and only if it has a minimal nonzero ideal and every ideal of R except the minimal one is idempotent.

Remarks.

1. (Theorem 1.6 of Blair-Tsutsui [2]) A ring with identity is a division ring if and only if it is fully right idempotent (every right ideal is idempotent) and the set of right ideals is linearly ordered. If the ring is right Noetherian, then the condition 'fully right idempotent' can be replaced by 'fully idempotent' since its Jacobson radical, being an idempotent ideal, must be zero by Nakayama's lemma.
2. (Hirano [7]) A ring is completely fully prime (every ideal is completely prime) if and only if $< a^2 >=< a >$ for every element a in the ring and the set of ideals is linearly ordered.
3. (Koh [8]) A ring R with identity (or with the property that every element a is in aR) is simple if and only if every right ideal of R is prime.
4. It is known that a prime right Goldie fully right idempotent ring is simple. On the other hand, there is an example of a fully prime Noetherian ring which is not simple. Thus every right ideal of a fully prime ring is not necessarily idempotent.
5. Every right ideal I of a fully idempotent ring (hence, in particular, a fully prime ring) with identity has the property that $I^2 = I^3$. For any idempotent right ideals I and J of a fully prime ring, either $IJ = I$ or $JI = J$, and every non-idempotent right ideal is contained in the maximal (two sided) ideal.

Problem. A ring in which every ideal is idempotent have received attention in the literature, as well as a ring all of whose right ideals are idempotent and a ring whose set of right ideals is linearly ordered. What can we say about a ring whose set of (two sided) ideals is linearly ordered ?

Theorem 4. (Theorem 1.3 of Blair-Tsutsui [2]) The center of a fully prime ring is either a field or zero. The center is a field if and only if the ring has an identity element.

We now list several examples and constructions of fully prime rings.

1. Let V be a right vector space over a division ring D. Then $\mathrm{End}_D V$ is a fully prime ring whose only nontrivial ideals are of the form $\{f \in \mathrm{End}_D V \mid \dim(f(V)) < C\}$, where C is any infinite cardinal number such that $C \leq \dim(V)$. Denote the cardinality of a denumerable set by $\aleph_0$, and for any integer $n \geq 1$, $\aleph_n$ will denote the smallest cardinal greater than $\aleph_{n-1}$. If $\dim_D V = \aleph_{n-1}$, then $\mathrm{End}_D V$ has *exactly n nonzero proper ideals.* This example can be extended to construct examples of fully prime rings whose sets of ideals have infinite cardinality. For example, if $\dim_D V = \aleph_{\omega_0}$, where ω_0 is the first limit ordinal, then $\mathrm{End}_D V$ is a fully prime ring with *countably many ideals.*

2. Any prime regular right self - injective ring with identity is a fully prime ring. In fact, an ideal P of a regular right self - injective ring R is prime if and only if the set of ideals of R/P is linearly ordered by Theorem 8.20 and Corollary 9.16 of Goodearl [6]. Thus for example, if R is a prime nonsingular ring with identity, then its maximal right quotient ring is a fully prime ring. For each positive integer n, let F_n be a field and $R_n = M_{n!}(F_n)$, the ring of $n!$ by $n!$ matrices over F_n, and set $T = \Pi R_n$. Then T is a regular self - injective ring with a prime ideal P such that the set of ideals of T/P is not well - ordered by Example 12.25 of Goodearl [6]. Hence T/P is a fully prime ring whose *set of ideals is not well-ordered.*

3. Let G be an algebraically closed or universal group and let K be a field. Then $K[G]$ is a primitive group ring whose only nonzero proper ideal is the augmentation ideal by Bonvallet, Hartley, Passman, and Smith [3], and hence it is fully prime.

4. (Example 3.2 of Blair-Tsutsui [2]) Let R be the set of infinite matrices over a field F that have the form

$$(A,a) = \begin{bmatrix} A & & & & \\ & a & & 0 & \\ & & a & & \\ & 0 & & a & \\ & & & & \ddots \end{bmatrix}$$

where A is an arbitrary finite matrix over F and a is any element of F. Then R is a fully prime ring which is *integral over its center but not simple*: The only nonzero proper ideal of R is the subset of all matrices of the form $(A,0)$ and it is idempotent. By the Cayley-Hamilton Theorem, each square matrix M over F satisfies a monic polynomial $f(x)$ with coefficients in F. Thus (M,a) satisfies the monic polynomial $g(x) = f(x)(f(x) - f(a))$.

