PROCEEDINGS OF THE 23RD SYMPOSIUM ON RING THEORY

HELD AT CHIBA UNIVERSITY, CHIBA

SEPTEMBER 23—25, 1990

EDITED BY

SHIGEO KOSHITANI

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OKAYAMA JAPAN

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PREFACE

The year, 1990, it has really been so excited for mathematicians, especially in Japan. We mean, of course, ICM90 Kyoto! Just one month later, the 23rd Symposium on Ring Theory was held at Chiba University, Chiba, on 23–25 September, 1990. This volume is the Proceedings of the symposium. It consisted of twelve talks, so that the proceedings contains twelve articles by the speakers. We would like to thank all of the speakers of the symposium for their contribution.

The symposium and the proceedings were financially supported by the Grant-in-Aid for Scientific Research from the Ministry of Education, Science and Cultue through the arrangements by Professor Y. Kitaoka, Nagoya University (Grant-in-Aid for Co-operative Research (A) No. 02302002). We would like to express our great gratitude to Professor Kitaoka for his kind arrangements, and also to Ms. Hayashi, a secretary of the Department of Mathematics, Nagoya University, for her kind arrangements to the symposium.

Finally, we wish to thank to Professor H. Tominaga for the publication of the proceedings.

Chiba University, Chiba, November 1990

Shigeo Koshitani

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ON COMMUTATIVITY OF RINGS

Tsunekazu NISHINAKA

Recently, in his paper [10], W. Streb gave a classification of non-commutative rings. By making use of this result, H. Komatsu, H. Tominaga and the present author have obtained a number of commutativity theorems for rings, in [6], [7], [8] and [9]. In the present paper, we shall exhibit several theorems which are especially interesting among those obtained in [6], [7] and [8].

Throughout, R will represent a ring with center C = C(R) (not necessarily with 1). As usual, we write $[x,y] = xy-yx$ for x, y $\in$ R, and D = D(R) will denote the commutator ideal of R.

In preparation for proving our theorems, we begin with a preliminary section.

1. Streb's theorem and Chacron's condition. In [7, Proposition 2], we have proved the following:

Proposition 1. Let R be a ring generated by two elements such that D is the heart of R and RD = DR = 0. Then R is nilpotent.

In view of Proposition 1, we see that the theorem of Streb [8, Theorem S] should be stated as follows:

Theorem S (see [7]). Let R be a non-commutative ring ($R \neq C$). Then there exists a factorsubring of R which is of type $a)_\ell$, $a)_r$, b), c), d) or e):

a)$_\ell$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

a)$_r$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

b) $M_\sigma(K) = \{\begin{pmatrix} \alpha & \beta \\ 0 & \sigma(\alpha) \end{pmatrix} \mid \alpha, \beta \in K\}$, where K is a finite field with a non-trivial automorphism σ.

c) A non-commutative division ring.

d) A simple radical ring with no non-zero divisors of zero.

e) A finite nilpotent ring S such that D(S) is the heart of S and SD(S) = D(S)S = 0.

Further, from the proof of [10, Korollar (1)], we can easily see

Theorem S^1. Let R be a non-commutative ring with 1. Then there exists a factorsubring of R which is of type a)^1, b), c), d)^1 or e)^1:

a)^1 $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

b) $M_\sigma(K)$.

c) A non-commutative division ring.

d)^1 S = <1> + T is an integral domain, where T is a simpie radical ring.

e)^1 S = <1> + T, where T is a finite nilpotent ring such that D(T) is the heart of T and TD(T) = D(T)T = 0.

Now, Theorem S and Theorem S^1 give the following Meta Theorem which plays an important role in our subsequent study.

Meta Theorem. Let P be a ring-property which is inherited by factorsubrings. If no rings of type a)$_\ell$, a)$_r$, b), c), d) or e) (resp. a)^1, b), c), d)^1 or e)^1) satisfy P, then every ring (resp. every ring with 1) satisfying P is commutative.

We consider here the following conditions:

(H) For each $x \in R$, there exists $f(X) \in X^2Z[X]$ such that $x-f(x) \in C$.

(H') For each $x, y \in R$, there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $[x-f(x),y] = 0$.

(C) For each $x, y \in R$, there exist $f(X), g(X) \in X^2\mathbb{Z}[X]$ such that $[x-f(x),y-g(y)] = 0$.

(S) For each $x, y \in R$, there exists $f(X,Y) \in \mathbb{Z}\langle X,Y\rangle[X,Y]\mathbb{Z}\langle X,Y\rangle$ each of whose monomial terms is of length ≥ 3 such that $[x,y] = f(x,y)$.

The conditions (H) and (H') were introduced by Herstein, and the conditions (C) and (S) have been introduced by Chacron and Streb, respectively. By a well-known theorem of Herstein (signified as Theorem H), every ring satisfying (H) is commutative. Obviously no rings of type e) satisfy (S). This together with Meta Theorem enables us to reduce the proof of Theorem H to the case that R is a division ring. Further, by making use of Theorem H, we can show that no rings of type c) or d) satisfy (C) (see [4, Corollary 1]). Thus, in view of Meta Theorem, we readily obtain another theorem of Herstein [3, Theorem 3]: Every ring satisfying (H') is commutative. As was claimed in [4], [4, Corollary 1] reproves [2, Theorem 2]: Every semiprime ring satisfying (C) is commutative. Also, note that no rings of type e) or $e)^1$ satisfy the condition (S).

Now, combining Theorems S and S^1 with [2,Theorem 2], we obtain

Proposition SC. Suppose that a ring R (resp. a ring R with 1) satisfies (C). If R is non-commutative then there exists a factorsubring of R which is of type $a)_\ell$, $a)_r$ or b) (resp. $a)^1$ or b)).

2. Commutativity theorems. We are now ready to state our theorems.

Theorem 1 ([6, Theorem 1]). A ring R is commutative if (and only if) R satisfies (C) and if for each $x, y \in R$, there exist integers $\ell \geq 0$, $m > 0$ and $f(X) \in X^2\mathbb{Z}[X]$ with $f(1) = \pm 1$ such that $[x,x^m y-f(y)x^\ell] = 0$.

Proof. We can easily see that each ring of type $a)_\ell$, $a)_r$ or b) fails to satisfy the latter condition in Theorem 1. Thus, in view of Proposition SC, R is commutative.

In Theorem 1, we cannot remove the hypothesis that $m > 0$ or $f(1) = \pm 1$. But, in case R has 1, we have the following:

Theorem 2 (see [9, Theorem 1]). Let R be a ring with 1. Then R is commutative if (and only if) R satisfies (C) and if for each $x, y \in R$, there exist non-negative integers ℓ, m, n and $f(X) \in X^2Z[X]$ such that $[x, x^m y - x^n f(y) x^\ell] = 0$.

Proof. Let $x, y \in R$, and $y^2 = 0$. Then we can easily see that $x^m[x,y] = 0$ for some non-negative integer m; so that $[x,y] = 0$. Hence y is in C. This shows that each ring of type $a)^1$ or b) fails to satisfy the latter condition in Theorem 2. Thus, in view of Proposition SC, R is commutative.

Next, for fixed non-negative integers ℓ, m and n, we consider the following conditions:

$(\dagger)_{(\ell,m,n)}$ For each $x, y \in R$, there exists $f(X) \in X^2Z[X]$ such that $[x, x^m y - x^n f(y) x^\ell] = 0$.

$(*)_{(\ell,m,n)}$ For each $x, y \in R$, either $[x,y] = 0$ or $x^m y = x^n f(y) x^\ell$ for some $f(X) \in X^2Z[X]$.

$(**)_{(\ell,m,n)}$ For each $x, y \in R$, either $[x,y] = 0$ or $x^m y - x^n f(y) x^\ell \in C$ for some $f(X) \in X^2Z[X]$.

As is easily seen, $(*)_{(\ell,m,n)}$ implies $(**)_{(\ell,m,n)}$, and $(**)_{(\ell,m,n)}$ does $(\dagger)_{(\ell,m,n)}$.

Theorem 3 (see [9, Theorem 1]). Let R be a ring with 1. If R satisfies $(\dagger)_{(\ell,m,n)}$ then R is commutative.

Proof. In view of Meta Theorem, it suffices to show that R cannot be of type $a)^1$, b), c), $d)^1$ or $e)^1$.

As is easily seen, R cannot be of type $a)^1$ or b).

If R is of type c) (resp. $d)^1$), then [9, Lemma 2] shows that for each x, y ∈ R (resp. x, y ∈ T), there exists $h(X) \in X^2Z[X]$ such that $[x,y-h(y)] = 0$. Hence R (resp. T) is commutative, by [3, Theorem 3]. This is impossible.

Finally, suppose that R is of type $e)^1$. For each s, t ∈ T, there exists $f(X) \in X^2Z[X]$ such that $[s,t] = (s+1)^m[s,t] = (s+1)^n [s,f(t)](s+1)^{\ell} = 0$, a contradiction.

In [5, Theorem], T.P. Kezlan has proved that every ring R with 1 and satisfying $(\dagger)_{(\ell,1,0)}$ is commutative.

Recently, H.E. Bell [1] announced that every ring R satisfying $(*)_{(1,1,0)}$ is commutative. Needless to say, if R has 1, this is a special case of Theorem 3. As an application of Theorem 3, we can prove the following:

Theorem 4 ([7, Theorem 1]). Let $\ell > 0$. If R satisfies $(*)_{(\ell,1,n)}$, then it is commutative.

Proof. In view of Meta Theorem, it suffices to show that R cannot be of type $a)_\ell$, $a)_r$, b), c), d) or e).

We can easily see that R cannot be of type $a)_\ell$ or $a)_r$. Further, by Theorem 3, no rings of type b) or c) satisfy $(*)_{(\ell,1,n)}$.

Now, suppose that R is of type d), and choose x, y ∈ R with $[x,y] \neq 0$. Then there exists $p(X) \in XZ[X]$ such that $xy = x^n p(y) y x^{\ell}$. If $[x,y^{\ell}] \neq 0$ and $[x^{\ell},y] \neq 0$, there exist $f(X)$, $g(X) \in X^2Z[X]$ such that $xy^{\ell} = x^n f(y^{\ell}) x^{\ell}$ and $yx^{\ell} = y^n g(x^{\ell}) y^{\ell}$. Putting $f(y^{\ell}) = f_0(y)y$ and $g(x^{\ell}) = g_0(x)x$ with some $f_0(X)$, $g_0(X) \in XZ[X]$, we obtain $xy^{\ell} = x^n f_0(y) y^n g_0(x^{\ell}) x y^{\ell}$. Since R is a radical ring, this forces a contradiction $xy^{\ell} = 0$. Next, if $[x^{\ell},y] = 0$, then $xy = x^n p(y) x^{\ell-1} xy$, which implies a contradiction $xy = 0$. Similarly, $[x,y^{\ell}] = 0$ forces a contradiction.

Finally, suppose that R is of type e). Then $R^2 \subseteq C$. Given x, y ∈ R with $[x,y] \neq 0$, we can take $p(X) \in XZ[X]$ such that $xy = x^n y p(y) x^{\ell} = xyp(y)x^{\ell+n-1}$, whence $xy = 0$ follows; similarly $yx = 0$. But this is impossible.

Corollary 1 ([7, corollary 1]). Let $\ell > 0$. If R satisfies (S) and $(**)_{(\ell,1,n)}$, then it is commutative.

For a positive integer n, we consider the following condition:
Q(n) If x, y ∈ R and $n[x,y] = 0$, then $[x,y] = 0$.

Finally, we state the following theorem without proof.

Theorem 5 ([6, Theorem 5]). Let R be a ring with 1. Suppose that R satisfies the polynomial identity

$$X^m[f(X),Y]+w(X,Y)[X,g(Y)]w^*(X,Y) = 0,$$

where m is a non-negative integer, $w(X,Y)$ and $w^*(X,Y)$ are monic monomials in $Z<X,Y>$, $f(X)$ and $g(X)$ are polynomials in $XZ[X]$ with $f(1) = \pm 1$ and $g(1) = \pm 1$, and every monomial term of $w(X,Y)g(Y)w^*(X,Y)$ has degree ≥ 2 in Y. Suppose that $d = (f'(1),g'(1))$ is non-zero, where $f'(X)$ and $g'(X)$ are the usual derivatives of $f(X)$ and $g(X)$, respectively. If R satisfies the condition Q(d), then R is commutative.

Corollary 2. Let ℓ, m, n be non-negative integers, k a positive integer, and $f(X) \in X^2Z[X]$. Let $d = (k,f'(1))$. If a ring R with 1 and the polynomial identity $[X^k,X^mY]-[X,X^nf(Y)X^\ell] = 0$ satisfies the condition Q(d), then R is commutative.

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THE COENDOMORPHISM BIALGEBRA OF AN ALGEBRA *

Daisuke TAMBARA

If A, B are algebras over a field k and $\dim A < \infty$, we have a k-algebra $a(A, B)$ with the adjoint property

$$\mathrm{Hom}_{k\text{-alg}}(B, A \otimes C) \cong \mathrm{Hom}_{k\text{-alg}}(a(A, B), C)$$

for any k-algebra C. The algebra $a(A, A)$ has a natural structure of a bialgebra and coacts on the algebra A through the adjunction map $A \to A \otimes a(A, A)$.

Though this is a similar construction to <u>hom</u> for graded algebras with quadratic relations [4], $a(A, B)$ has some nice properties. This report consists of three parts.

1. The monoidal equivalence between $a(A, A)$-comodules and chain complexes.
2. Ext of $a(A, B)$-modules from the viewpoint of the fibre product of module categories.
3. The algebra structure of $a(A, M_n(k))$.

1. One of main results of [6] is

Theorem *If* $\dim A > 1$, *then* {*right* $a(A, A)$-*comodules*} *and* {*chain complexes of* k-*modules*} *are equivalent as monoidal categories.*

This is a consequence of the wellknown equivalence of simplicial complexes and chain complexes. We make the functor as follows. Let C_{-}(resp.

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* The detailed version of this paper will be submitted for publication elsewhere.

$\mathcal{C}_+$) be the category of chain(resp. cochain) complexes of k-modules. Let $\diamond: \mathcal{C}_- \times \mathcal{C}_+ \to k$-Mod be the functor defined as follows.

$$X \diamond Y \to \bigoplus_n X_n \otimes Y_n \rightrightarrows \bigoplus_n X_n \otimes Y_{n+1} \qquad \text{(exact)}.$$

Let A be a k-algebra. Let $\Omega = \Omega_A = \mathrm{Ker}(A \otimes A \xrightarrow{\text{mult.}} A)$ and $\delta: A \to \Omega$ the map $a \mapsto 1 \otimes a - a \otimes 1$. The tensor graded algebra $T_A(\Omega)$ has a differential d of degree 1 defined by

$$d(a_0\delta(a_1) \otimes \delta(a_2) \otimes \cdots \otimes \delta(a_n)) = \delta(a_0) \otimes \delta(a_1) \otimes \cdots \otimes \delta(a_n).$$

Then $Q := (T_A(\Omega), d)$ is a differential graded algebra, i.e., a monoid object of $\mathcal{C}_+$. The bialgebra $a(A, A)$ coacts on Q naturally and so the functor $(-) \diamond Q: \mathcal{C}_- \to k$-Mod has values in Comod-$a(A, A)$. The monoid structure of Q makes $(-) \diamond Q: \mathcal{C}_- \to$ Comod-$a(A, A)$ a monoidal functor. This gives the equivalence of the theorem.

The equivalence treated in Pareigis [5] is the case $A = k[t]/(t^2)$. We also have

Theorem *A bialgebra B such that {right B-comodules} is monoidally equivalent to {chain complexes of k-modules} is isomorphic to $a(A, A)$ for an algebra A.*

2. Ext of $a(A, B)$-modules were computed in [6]. We take here a different approach. A functor $f: A$-Mod $\to B$-Mod is called a restriction functor if $f \cong P \otimes_A (-)$ for a (B, A)-bimodule P which is finitely generated projective as an A-module.

Proposition *Let $f_i : A_i$-Mod $\to A_0$-Mod, $i = 1, 2$ be restriction functors. Then there are restriction functors $g_i : A$-Mod $\to A_i$-Mod such that the following diagram is a fibre square.*

$$\begin{array}{ccc} A\text{-Mod} & \xrightarrow{g_1} & A_1\text{-Mod} \\ g_2 \downarrow & & \downarrow f_1 \\ A_2\text{-Mod} & \xrightarrow{f_2} & A_0\text{-Mod}. \end{array}$$

Indeed, let $f_i \cong P_i \otimes_{A_i} (-)$ and set $P_i^* = \mathrm{Hom}_{-A_i}(P_i, A_i)$. Then we can take A to be the quotient ring of

$$T_{A_1 \times A_0 \times A_2}(P_1 \oplus P_2 \oplus P_1^* \oplus P_2^*)$$

by the ideal generated by the graphs of the canonical maps

$$A_0 \to P_i \otimes_{A_i} P_i^*, \qquad P_i^* \otimes_{A_0} P_i \to A_i.$$

Suppose given an arbitrary fibre square diagram as above with f_i, g_i restriction functors and set $g_0 = f_i \circ g_i$. Let g_i^l be the left adjoints of g_i.

Proposition *For any A-module X, the natural sequence of A-modules*

$$0 \to g_0^l g_0 X \to g_1^l g_1 X \oplus g_2^l g_2 X \to X \to 0$$

is exact.

This is a generalization of Dicks' Mayer–Vietoris presentation [3] and the proof is also based on it.

Proposition *Suppose $f_i = P_i \otimes_{A_i} (-)$ with P_i faithfully flat A_0-modules for $i = 1, 2$. If X is an A-module such that the A_0-module $g_0 X$ is flat, then we have natural isomorphisms*

$$\mathrm{Ext}_A^n(g_i^l g_i X, Y) \cong \mathrm{Ext}_{A_i}^n(g_i X, g_i Y).$$

for $i = 0, 1, 2$.

The proof relies on a structure theorem of coproducts [2]. The above two propositions yield the Mayer–Vietoris sequence connecting $\mathrm{Ext}_A(X, Y)$ with $\mathrm{Ext}_{A_i}(g_i X, g_i Y)$. Applying it to the fibre square

$$\begin{array}{ccc} a(A,B)\text{-Mod} & \xrightarrow{\text{forget}} & k\text{-Mod} \\ (-)\otimes A \downarrow & & \downarrow (-)\otimes A \\ B \otimes A^{\mathrm{op}}\text{-Mod} & \xrightarrow{\text{forget}} & A^{\mathrm{op}}\text{-Mod}, \end{array}$$

we get

Theorem *For $a(A, B)$-modules X, Y and $n \geq 2$, we have*

$$\mathrm{Ext}_{a(A,B)}^n(X, Y) \cong \mathrm{Ext}_{B \otimes A^{\mathrm{op}}}^n(X \otimes A, Y \otimes A).$$

3. We assume that k is algebraically closed and B is a full matrix algebra. For a B-bimodule M we write $Z_B(M) = \{m \in M \mid bm = mb \text{ for all } b \in B\}$. Let A be a finite dimensional algebra and set $N = \mathrm{rad}(A)$, $S = A/N \cong$

$\prod_i \mathrm{End}(V_i)$, $E = \mathrm{End}(\oplus_i V_i)$. We consider the centralizer of the image of the composite

$$B \xrightarrow{\sigma} A \otimes a(A,B) \to S \otimes a(A,B) \hookrightarrow E \otimes a(A,B)$$

where σ is the canonical map. Put $R = \prod_i k$. Then $R \subset S \subset E$. Define an algebra $a(B, R \to E)$ by the pushout diagram

$$\begin{array}{ccc} a(B,R) & \to & a(B,E) \\ \downarrow & & \downarrow \\ R & \to & a(B, R \to E) \end{array}$$

where the left vertical arrow is induced by $k \to B$. Since $A \cong S \oplus N$, N is an S-bimodule. By the matrix-notation of a bimodule we write

$$N = (V_i \otimes V_j^* \otimes N'_{ij})_{i,j}$$

where $N' = (N'_{ij})$ is an R-bimodule. Set $N'^* = (N'^*_{ij})$.