5. (Theorem 4.1 and 4.4 of Blair-Tsutsui [2]) If P is a proper ideal of a fully prime ring R with identity, and $Z(R)$ is the center of R, then $S_P = P + Z(R)$ is a fully prime ring whose maximal ideal P is also a maximal right and left ideal.
   Further, proper ideals of S_P are precisely those ideals of R that are contained in P.
   If P is a nonzero ideal of R, then R is right primitive if and only if S_P is right primitive.

6. (Theorem 4.5 and 4.6 of Blair-Tsutsui [2]) Whenever a simple ring R with identity has a right ideal I whose left annihilator is zero, a non-simple fully prime ring with exactly one nonzero proper ideal can be constructed. In fact, let R be a domain with identity whose center is denoted by $Z(R)$. Then, the following conditions are equivalent:
   (1) R is a simple ring.
   (2) $S = I + Z(R)$ is a fully prime ring for every right ideal I of R.
   (3) $S = aR + Z(R)$ is a fully prime ring for every principal right ideal aR.

   In particular, if k is a field of characteristic 0 and $A_1(k)$ is the Weyl algebra, the k-algebra with two generators x, y and the relation $xy - yx=1$, then $xA_1(k) + k$ is a *right Noetherian fully prime ring which is not simple.*

Problem. A right Noetherian fully prime ring with identity is semiprimitive, but it is not necessarily a simple ring by the example above. Is a right Noetherian fully prime ring primitive ?

7. (Section 5 of Blair - Tsutsui [2]) Let R be a simple radical ring (A simple ring, necessarily without identity, such that $R = R^2 = J(R) \neq 0$, where $J(R)$ denotes the Jacobson radical of R.), and let F be the field of rational numbers if R has characteristic zero; choose the field of integer modulo p for F if R has a nonzero characteristic p.
   Set $S = R \oplus F$ where addition is defined componentwise and multiplication is given by $(r_1,k_1)(r_2,k_2) = (r_1r_2 + k_1r_2 + k_2r_1,\ k_1k_2)$, where $r_1,r_2 \in R,\ k_1,k_2 \in F$. Then S is a fully prime ring *with identity which is not semiprimitive,* whose only nonzero proper ideal is $R \oplus 0$.

Problem. Is a semiprimitive fully prime ring primitive?

8. (Theorem 2.5 of Blair-Tsutsui [2]) Every ideal of a fully prime ring is fully prime when it is considered as a ring. Every ideal of an ideal of a fully prime ring is an ideal of the ring.

9. (Theorem 2.1, 2.3, and Corollary 2.4 of Blair-Tsutsui [2]) If R is a fully prime ring, then n by n matrix ring over R is a fully prime ring. If e is an idempotent element in R, then eRe is also a fully prime ring. Thus fully prime is a *Morita invariant property* for rings with identity.

10. Any ring with at most one nonzero proper ideal is almost fully prime. For example, the ring Z_{p^2} of integer modulo p^2, where p is a prime number, has a unique nonzero proper ideal $I = \left\{ pn \middle| n \in Z_{p^2} \right\}$ and $I^2 = 0$. If F_1 and F_2 are fields, then $F_1 \oplus F_2$ has exactly two nonzero proper ideals $F_1 \oplus 0$ and $0 \oplus F_2$, both of which are prime. By Theorem 2.3 of Tsutsui [10], a commutative almost fully prime ring is either a field, isomorphic to a direct sum of two fields, or a ring with exactly one nonzero proper ideal whose square is zero.

11. (Example 2.4 of Tsutsui [10]) Almost fully prime rings can be constructed from a fully prime ring with identity having minimal (nonzero) ideal: Let R be a fully prime ring with nonzero minimal ideal P. For $p_1, p_2 \in P, r_1, r_2 \in R$, let S_1 be the additive abelian group $P \oplus R$ with multiplication defined by $(p_1, r_1) \cdot (p_2, r_2) = (p_1p_2 + p_1r_2 + r_1p_2,\ r_1r_2)$, and S_2 be the same additive abelian group with multiplication defined by $(p_1, r_1) \cdot (p_2, r_2) = (p_1r_2 + r_1p_2,\ r_1r_2)$. Further, let $Q_1 = \{(p,0) | p \in P\}$, and $Q_2 = \{(p,-p) | p \in P\}$. Then;
   (a) S_1 is an almost fully prime ring with two minimal ideals Q_1 and Q_2.
   (b) S_2 is an almost fully prime ring with unique minimal ideal Q_1.