Theorem *We have an isomorphism of algebras*

$$a(B, R \hookrightarrow E) \amalg_R T_R(N'^* \otimes Z_B(\Omega_B)) \cong Z_B(E \otimes a(A,B)).$$

The isomorphism can be made explicit but not canonical. When A is local, we have

$$a(A,B) \cong T(N^* \otimes Z_B(\Omega_B)) \otimes B \cong T((A/k1)^* \otimes (B/k1)) \otimes B.$$

as algebras. When A is a full matrix algebra, the theorem says

$$A \otimes a(A,B) \cong B \otimes a(B,A),$$

which is also clear from the fact that $A \otimes a(A,B) \cong A \amalg B$ [1].

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A NOTE ON HOPF GALOIS EXTENSIONS

ATSUSHI NAKAJIMA

Let R be a commutative ring with identity and let J be a finite commutative cocommutative Hopf algebra over R. A commutative R-algebra S is called a *right J-comodule algebra* if the right J-comodule structure morphism $\rho_S : S \to S \otimes J$ is an R-algebra morphism,where the tensor product is taken over R. A right J-comodule algebra S is called a *J-Galois extension* of R if the morphism γ_S : $S \otimes S \to S \otimes J$ defined by $\gamma_S(x \otimes y) = (x \otimes 1)\rho_S(y) = \sum_{(y)} xy_{(0)} \otimes y_{(1)}$ is an isomorphism, where $\rho_S(y) = \sum_{(y)} y_{(0)} \otimes y_{(1)}$ in $S \otimes J$. J is called a *Galois Hopf algebra* of S/R. Two J-Galois extensions S and T are *isomorphic* if there exists an R-algebra morphism $f : S \to T$ such that $\rho_T f = (f \otimes 1)\rho_S$. We denote the set of isomorphism classes of J-Galois extensions of R by $Gal(R, J)$.

Now we define a product on $Gal(R, J)$. For J-Galois extensions S and T , we consider the following

$$(1 \otimes \tau)(\rho_S \otimes 1), \quad 1 \otimes \rho_T : S \otimes T \to S \otimes T \otimes J,$$

where τ is the twist morphism ($\tau : x \otimes y \to y \otimes x$). Then the difference kernel $ker((1 \otimes \tau)(\rho_S \otimes 1) - 1 \otimes \rho_T)$ is an R-subalgebra of $S \otimes T$ and it is a J-Galois extension of R. We denote the above subalgebra by $S \cdot T$. Let (S) be the isomorphism class of J-Galois extensions of R which are isomorphic to S. Then $Gal(R, J)$ is an abelian group with identity element (J) under the product $(S)(T)$ = $(S \cdot T)$. These were discussed in [1] and there are many related results for the group $Gal(R, J)$ since [3].

Let k be a field with characteristic p ($p \neq 0$). In [4], late Professor A. Hattori pointed out that a purely inseparable field extension $k[x]$

This note is derived from [5].

$= k[X]/(X^p - r)$ over a field k has two different type of Galois Hopf algebra J_1 and J_2 as follows.

(1) $J_1 = k\langle\sigma\rangle$ is the group algebra, where $\langle\sigma\rangle$ is a cyclic group of order p. Then $k[x]$ is a right J_1-comodule algebra with structure morphism $\rho(x) = x \otimes \sigma$ and $k[x]$ is a J_1-Galois extension of k.

(2) $J_2 = k\langle\delta\rangle$ is a free k-module with basis $\{1, \delta, ..., \delta^{p-1}\}$ and the Hopf algebra structure is given by $\delta^p = 0$, $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta$, $\epsilon(\delta) = 0$ and $\lambda(\delta) = -\delta$. Then $k[x]$ is a right J_2-comodule algebra with structure morphism $\rho(x) = x \otimes 1 + 1 \otimes \delta$ and $k[x]$ is a J_2-Galois extension of k.

It is easy to see that J_1 and J_2 are not isomorphic as Hopf algebras and we also have the following

THEOREM 1. *Let S and T be commutative k-algebras. Then*

(1) *S/k is a $k\langle\sigma\rangle$-Galois extension if and only if S is isomorphic to $k[X]/(X^p - s)$ for some $s \in k$.*
(2) *T/k is a $k\langle\delta\rangle$-Galois extension if and only if T is isomorphic to $k[X]/(X^p - t)$ for some $t \in k$.*

It seems that the $k\langle\sigma\rangle$-Galois extensions and the $k\langle\delta\rangle$-Galois extensions are equal. What is the difference of the $k\langle\sigma\rangle$-Galois extensions and the $k\langle\delta\rangle$-Galois extensions ? For the question, we have the following which is proved by Th.1 and the definition of product.

THEOREM 2. *Under the above notations,*

(1) *$Gal(k, k\langle\sigma\rangle)$ is isomorphic to $U(k)/U(k)^p$ as multiplicative groups, where $U(k)$ is the set of invertible elements of k.*
(2) *$Gal(k, k\langle\delta\rangle)$ is isomorphic to k/k^p as additive groups.*

COROLLARY 3. *k is a perfect field if and only if $Gal(k, k\langle\sigma\rangle) = 1$, or equivalently, $Gal(k, k\langle\delta\rangle) = 0$.*

Ths.1 and 2 are generalized to a commutative ring and of course $Gal(k, k\langle\sigma\rangle)$ and $Gal(k, k\langle\delta\rangle)$ are non-trivial in general.

REMARK. In [2], they showed that the field extension $Q(\sqrt[4]{2})/Q$ has two different type of Galois Hopf algebras, where Q is the field of rational integers. When this is the case, it is not known that the two isomorphism class groups are isomorphic or not. But there exists a separable field extension over $k = GF(2)$ which has two different type of Galois Hopf algebras H_1 and H_2. And when this is the case, we can show that groups $Gal(GF(2), H_1)$ and $Gal(GF(2), H_2)$ are not isomorphic.

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MINIMAL INJECTIVE RESOLUTION OF A NOETHER RING

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This is an expository article of our papers [1, 2] and their related topics.

Let us start by fixing some notations.

<u>NOTATION</u> In almost cases in our consideration rings are left and right noether rings. Let

R : a left and right noether ring,

(*) $0 \to {}_RR \to E_0 \to E_1 \to \cdots \to E_n \to \cdots$

: a fixed minimal injective resolution of ${}_RR$.

For an R-module M, we denote

$\mathrm{pd}(M)$ = projective dimension of M, id(M) = injective dimension of M,

fd(M) = flat(weak) dimension of M,

and self-injective dimension of R is defined as

l.inj.dim R = id($_R R$), r.inj.dim R = id(R_R)

and further, if they coincide, inj.dim R is used.

Our main concern in this article is about a minimal injective resolution (*). More precisely, we will discuss on the following conjectures and problems.

(I) *Cogenerator problems*

(1) Let R be an artin algebra, then is $\bigoplus_{k \geqq 0} E_k$ a cogenerator ?

(This is the AUSLANDER-REITEN conjecture [4], which is a generalized version of the NAKAYAMA conjecture.)

If the Auslander-Reiten conjecture is true, the direct sum of the terms E_k's may be finite since the number of non-isomorphic injective indecomposables is finite over an artin rings. Thus it's reasonable to raise the following problem.

(2) Let R be a noether ring. When is $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ for some $n \geqq 0$ a cogenerator ?

A module W is a cogenerator if any module embeds in a direct product of copies of W. But this notion is not fittable in considering finitely generated modules as embedded modules. Thus we introduce a slightly stronger notion of a cogenerator: A module W is called a finitely embedding cogenerator if any finitely generated modules X is embedded in a direct sum of copies of W. (In this case, X embeds in a finite direct sum of W.) These two notions coincide for an artin rings, in fact, let W be a cogenerator over an artin ring R and X a finitely generated R-module, then Soc(X) is a finitely generated essential submodule of X. Hence Soc(X) is a direct sum of a finite number of simples such as Soc$(X) = S_1 \oplus \cdots \oplus S_r$, and thus $X \subseteq$ E$(S_1) \oplus \cdots \oplus$ E(S_r), which is contained in $W^{(r)}$, direct sum of r-copies of W since each E(S_i) embeds in W. (See Lemma by OSOFSKY below.)

For noether rings, however, they are different.

For example, take $R = \mathbb{Z}$ the ring of integers as a ring and let $\mathbb{Q}$ be the additive group of all rational numbers, then $\mathbb{Q}/\mathbb{Z}$ is a cogenerator but not a finitely embedding cogenerator. On the one hand, $\mathbb{Z} \oplus (\mathbb{Q}/\mathbb{Z})$ is a finitely embedding cogenerator.

Now we have a problem concerning the new notion.

(3) When is $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ for some $n \geqq 0$ a finitely embedding cogenerator ?

(II) _Terms E_k in a minimal injective resolution (*)_

Let R be a noether ring with inj.dim $R = n < \infty$.

(1) Characterize indecomposable summands in E_k, at least, in the last term E_n.

(2) Does the last term E_n have nonzero socle ?

(I-1) _AUSLANDR-REITEN Conjecture_

Let R be an artin algebra. <u>Nakayama conjecture</u> says that if each E_k ($k \geqq 0$) is projective, then R will be QF, that is, a quasi-Frobenius ring.

The Auslander-Reiten conjecture implies the Nakayama conjecture:

(Proof) If the Auslander-Reiten conjecture is true, $\bigoplus_{k \geqq 0} E_k$ is a cogenerator and hence any injective indecomposable module E(S) with S simple is a direct summand of some E_k which is projective, and thus E(S) is projective. That is, it holds that injectivity implies projectivity over R, which is equivalent to R QF.

Presently the following result is the best one for the Auslander-Reiten conjecture.

(WILSON [19]) The *Auslander-Reiten conjecture is true for positively graded finite dimensional algebras over fields.*

As another articles concerning Nakayama conjecture and Auslander-Reiten conjecture, we refer ASASHIBA [3], COLBY and FULLER [7].

A useful criterion for a cogenerator is

<u>LEMMA</u>(OSOFSKY [16]) *For a module W over any ring R, W is a cogenerator if and only if W embeds all indecomposable injectives $E(S)$ with S simple.*

Thus $\bigoplus_{k\geq 0} E_k$ *is a cogenerator if and only if for any simple module S there is a k with* $\mathrm{Ext}_R^k(S, R)\neq 0$.

Next we consider the case when $E_0\oplus E_1\oplus \cdots \oplus E_n$ is a cogenerator or a finitely embedding cogenerator. In order that, we will consider two notions for the investigation.

(I-2) *Gorenstein Ring and Dominant Dimension*

<u>GORENSTEIN RING</u>: A (left and right) noether ring is called an <u>n-Gorenstein</u> ring for an $n\geqq 0$ provided that it has self-injective dimension n on both sides.

A 0-Gorenstein ring is just a QF ring, and generally if R is an n-Gorenstein ring, then the ring of upper triangular matrices over R is an (n+1)-Gorenstein ring.

Moreover, the group ring $\mathbb{Z}[G]$ of a finite group G over the ring of integers $\mathbb{Z}$ is a 1-Gorenstein ring but not hereditary.

<u>THEOREM</u> [9] *If R is an n-Gorenstein ring, then $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a cogenerator.*

Concerning this theorem we expect to strengthen the conclusion as follows.

<u>PROBLEM</u> Under the assumption same with the theorem above, does it actually hold $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding cogenerator ?

As for the condition that $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a cogenerator,. which is on the module itself but not on the ring, we refer COLBY-FULLER [7].

The following is well known and mentioned in [16] without proof.

<u>THEOREM</u> *Let R be a noether ring and assume ${}_RR$ is a cogenerator, then R is a QF ring.*

In the theorem above, we can't weaken the condition ${}_RR$ a cogenerator to E_0 a cogenerator. In fact, let R be a local noether ring with radical J, which is not QF and consider the trivial extension ring of R with the bimodule ${}_RR/J_R$. However, if we assume that R is a Gorenstein ring, we may raise the following problem, which will be affirmative if the conjecture (II-2) is true.

<u>PROBLEM</u> If R is an n-Gorenstein ring and E_0 is a cogenerator, then is R a QF ring ?

For the problem (I-3), YOSHINO informed us the following commutative case.

<u>THEOREM</u> *If R is a commutative noether ring with Krull dimension n, then $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a finitely embedding cogenerator.*

(Proof) Any finitely generated module is embedded in an injective module, which is a direct sum of injective indecomposables over a noether ring, and we recall that any injective indecomposable module over a commutative noether ring R is of the form $E(R/P)$ with P a prime ideal. (MATLIS [11]) Further, by Bass's theorem, $E(R/P)$ appears in E_k as a direct summand if and only if $[\mathrm{Ext}_R^k(R/P, R)]_P \neq 0$. On the one hand, it is known that depth $R_P = \min\{ i \mid \mathrm{Ext}_{R_P}^i(R_P/PR_P, R_P) \neq 0\}$ is at most the Krull dimension of R_P. Hence any injective indecomposable embeds in $W = E_0 \oplus E_1 \oplus \cdots \oplus E_n$ and it turns out that any finitely generated module embeds in a direct sum of copies of W, which is actually finite.

Further to investigate our problem (II), we'd like to introduce the following notion.

DOMINANT DIMENSION OF A RING: Any ring R has dominant dimension at least n (denoted by dom.dim $R \geqq n$) provided that each E_k is flat for all k $(0 \leqq k < n)$. (See [8]) Dominant dimension was first defined by NAKAYAMA for finite dimensional algebras over fields, and has been investigated by TACHIKAWA and MÜLLER. Thus we should mention some of their results.

REMARKS *Now we assume R is a noether ring.*

(1) For $n = 1$, this is nothing but R a QF-3 ring in the sense of MORITA and in this case, it is shown that the definition of dominant dimension is left-right symmetric. (MORITA [14])

(2) If dom.dim $R > 1$, then R is an artin ring. [1]

(3) It is shown that dominant dimensions defined on the left and right sides are coincident for artin rings.

(MÜLLER and TACHIKAWA, See [18].)

(4) Combining these results above, dominant dimensions defined on the left and right sides are coincident for any noether ring.

(5) If r.inj.dim $R <$ dom.dim R, R is a QF ring.

Now we can answer for the problem (I-2 and 3) in the case $n=1$, that is, $E_0 \oplus E_1$ is a cogenerator or a finitely embedding cogenerator.

<u>THEOREM</u> [2] *Let R be a noether ring.*

(1) If every dense maximal left ideal is reflexive, then $E_0 \oplus E_1$ is a cogenerator.

(2) If we assume dom.dim $R \geqq 1$, then $E_0 \oplus E_1$ is a cogenerator if and only if every maximal left ideal is reflexive.

(3) Assume dom.dim $R \geqq 1$, then $E_0 \oplus E_1$ is a finitely embedding cogenerator if and only if any finitely generated uniform, torsion left R-module U has a non-zero submodule V for which there exists an exact sequence $0 \to L \to F \to V \to 0$ with F finitely generated free and L reflexive.

(Here a torsion theory follows the Lambek's sense.)

<u>EXAMPLE</u> As a ring with $E_0 \oplus E_1$ a cogenerator but of infinite self-injective dimension, we give a serial ring with the admissible sequence 3, 4.

(II) *Terms E_k*

For commutative case, we have a fascinating result by Bass as follows.

THEOREM (BASS [5]) *Let R be a commutative n-Gorenstein ring.*

(1) An injective indecomposable module $E(R/P)$ with P a prime ideal appears as a direct summand of E_k if and only if $ht(P)=k$.

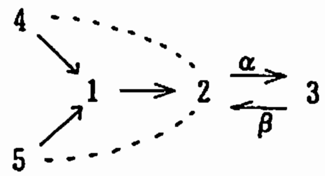
(2) $fd(E_k)=k$ for each $k \geqq 0$. Hence a commutative Gorenstein ring has dominant dimension $\geqq 1$.

(3) The multiplicity of $E(R/P)$ in E_k is given by

$$\dim_{K(P)} \operatorname{Ext}_R^k (R/P,\ R)_P, \quad \text{where } K(P)=R_P/PR_P.$$

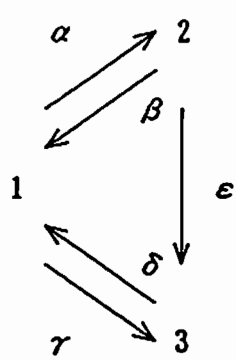
For non-commutative case, we can't expect such a beautiful result in a general setting as the following examples show.

<u>EXAMPLES</u> (1) Let R be a path algebra given by the quiver:



bounded by $\alpha\beta=\beta\alpha=0$ and zero-relations denoted by dotted lines. Then R is a 2-Gorenstein ring and $\mathrm{pd}(E_0)=\mathrm{pd}(E_1)=\mathrm{pd}(E_2)=2$, that is, all terms E_0, E_1 and E_2 have the possible highest projective dimension. (See Proposition below.)

(2) Let R be a path algebra given by the quiver:



bounded by

$$\alpha\beta=\gamma\delta=\alpha\delta=\gamma\beta=\varepsilon\alpha=0.$$

Then R is a quasi-hereditary ring of global dimension 3, and all injective indecomposable R-modules have the highest projective dimension 3. We will see below that all of them do not necessarily appear in the last term.

(3) Let R be a serial ring with admissible sequence 2, 2, 3, 4 and Re_1, Re_2, Re_3, Re_4 the corresponding Kupisch series, then R is a 3-Gorenstein ring. The injective indecomposable Re_4 appears both in E_0 and E_2, but neither in E_1 nor E_3.

For non-commutative Gorenstein rings, the following seems the best information concerning minimal injective resolution (*) at present, which we can know in general.

THEOREM [1] *Let* R *be an n-Gorenstein ring and* $n \geq 1$, *then* E_0 *and* E_n *have no isomorphic indecomposable direct summand.*

Concerning the last term,

PROPOSITION [10] *Let* R *be an n-Gorenstein ring.*

(1) $pd(E_k) \leq n$ *for all* $k \geq 0$ *and every direct summand of* E_n *has the highest projective dimension* n;

(2) If S *is a simple left* R*-module of projective dimension* n, S *appears in* E_n. *Conversely,* S *is a simple left* R*-module embedded in* E_n, $pd(S) = n$ *or* ∞.

Now we can characterize a direct summand in E_0 or E_1 by using reflexivity of irreducible left ideals.

<u>THEOREM</u> [2] *Let R be a noether ring with dominant dimension ≥ 1, then an injective indecomposable left module U is isomorphic to a direct summand of $E_0 \oplus E_1$ if and only if there exists a reflexive left ideal I with $U \cong E(R/I)$.*

This is considered as a non-commutative version of Matlis's result [13] for commutative case, and he actually shows that every prime ideal of height 1 in a commutative noetherian domain is reflexive.

<u>The socle of E_n</u> Finally we will mention about the conjecture which says that for an n-Gorenstein ring, the socle of E_n will be nonzero. The cases which are presently known to be true are the following.

(1) By Bass's theorem, we have $Soc(E_n) \neq 0$ for R a commutative n-Gorenstein ring.

(2) Let R be a 1-Gorenstein ring, then $Soc(E_1) \neq 0$.

(See Iwanaga [9] and Sato [17].)

(3) *Let R be a fully bounded noether ring. If R is n-Gorenstein, then E_n is an artinian module. Thus, in particular, $Soc(E_n) \neq 0$.*

Proof Recall that a ring R is bounded if essential onesided ideals in R contain nonzero twosided ideals, and R is fully bounded provided that every prime factor ring of R is bounded.

The proof depends on the work by Jategaonkar [11]. First of all, we have the following for a fully bounded noether ring R by [Theorems 3.5 and 5.3 in 11] and the general property of Krull dimension:

"$K\text{-}dim\ E_k \geq K\text{-}dim\ E_{k+1}$ *for any* $k = 0, 1, \cdots$",

where K-dim stands for Krull dimension.

Now contrary to our claim, assume K-dim $E_n > 0$, then we have K-dim$(E_0 \oplus E_1 \oplus \cdots \oplus E_n) > 0$ by the fact just mentioned above. Since R is an n-Gorenstein ring, the injective module $E_0 \oplus E_1 \oplus \cdots \oplus E_n$ is a cogenerator and hence an injective indecomposable left R-module E(S) with S simple embeds in some E_k $(0 \leqq k \leqq n)$.

This therefore implies K-dim $E_k = 0$ by Jategaonkar's results, which is a contradiction. Thus K-dim $E_n = 0$, that is, E_n is artinian.