As has been noted, a commutative fully prime ring with identity is a field, and a commutative fully idempotent ring is a von Neumann regular ring. It turns out that for several classes of rings which are natural generalizations of commutative rings, fully prime rings are simple Artinian, and fully idempotent rings are semisimple Artinian. Applying the structure theory of fully prime rings, the structure of almost fully prime rings can also be determined. Hereafter, we will assume that all rings contain a multiplicative identity element.

Theorem 5. (Theorem 3.1 of Blair-Tsutsui [2]) If R is a right Goldie fully prime ring which is integral over its center, then R is simple Artinian.

Corollary. If R is a right Goldie fully idempotent ring which is integral over its center, then R is semisimple Artinian.

Proof. Since a minor modification of the proof of Theorem 1.3 of Blair-Tsutsui [2] yields that the center of a prime fully idempotent ring is a field, a right Goldie prime fully idempotent ring which is integral over its center is simple Artinian. Thus, the result follows by the Chinese Remainder Theorem since R has a finite set of prime ideals whose intersection is zero.

A minor modification of the proof of Theorem 2.3 of Blair-Tsutsui [2] yields the following structure theorem of almost fully prime rings.

Corollary. Let R be a right Goldie almost fully prime ring which is integral over its center. Then either

1. R is a simple Artinian ring,
2. R is isomorphic to a direct sum of two simple Artinian rings, or
3. R is a ring with exactly one proper nonzero ideal whose square is zero.

The most natural generalization of commutative rings are rings which satisfy a polynomial identity. For those, the following results hold.

Theorem 6. (Theorem 3.3 of Blair-Tsutsui [2]) A fully prime ring which satisfies a polynomial identity is a finite dimensional central simple algebra.

A minor modification of the proof of Theorem 2.3 of Blair-Tsutsui [2] again yields the following structure theorem of almost fully prime rings.

Corollary. Let R be an almost fully prime ring which satisfies a polynomial identity. Then either

1. R is a finite dimensional central simple algebra,
2. R is isomorphic to a direct sum of two finite dimensional central simple algebras, or
3. R is a ring with exactly one proper nonzero ideal whose square is zero.

Theorem 7. (Armendariz and Fisher [1]) A fully idempotent ring which satisfies a polynomial identity is a von Neumann regular ring.

More general than the class of rings with a polynomial identity is the class of fully bounded rings. We used the following lemma to prove Theorem 8 below.

Lemma. (P.F. Smith [9]) Let R be a fully right bounded ring satisfying ascending chain condition on two sided ideals. Then an R-module M is injective if and only if for every prme ideal P, each right R - homomorphism $f: P \rightarrow M$ can be extended to $g: R \rightarrow M$.

Theorem 8. (Theorem 1.5 of Tsutsui [10]) A right FBN fully idempotent ring is semisimple Artinian.

Corollary. (Theorem 3.4 of Blair-Tsutsui [2]) A right FBN fully prime ring is simple Artinian.

Corollary. (Theorem 2.5 of Tsutsui [10]) Let R be a right FBN almost fully prime ring . Then either

1. R is a simple Artinian ring,
2. R is isomorphic to a direct sum of two simple Artinian rings, or
3. R is a ring with exactly one proper nonzero ideal whose square is zero.

Theorem 9. (Theorem 3.5 of Tsutsui [10]) A right FBN fully $(*)$-prime ring is $(*)$-simple Artinian.

Remark. If R is a prime right bounded ring, then every essential submodule of a finitely generated faithful right R-module is faithful. Using this fact, one can show that every right ideal of a fully prime fully right bounded ring is idempotent. As was mentioned, a prime right Goldie fully right idempotent ring is simple, and a right bounded right Goldie simple ring is Artinian. Hence, the right Noetherian condition of Corollaries to Theorem 8, as those can be proved without using the lemma, can be relaxed to the right Goldie condition.

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