<u>EXAMPLES</u> For $R = Z$ a 1-Gorenstein ring, $E_1 = Q/Z$ is artinian and $Soc(E_1) = Soc(Q/Z) \neq 0$.

Similarly $R = Z[G]$ a group ring of a finite group G over Z is a 1-Gorenstein ring and $E_1 = Q[G]/Z[G]$ is artinian and $Soc(E_1) = Soc(Q[G]/Z[G]) \neq 0$.

More generally *let R be a Gorenstein order over a local commutative n-dimensional Gorenstein ring* (in the sense of NISHIDA [15]), *then R is an n-Gorenstein ring, which is fully bounded and the last term E_n is a cogenerator.* (See Theorem 1.1 in [15].) *Thus $Soc(E_n) \neq 0$.*

These examples are all in the case E_n a cogenerator and obviously $Soc(E_n) \neq 0$. However in the following example by FUJITA which is mentioned in [15], E_n is not a cogenerator but has a nonzero socle. Let R be a discrete valuation domain with J its radical, and consider an order over R:

$$\Lambda = \begin{bmatrix} R & J & J & J & J \\ R & R & J & R & J \\ R & J & R & R & R \\ J & J & J & R & J \\ J & J & J & R & R \end{bmatrix}$$

Λ is a 2-Gorenstein ring but not a Gorenstein order. In a minimal injective resolution $0 \to {}_{\Lambda}\Lambda \to E_0 \to E_1 \to E_2 \to 0$, we can see that $\mathrm{Soc}(E_0)=0$, four non-isomorphic simples appear in E_1 and one simple in E_2.

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ON-LEFT-EXACT RADICALS

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In his paper [7], V. S. Ramamurthi has studied the smallest left exact radical J^* larger than the Jacobson radical J. However if we take $\mathscr{C}$ as the class of cosingular modules, then J^* coincides with $t_{\mathscr{C}}$. In this note, first we shall study smallest left exact radical r^* larger than a preradical r and show that r^* can be described by various methods. Also we shall show that $(r_1 \wedge \cdots \wedge r_n)^* = r_1^* \wedge \cdots \wedge r_n^*$. Then we shall treat the largest left exact radical r_* smaller than a preradical r. Finally, we shall investigate a module M such that $(k_M)^*(M) = M$. In consequence, we can prove that every direct product M^Λ of copies of a module M which is non-singular and $(k_M)^*(M) = M$ has no nonzero injective submodule for any index set Λ.

Throughout this note R is a ring with identity and modules

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are unitary left R-modules unless otherwise stated. We denote the category of modules by R-mod and the injective hull of a module M by E(M). As for terminologies and basic properties concerning torsion theories and preradicals, we refer to [8]. For each preradical r, we denote the r-torsion (resp. r-torsionfree) class by $T(r)$ (resp. $F(r)$). Also the left linear topology corresponding to a left exact preradical r is denoted by $\mathcal{L}(r)$. Now for two preradicals r and s, we shall say that r *is larger than* s if $r(M) \supseteq s(M)$ for all modules M. For a preradical r, we put $\bar{r}(M) = \bigcap \{{}_RN \subseteq M \mid r(M/N) = 0\}$, $\tilde{r}(M) = r(E(M)) \cap M$ and $\hat{r}(M) = \sum \{{}_RN \subseteq M \mid r(N) = N\}$ for each module M. Then $\bar{r}$ (resp. $\tilde{r}$) is the smallest radical (resp. left exact preradical) larger than r and $\hat{r}$ is the largest idempotent preradical smaller than r.

Lemma 1. *Let r and s be preradicals. Then the following statements hold.*

(1) $F(r) = F(\bar{r})$ *and* $T(r) = T(\hat{r})$.

(2) $\bar{r} = \bar{s}$ *if and only if* $F(r) = F(s)$.

(3) $\hat{r} = \hat{s}$ *if and only if* $T(r) = T(s)$.

For a class $\mathcal{C}$ of modules, $k_{\mathcal{C}}$ denotes the largest one of those preradicals r such that $r(C) = 0$ for all modules C in $\mathcal{C}$. As is well-known, $k_{\mathcal{C}}$ is a radical.

Lemma 2. *Let r be a preradical. Then* $\bar{r} = k_{F(r)}$.

Proposition 3. *For a preradical* r, $\bar{\bar{r}}$ *is the smallest left exact radical larger than* r *and coincides with* $\tilde{\bar{r}}$.

Hereafter we denote $\bar{\bar{r}}$ simply by r^*.

We call a class of modules $\mathcal{C}$ a *Serre class* if it is closed under homomorphic images, submodules and group extensions.

Example 4. Let $\mathcal{C}$ be a Serre class. We define a preradical t by $t(M) = \sum \{ {}_RN \subseteq M \mid N \in \mathcal{C} \}$ for each module M. Then t is a left exact preradical and thus $t^* = \bar{t}$. For example, the class $\mathcal{C}$ of artinian modules is a Serre class and t^* is the artinian radical. Furthermore, t^* coincides with $\overline{Soc}$, since $F(t) = \{ {}_RM \mid Soc(M) = 0 \}$ [3, Example 8].

For a class $\mathcal{P}$ of preradicals we define a preradical $\wedge\{r \mid r \in \mathcal{P}\}$ as $(\wedge\{ r \mid r \in \mathcal{P}\})(M) = \cap\{r(M) \mid r \in \mathcal{P}\}$ for each module M.

Lemma 5. *Let* $\mathfrak{L}$, $\mathfrak{R}$ *and* $\mathfrak{S}$ *be classes of left exact preradicals, radicals and left exact radicals respectively. Then the following statements hold.*

(1) $\wedge\{r \mid r \in \mathfrak{L}\}$ *is a left exact preradical.*

(2) $\wedge\{r \mid r \in \mathfrak{R}\}$ *is a radical.*

(3) $\wedge\{r \mid r \in \mathfrak{S}\}$ *is a left exact radical.*

A module M is called *QF-3'* if $k_M(E(M)) = 0$. As is easily seen, M is QF-3' if and only if $k_M = (k_M)^*$.

The r^* can be described by various methods as follows.

Proposition 6. *Let r be a preradical. Then the following assertions hold.*

(1) $r^* = k_{\mathcal{D}}$ *, where* $\mathcal{D}$ *is the class of all r-torsionfree injective modules.*

(2) $r^* = k_{\mathcal{Q}}$ *, where* $\mathcal{Q}$ *is the class of all r-torsionfree QF-3' modules.*

(3) $r^* = k_{\mathcal{D}'}$ *, where* $\mathcal{D}' = \{ {}_R X \mid E(X) \in F(r)\}$.

Now we shall prove one of the main theorems of this note.

Theorem 7. *Let* $r_1, r_2, \cdots\cdots, r_n$ *be preradicals for* R-mod. *Then* $(r_1 \wedge r_2 \wedge \cdots\cdot \wedge r_n)^* = r_1^* \wedge r_2^* \wedge \cdots\cdot \wedge r_n^*$.

In contrast with r^*, we now treat the largest left exact radical r_* smaller than a preradical r.

Proposition 8. [3, Corollary 3.13]. *Let r be a preradical for* R-mod. *We put* $\mathcal{F} = \{E(X) \mid X \in F(r)\}$. *Then* $k_{\mathcal{F}}$ *is the largest left exact radical smaller than* $\bar{r}$.

For a preradical r, we denote by r_* the largest left exact radical smaller than r, if it exists.

Corollary 9. *Let r be a preradical for R-mod. Then*

(1) *If r is a radical, then r_* exists.*

(2) *r is a left exact radical if and only if there exists r_* and $r^* = r = r_*$ holds.*

The following example shows that if r is not a radical, then r_* need not exist in general.

Example 10. Let K be a field and let R be the ring of all 2 × 2 upper triangular matrices over K. Let r be the left exact preradical corresponding to the left linear topology having the smallest element

$$\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}.$$

Also r_1 and r_2 mean the left exact radicals corresponding to the left Gabriel topologies having the smallest elements

$$\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$$

respectively. Then r is larger than r_1 and r_2 properly. If r_* exists, then r_* is larger than r_1 and r_2. Thus we obtain

$$\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(r_*) \text{ and so } \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}^2 = 0$$

which belongs to $\mathcal{L}(r_*)$. Hence $r_* = 1$. This is a contradiction.

Proposition 11. *Let r be a radical for R-mod. If each nonzero cyclic module in $T(r_*)$ has a nonzero factor in $F(r)$,* *then* $r_* = 0$. *In particular,* $J_* = 0$, *where* J *is the Jacobson radical.*

Proposition 12. *Let* S *be a simple module. Then* S *is injective if and only if* $J(E(S)) = 0$. *In particular,* J *is left exact if and only if* R *is a left* V*-ring* [5, Proposition 5.3].

Proposition 13. *Let* $(r_i)_{i\in I}$ *be a family of preradicals for* R*-mod. We put* $\mathcal{F} = \bigcap(F(r_i) \mid i \in I)$. *Then* $(k_{\mathcal{F}})_*$ *is the largest one of those left exact radical* r *for which* $r(X) = 0$ *for all* $X \in \mathcal{F}$. *Furthermore* $(k_{\mathcal{F}})_* = (\overline{\sum_i r_i})_*$, *where* $(\sum_i r_i)(M) = \sum_i r_i(M)$ *for each module* M.

Proposition 14. *Let* $(r_i)_{i\in I}$ *be a family of left exact preradicals. We put* $\mathcal{F} = \bigcap(F(r_i) \mid i \in I)$. *Then* $k_{\mathcal{F}}$ *is a left exact radical.*

As above, k_M is the largest one of those preradical r for which $r(M) = 0$. Next we shall study $(k_M)^*$ for each module M and characterize those modules M for which $(k_M)^*(M) = M$.

Lemma 15. *Let* $\mathcal{A} = (M_\lambda)_{\lambda\in\Lambda}$ *be a family of modules. We put* $M = \sum_{\lambda\in\Lambda} \oplus M_\lambda$ *and* $M' = \prod_{\lambda\in\Lambda} M_\lambda$. *Then the following assertions hold.*

(1) $\wedge(k_{M_\lambda} \mid \lambda \in \Lambda) = k_M = k_{M'}$.

(2) $(k_M)^* = (k_{M'})^* \leq \wedge((k_{M_\lambda})^* \mid \lambda \in \Lambda)$.

(3) $(k_A)^* = (k_{A^{(I)}})^* = (k_{A^I})^*$ *for all modules* A *and all index set* I.

Corollary 16. [4, Proposition 2.2]. *Let* $(Q_\lambda)_{\lambda\in\Lambda}$ *be a family of QF-3′ modules. We put* $Q = \sum_{\lambda\in\Lambda} \oplus Q_\lambda$ *and* $Q' = \Pi_{\lambda\in\Lambda} Q_\lambda$. *Then both* Q *and* Q′ *are QF-3′.*

Proposition 17. *A module* M *is nonsingular, faithful and QF-3′ if and only if* $Z = k_{E(R)} = k_M$, *where* Z *is the singular torsion functor.*

Corollary 18. *For a ring* R, *the following conditions are equivalent:*

1) R *is a left nonsingular ring.*
2) *There exists a faithful nonsingular module.*
3) $Z = k_{E(R)}$.

A module M is called U-*torsionless* if $k_U(M) = 0$. Clearly, M is QF-3′ if and only if E(M) is M-torsionless.

Proposition 19. *Let* M *be a faithful module. If* E(M) *is* R-*torsionless, then* M *is QF-3′.*

However the converse is false. Take for example $R = \mathbb{Z}$ and $M = \mathbb{Q}$. Since ${}_{\mathbb{Z}}\mathbb{Q}$ is injective, ${}_{\mathbb{Z}}\mathbb{Q}$ is QF-3′ and so M is faithful and QF-3′. But since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = 0$, $k_R(M) = M$, namely, $E(M) = M$

is not R-torsionless.

Next we consider those modules M for which $(k_M)^*(M) = M$.

Proposition 20. *Let M be a module. Then the following assertions hold.*

(1) *If $(k_M)^*(M) = M$, then M has no nonzero injective submodule.*

(2) *$(k_M)^* = 1$ if and only if M^Λ has no nonzero injective submodule for any index set Λ.*

If $R = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z}$, where p is a prime number, then M^Λ has no nonzero injective submodule for any index set Λ.

Corollary 21. *Let M be a module. If $(k_M)^*(M) = M$, then $M^{(\Lambda)}$ has no nonzero injective submodule for all index sets Λ.*

Lemma 22. [2, Lemma 0.2] *Let E be an injective module and M a nonsingular module. If $f \in Hom_R(E,M)$, then both $Im(f)$ and $Ker(f)$ are injective.*

For a nonsingular module M, we have

Theorem 23. *For a nonzero nonsingular module M, the following conditions are equivalent:*

1) $(k_M)^*(M) = M$.

2) *M has no nonzero injective submodule.*

3) $M^{(\Lambda)}$ *has no nonzero injective submodule for all index sets* Λ.

4) M^{Λ} *has no nonzero injective submodule for all index sets* Λ.

5) $(k_M)^* = 1$.

Example 24. Let R be the ring of 2 × 2 upper triangular matrices over a field K. We put

$$M = \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}.$$

Then M is a simple projective module and so M is nonsingular. Clearly M is not injective, namely, $(k_M)^*(M) = M$.

Note that if $R = \mathbb{Z}$ and $M = \mathbb{Z}/p\mathbb{Z}$, as above. Then M is singular, noninjective and simple, but satisfies the equivalent condition of Theorem 23.

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On Rings of Finite Buchsbaum-Representation Type.

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1. Introduction.

This note aims to report the joint work with S.Goto (Meiji Univ.) on commutative Noetherian local rings with only finitely many isomorphism classes of indecomposable maximal Buchsbaum modules. The theory on rings of finite CM-representation type is not only important as the background but also very helpful for this research. After clarifying the relation between these theories we shall state the main results and investigate a typical example.

Throughout this note R is a commutative complete Noetherian local ring with unit and the maximal ideal $\mathfrak{m}$. The Krull dimension of R (resp. an R-module M) is denoted by $\dim R$ (resp. $\dim_R M$) and we put $d = \dim R$.

2. Definitions.

We first recall some definitions which are fundamental in commutative algebra.

Let M be a finitely generated R-module and $a_1, a_2, \dots, a_s$ be the elements in $\mathfrak{m}$. We say that $a_1, a_2, \dots, a_s$ is a system of parameters (We abbreviate it to s.o.p..) for M (resp. regular M-sequence) if $s = \dim_R M$ and $\ell_R(M/(a_1, \dots, a_s)M)$ is finite (resp. a_i is a non-zerodivisor on $M/(a_1, \dots, a_{i-1})M$ for $1 \leq i \leq s$, when this is the case s is called the length of this sequence.) and we define

$$\operatorname{depth}_R M = \max\{\, n \mid \begin{array}{l}\text{There exists a regular}\\ \text{M-sequence of length } n\,.\end{array}\} .$$

As any regular M-sequence forms a part of a s.o.p. for M, we have $\operatorname{depth}_R M \leq \dim_R M$. In particular if the equality holds, i.e. $\operatorname{depth}_R M = \dim_R M$, then M is called a Cohen-Macaulay R-module. Furthermore a Cohen-Macaulay R-module M such that $\dim_R M = d$ is called a maximal Cohen-Macaulay (We abbreviate it to MCM.) R-module. We say that a system of elements $a_1, \dots, a_r \in \mathfrak{m}$ is a weak M-sequence if the kernel of the R-homomorphism

$$M/(a_1, \dots, a_{i-1})M \xrightarrow{a_i} M/(a_1, \dots, a_{i-1})M$$

is annihilated by $\mathfrak{m}$ for any $1 \leq i \leq r$ and M is called a Buchsbaum R-module if every s.o.p. for M is a weak M-sequence. Maximal Buchsbaum R-module is a Buchsbaum R-module whose Krull dimension is equal to d.

If M is a Cohen-Macaulay R-module, then every s.o.p. for M is a regular M-sequence and so it is a weak M-sequence by definition. Therefore a Cohen-Macaulay R-module is a Buchsbaum R-module.

Though there are several criterions for M to be Cohen-Macaulay or Buchsbaum, we state only the followings which is based on the local cohomology module

$$H_{\mathfrak{m}}^{i}(M) := \varinjlim_{n} \mathrm{Ext}_{R}^{i}(R / \mathfrak{m}^{n}, M).$$

Theorem(1.1)(cf.[6]) A finitely generated R-module M is a MCM R-module if and only if $H_{\mathfrak{m}}^{i}(M) = 0$ for any $i \neq d$.

Theorem(1.2)(cf.[9]) Let M be a finitely generated R-module and suppose $s := \mathrm{depth}_R M \leq \dim_R M = d$. Then M is a maximal Buchsbaum R-module if $H_{\mathfrak{m}}^{i}(M) = 0$ for any $i \neq s, d$ and if $\mathfrak{m} \cdot H_{\mathfrak{m}}^{i}(M) = (0)$.

We denote by $n(R)$ (resp. $n_B(R)$) the number of the isomorphism classes of indecomposable MCM (resp. maximal Buchsbaum) R-modules and we say that R has finite CM-representation (resp. Buchsbaum-representation) type if $n(R)$ (resp. $n_B(R)$) is finite. As we have seen in the above, any MCM R-module is a maximal Buchsbaum R-module and so we have $n(R) \leq n_B(R)$, which means that R has finite CM-representation type if it has finite Buchsbaum-representation type.

4. Main Results.

The next theorem due to D. Eisenbud and S. Goto is the starting-point of the research on rings of finite Buchsbaum-representation type.

Theorem(3.1)(cf.[1], [2]) Let R be a regular local ring of $\dim R = d \geq 1$. Then $\mathrm{Syz}_R^n(R / \mathfrak{m})$ $(1 \leq n \leq d)$ are the representatives of indecomposable maximal Buchsbaum R-modules. Hence R has finite Buchsbaum-representation type and $n_B(R) = d$.

The converse of Theorem(3.1) holds if R satisfies certain conditions as follows.

Theorem(3.2)([5]) Let $P = k[[X_1, \ldots , X_n]]$ be a formal power series ring over an algebraically closed field k with $\mathrm{ch}\, k \neq 2$ and let I be an ideal of P . We put $R = P / I$. If R is Cohen-Macaulay and $\dim R \geq 2$, then R has finite Buchsbaum-representation type if and only if R is a regular local ring.

As is shown in [5], Theorem(3.2) doesn't hold if $\dim R = 1$. In his paper [3] S. Goto determined all one-dimensional complete local rings R of finite Buchsbaum-representation type and his result is summarized into the following

Theorem(3.3)([3]) Let R be a complete Noetherian local ring of $\dim R = 1$. Then the following conditions are equivalent.

(1) R has finite Buchsbaum-representation type.

(2) $e(R) \leq 2$, $v(R) \leq 2$ and $R / H_{\mathfrak{m}}^0(R)$ is reduced, where $e(R)$ and $v(R)$ respectively denote the multiplicity and the embedding dimension of R .

(3) $R \cong P / fI$, where P is a 2-dimensional complete regular local ring with the maximal ideal n , $f \in n \setminus n^3$ and I is an ideal of P such that $n^s \subseteq I$ for some $s \geq 0$.

Remark. Theorem(3.3) was first proved by S. Goto in the case where R / m is an infinite field. But we can avoid the restriction on the residue class field by the technique to construct infinitely many indecomposable maximal Buchsbaum R-modules stated in the next section.

Corollary(3.4)([3]) Let R be Cohen-Macaulay and $\dim R = 1$. Then R has finite Buchsbaum-representation type if and only if R is reduced and $e(R) \leq 2$.

Corollary(3.5)([3]) Let $P = k[[X_1, \ldots , X_n]]$ be a formal power series ring over an algebraically closed field k and let I be an ideal of P . We put $R = P / I$. If R is Cohen-Macaulay and $\dim R = 1$, then R has finite Buchsbaum-representation type if and only if R is a simple curve singularity of type (A_n).

Furthermore S. Goto succeeded to determine all 2-dimensional complete local rings R of finite Buchsbaum-representation type.

Theorem(3.6)([4]) Let R be a complete Noetherian local ring of $\dim R = 2$ and suppose that R / m is algebraically closed. Then the following conditions are equivalent.

(1) R has finite Buchsbaum-representation type.

(2) $e(R) = 1$ and $v(R) \leq 3$.

(3) $R \cong P / xI$, where P is a 3-dimensional complete regular local ring with the maximal ideal $\mathfrak{n}$, $x \in \mathfrak{n} \setminus \mathfrak{n}^2$ and I is an ideal of P such that $\mathrm{ht}\, I \geq 2$.

Corollary(3.7)([4]) Let $\dim R = 2$ and $R / \mathfrak{m}$ be algebraically closed. If R is unmixed, then R has finite Buchsbaum-representation type if and only if R is a regular local ring.

The next theorem is the generalization of the implication (3) ⇒ (1) of Theorem(3.6), which insists that Theorem(3.2) doesn't hold if we remove the assumption that R is Cohen-Macaulay.

Theorem(3.8)([8]) Let P be a $(d + 1)$-dimensional $(d \geq 1)$ regular local ring with the maximal ideal $\mathfrak{n}$. Let $X \in \mathfrak{n} \setminus \mathfrak{n}^2$ and let I be a proper ideal of P with $\mathrm{ht}_P I \geq 2$. We put $R = P / XI$, $\mathfrak{p} = XR$ and $\mathfrak{m} = \mathfrak{n}R$. Then R has finite Buchsbaum-representation type and

$$\mathrm{Syz}^n_{R/\mathfrak{p}}(R / \mathfrak{m}) \ (1 \leq n \leq d) \ , \ R / \mathfrak{m}\mathfrak{p}$$

are the representatives of the indecomposable maximal Buchsbaum R-modules.

4. A construction of indecomposable modules.

In this section we explain a method to construct countably many indecomposable maximal Buchsbaum R-modules and applying it, we shall investigate a concrete example.

Let C be a finitely generated R-module and let

$$\sigma : 0 \to L \to F \xrightarrow{\varepsilon} C \to 0$$

be the initial part of a minimal free resolution of C. We define a homomorphism

$$\rho : \mathrm{End}_R(C) \to \mathrm{End}_R(L / \mathfrak{m}L)$$

of algebras by

$$\rho(\phi)(z') = \psi(z)'$$

for any $\phi \in \mathrm{End}_R(C)$ and $z \in L$, where $\cdot'$ denotes the reduction mod $\mathfrak{m}L$ and ψ is an R-endomorphism over F with $\varepsilon \cdot \psi = \phi \cdot \varepsilon$. We put $A_\sigma = \mathrm{Im}\ \rho$ and we regard $L / \mathfrak{m}L$ as a left A_σ-module. Then we have the following

Theorem(4.1)([7]) Let C be indecomposable and $\mathrm{depth}_R C \geq 1$. Suppose that there exist elements x and y of L such that x' and y' are linearly independent over A_σ. We denote, for each integer $n \geq 1$, by N_n the R-submodule of L^n generated by

$$\begin{pmatrix} x \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ x \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots \quad, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ y \\ x \end{pmatrix}$$

and $\mathfrak{m}L^n$. We put $M_n = F^n / N_n$. Then the following statements hold.

(1) M_n is indecomposable if A_σ is commutative.

(2) $M_n \not\cong M_m$ if $n \neq m$.

(3) M_n is a maximal Buchsbaum R-module if C is MCM.

As an application of the above theorem we show the next

Example(4.2) Let $R := k[[t^3, t^4, t^5]] \subseteq k[[t]]$, where k is a field and t is an indeterminate over k . Then R is not of finite Buchsbaum-representation type.

Proof. Put $S = k[[t]] = R + Rt + Rt^2$ and let

$$\sigma : 0 \to L \to R^3 \xrightarrow{\varepsilon} S \to 0$$

be the initial part of a minimal free resolution of S with

$$\varepsilon(e_1) = 1 \ , \ \varepsilon(e_2) = t \ \text{ and } \ \varepsilon(e_3) = t^2 \ ,$$

where e_1 , e_2 and e_3 are the canonical basis of R^3 . Since S is an indecomposable MCM R-module, by Theorem(4.1) it is sufficient to show that A_σ is commutative and there exist elements x and y in L such that x' and y' are linearly independent over A_σ , where $\cdot'$ denotes the reduction mod $\mathfrak{m}L$ ($\mathfrak{m} = t^3S$). As $\mathrm{End}_R(S)$ is a commutative R-algebra which is generated by 1_S , $t1_S$ and $t^2 1_S$ as R-module, so A_σ is commutative and is generated by $\rho(1_S) = 1_{L/\mathfrak{m}L}$, $\rho(t1_S)$ and $\rho(t^2 1_S)$ over k . We put $\xi_i = \rho(t^i 1_S)$ for $i = 1,2$. Let α_1

and α_2 be the R-endomorphisms over R^3 defined respectively by the matrices

$$\begin{pmatrix} 0 & 0 & t^3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & t^3 & 0 \\ 0 & 0 & t^3 \\ 1 & 0 & 0 \end{pmatrix} .$$

Then $\varepsilon \cdot \alpha_i = (t^i 1_S) \cdot \varepsilon$ for $i = 1, 2$. Hence ξ_i is induced from α_i. Put

$$x = \begin{pmatrix} t^4 \\ -t^3 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ t^4 \\ -t^3 \end{pmatrix}, \quad x_2 = \begin{pmatrix} -t^6 \\ 0 \\ t^4 \end{pmatrix}, \quad y = \begin{pmatrix} t^5 \\ -t^4 \\ 0 \end{pmatrix}, \quad y_1 = \begin{pmatrix} 0 \\ t^5 \\ -t^4 \end{pmatrix}, \quad y_2 = \begin{pmatrix} -t^7 \\ 0 \\ t^5 \end{pmatrix}.$$

Then we have $\xi_i x' = x_i'$ and $\xi_i y' = y_i'$. Assume

$$(a_0 1_{L/mL} + a_1 \xi_1 + a_2 \xi_2) x' + (b_0 1_{L/mL} + b_1 \xi_1 + b_2 \xi_2) y' = 0$$

with $a_i \in k$ and $b_j \in k$. Then we get

$$a_0 x' + a_1 x_1' + a_2 x_2' + b_0 y' + b_1 y_1' + b_2 y_2' = 0 .$$

Since x', x_1', x_2', y', y_1' and y_2' are linearly independent over k, so

$$a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 0 .$$

Hence x' and y' are linearly independent over Λ_σ.

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On periodicity of the cohomology of Frobenius algebras [†]

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Introduction.

The purpose of this note is to propose some results concerning the cohomology of Frobenius algebras, especially its periodicity.

In §1.1, we review some facts about the complete cohomology of Frobenius algebras after [N]. In §1.2 we define *a cup product* on the complete cohomology group, and in §1.3 we have a result with respect to periodicity of the cohomology which is obtained from the existence of the cup product. In §1.4, as preliminaries to §2, we provide *a restriction map* and *a corestriction map*, and we give a relation between the restriction map and the cup product. In §2, as applications of §1, we deal with crossed product algebras and twisted group algebras. More precisely, periodicity of their cohomology groups and existence of an invertible element in the cohomology ring are studied. In §3, we investigate the structure of the cohomology group of the quaternion algebra over **Z** using the spectral sequence, which can not be treated in §2. §1 and §2 in this note is a summary of [S1], [S2], [S3] and [S4].

§1. General theory of periodic cohomology.

1.1. Complete cohomology theory of Frobenius algebras.

Let Λ be a finitely generated free Frobenius algebra over a commutative ring R with identity. Namely there exists a left Λ-isomorphism $\varphi : \Lambda \xrightarrow{\sim} \Lambda^* = \mathrm{Hom}_R(\Lambda, R)$ and there exists a pair of R-basis of Λ; $\{u_i\}, \{v_i\}$ $(1 \leq i \leq n)$ such that $\varphi(u_i)(v_j) = \delta_{ij}$. We set $\mu = \varphi(1)$. A map $(x \mapsto x^{\Delta} := \sum_{i=1}^{n} \mu(u_i x) v_i)$ is an automorphism of Λ over the center Z of Λ, which is called *the Nakayama automorphism.*

[†] The detailed version of this paper has been submitted for publication elsewhere.

First, we construct a complete Λ^e-projective resolution of Λ. Let

$$(*)\qquad \ldots \to X_p \xrightarrow{d_p} X_{p-1} \to \ldots \to X_0 \xrightarrow{\epsilon} \Lambda \to 0$$

be the standard free resolution of Λ, where $X_p = \Lambda \otimes_R \ldots \otimes_R \Lambda$ $(p+2$ times). We regard $X_{q-1}^* = \mathrm{Hom}_R(X_{q-1}, R)$ (for $q \geq 1$) as a left Λ^e-module by setting $(x \otimes y^\circ \cdot f)(w) = f(y^\Delta wx)$ for $x \otimes y^\circ \in \Lambda^e, w \in X_{q-1}$ and $f \in X_{q-1}^*$, and denote it by X_{-q}. Then, taking the R-dual of $(*)$, we have the following complete Λ^e-projective resolution of Λ;

$$(**)\qquad \begin{array}{ccccccc} \ldots \to X_p \xrightarrow{d_p} \ldots \to & X_0 & \xrightarrow{d_0} & X_{-1} & \to \ldots \to X_{-q} \xrightarrow{d_{-q}} \ldots . \\ & \downarrow & & \uparrow & \\ & \Lambda & \xrightarrow{\varphi} & \Lambda^* & \end{array}$$

The sequence $(**)$ derives the *r-th complete cohomology group* $\hat{H}^r(\Lambda, A)$ of Λ, which is a Z-module, for any left Λ^e-module A and for every integer r. In particular, $\hat{H}^0(\Lambda, A) = A^\Lambda/N_\Lambda(A)$, $\hat{H}^{-1}(\Lambda, A) = {}_{N_\Lambda} A/I_\Lambda(A)$, where A^Λ denotes the commutator of Λ in A, $N_\Lambda(a) = \sum_{i=1}^n u_i a v_i$ for $a \in A$, ${}_{N_\Lambda}A = \{a \in A \mid N_\Lambda(a) = 0\}$ and $I_\Lambda(A) = \{\sum_k a_k \lambda_k^\Delta - \lambda_k a_k \mid \lambda_k \in \Lambda, a_k \in A\}$.

Next, we give a sufficient condition for A such that $\hat{H}^r(\Lambda, A)$ vanishes for any r. A left Λ^e-module A is called *weakly projective* if there exists $g \in \mathrm{Hom}_\Lambda^l(A, A)$ (or $g \in \mathrm{Hom}_\Lambda^r(A, A)$) such that $N_\Lambda(g) =$ *identity*. If A is a weakly projective module then we have $\hat{H}^r(\Lambda, A) = 0$. For example, $\Lambda \otimes_R B(\simeq \mathrm{Hom}_R(\Lambda_\Lambda, B_\Lambda))$ and $B \otimes_R \Lambda(\simeq \mathrm{Hom}_R({}_\Lambda\Lambda, {}_\Lambda B))$ are weakly projective for left Λ^e-modules B. Thus we can take A' (and A'') such that $\hat{H}^r(\Lambda, A)$ is isomorphic to $\hat{H}^{r+1}(\Lambda, A')$ (and $\hat{H}^{r-1}(\Lambda, A'')$). This is quoted as *dimension-shifting.*

1.2. Cup product.

We define a Z-homomorphism $\cup : \hat{H}^r(\Lambda, A) \otimes_Z \hat{H}^s(\Lambda, B) \to \hat{H}^{r+s}(\Lambda, A \otimes_\Lambda B)\,; \alpha \otimes \beta \mapsto \alpha \cup \beta$ by giving an explicit diagonal approximation $\Delta_{r,s} : X_{r+s} \to X_r \otimes_\Lambda X_s$ for every r, s. This map, called *a cup product,* has some properties. For example, in case of $r = 0$, the cup product coincides with a map $A^\Lambda/N_\Lambda(A) \otimes_Z B^\Lambda/N_\Lambda(B) \to (A \otimes_\Lambda B)^\Lambda/N_\Lambda(A \otimes_\Lambda B)$; $\bar{a} \otimes_Z \bar{b} \mapsto \overline{a \otimes_\Lambda b}$, and satisfies *anti-commutativity*; $\alpha \cup \beta = (-1)^{rs} \beta \cup \alpha$ for $\alpha \in \hat{H}^r(\Lambda, \Lambda), \beta \in \hat{H}^s(\Lambda, A)$ and *associativity*; $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$ for $\alpha \in \hat{H}^r(\Lambda, A), \beta \in \hat{H}^s(\Lambda, B)$ and $\gamma \in \hat{H}^t(\Lambda, C)$. Hence the cohomology ring $\hat{H}^*(\Lambda, \Lambda) = \sum_{r \in Z} \oplus \hat{H}^r(\Lambda, \Lambda)$ is defined.

1.3. Periodicity of the cohomology.

The existence of a cup product gives us the implications $(I) \Rightarrow (II) \Rightarrow (III)$ with respect to the following properties;

(I) for some $d \neq 0$ there is an element $\alpha \in \hat{H}^d(\Lambda, \Lambda)$ which is invertible in the ring $\hat{H}^*(\Lambda, \Lambda)$,

(II) there exists an integer $d \neq 0$ such that $\hat{H}^n(\Lambda, A) \simeq \hat{H}^{n+d}(\Lambda, A)$ as Z-modules for all left Λ^e-modules A and all integers n,

and

(III) for some $d \neq 0$, $\hat{H}^d(\Lambda, \Lambda) \simeq Z/N_\Lambda(\Lambda)$ as Z-modules.

If Λ has the property (II), then we say that Λ has *periodic cohomology* of period d.

Remark. We can take a common integer d in $(I), (II)$ and (III).

I do not know whether the implication $(III) \Rightarrow (I)$ is, in general, true or not. But we can prove it under a certain strong assumption.

Theorem. *Let R be a Noetherian integrally closed domain and let K be the quotient field of R. Suppose that* A *is a finite dimensional commutative separable K-algebra and Λ is the maximal R-order in* A *which is a free Frobenius R-algebra. Then (III) implies (I).*

Outline of the proof. By means of dimension-shifting, the proof is reduced to the assertion that a restriction map

$$\gamma : \mathrm{Hom}_{\Lambda^e}(M, \mathrm{A}/\Lambda)/N_\Lambda(\mathrm{Hom}_\Lambda^r(M, \mathrm{A}/\Lambda)) \rightarrow \mathrm{Hom}_\Lambda({}_{N_\Lambda}M/I_\Lambda(M), {}_{N_\Lambda}(\mathrm{A}/\Lambda)/I_\Lambda(\mathrm{A}/\Lambda))$$

is an isomorphism for every left Λ^e-module M.

Example. We put $\mathrm{A} = \mathrm{Q}(\sqrt{m})$. Let Λ be the ring of integers of A under the notations of Theorem. Then we have the isomorphism $\hat{H}^{-2}(\Lambda, \Lambda) \simeq \Lambda/N_\Lambda(\Lambda)$ as Λ-modules by direct calculations. This implies that $\hat{H}^*(\Lambda, \Lambda)$ has an invertible element of degree 2. In [B-F], it is, however, shown that Λ has periodic cohomology of period 2.

1.4. Restriction and Corestriction.

Let Γ, Λ be Frobenius R-algebras such that Γ/Λ is a Frobenius extension. Then two kind of homomorphisms; $Res : \hat{H}^r(\Gamma, A) \rightarrow \hat{H}^r(\Lambda, A)$ and $Cor : \hat{H}^r(\Lambda, A) \rightarrow \hat{H}^r(\Gamma, A)$ are defined for left Γ^e-modules A and for every integer r, which is called *a restriction map* and *a corestriction map* respectively. The map Res commutes with a modified cup product $\cup_t$: $\hat{H}^r(\Lambda, A) \otimes_{Z\Lambda} \hat{H}^s(\Lambda, B) \rightarrow \hat{H}^{r+s}(\Lambda, A \otimes_\Gamma B)$, namely $Res(\alpha \cup \beta) = Res(\alpha) \cup_t Res(\beta)$. Hence Res induces a ring homomorphism $\hat{H}^*(\Gamma, \Gamma) \rightarrow \hat{H}^*(\Lambda, \Gamma)$. And $Cor \cdot Res = N_{\Gamma/\Lambda}(1)(\in Z\Gamma)$ holds.

§2. Applications.

2.1. Cohomology of crossed product algebras.

Let R be a commutative ring and Λ a Frobenius commutative R-algebra with no zero-divisor. Let G be a finite group of automorphisms of Λ over R with $\Lambda^G = R$. We consider a crossed product $\Gamma = (\Lambda, G, \theta) = \sum_{\sigma \in G} \oplus \Lambda w_\sigma$ with any normalized factor set $\theta : G \times G \to U(\Lambda)$; $w_\sigma \lambda = \sigma(\lambda) w_\sigma$, $w_\sigma w_\tau = \theta(\sigma, \tau) w_{\sigma\tau}$ for $\lambda \in \Lambda$ and for $\tau, \tau \in G$. Then Γ/Λ is a Frobenius extension and the center of Λ coincides with R. A map γ_σ : $\hat{H}^r(\Lambda, A) \to \hat{H}^r(\Lambda, A)$ is defined for left Γ^e-modules A and for $\sigma \in G$, which coincides with a map $A^\Lambda / N_\Lambda(A) \to A^\Lambda / N_\Lambda(A)$; $\overline{a} \mapsto \overline{w_\sigma a w_\sigma^{-1}}$ in case of $r = 0$. This map is called *a conjugation map*. The conjugation map has the properties that $\gamma_\sigma \gamma_\tau = \gamma_{\sigma\tau}, \gamma_1 =$ *identity* and $\gamma_\sigma(\alpha \cup_t \beta) = \gamma_\sigma(\alpha) \cup_t \gamma_\sigma(\beta)$ for the modified cup product $\cup_t$. It follows that γ_σ is a ring automorphism on $\hat{H}^*(\Lambda, \Gamma)$.

In the followings, we set $R = \mathbf{Z}$. If $N_\Gamma(\Gamma) = z\mathbf{Z}$ then, since z annihilates $\hat{H}^r(\Gamma, A)$, we have a primary decomposition; $\hat{H}^r(\Gamma, A) = \bigoplus_{p|z} \hat{H}^r(\Gamma, A)_{(p)}$. Let p be a prime divisor of z which does not divides $|G|$. Then the endomorphism $Cor \cdot Res = |G|$ on $\hat{H}^r(\Gamma, A)_{(p)}$ is an isomorphism. It follows that $Res : \hat{H}^r(\Gamma, A)_{(p)} \to \hat{H}^r(\Lambda, A)_{(p)}$ is a monomorphism. Moreover, indeed, we have an isomorphism;

$$Res : \hat{H}^r(\Gamma, A)_{(p)} \simeq [\hat{H}^r(\Lambda, A)_{(p)}]^G \stackrel{\text{def}}{=} \{\alpha \in \hat{H}^r(\Lambda, A)_{(p)} \mid \gamma_\sigma(\alpha) = \alpha \text{ for all } \sigma \in G\}$$

The elements of the right hand side are called *G-invariant*. This map Res derives a ring isomorphism $Res : \hat{H}^*(\Gamma, \Gamma)_{(p)} \xrightarrow{\sim} [\hat{H}^*(\Lambda, \Gamma)_{(p)}]^G$.

Theorem. *Let p be a prime integer such that $p \mid z$ and $p \nmid |G|$. Assume that the group of unit elements in $(\Lambda/N_\Lambda(\Gamma))_{(p)}$ is a finite group. Then the graded ring $\hat{H}^*(\Gamma, \Gamma)_{(p)}$ has an invertible element of non-zero degree if and only if so does $\hat{H}^*(\Lambda, \Gamma)_{(p)}$. Hence, in such a case, $\hat{H}^r(\Gamma, A)_{(p)}$ is periodic.*

Example. Let Λ be the ring of integers of $\mathbf{Q}(\sqrt{m})$ where m is a square-free integer. G denotes the Galois group of $\mathbf{Q}(\sqrt{m})/\mathbf{Q}$. We set $\Gamma = \Lambda 1 \oplus \Lambda w_\sigma; w_\sigma^2 = \pm 1$. Note that $N_\Gamma(\Gamma) = 4m\mathbf{Z}$ (if $m \equiv 2, 3 \pmod 4$), $= m\mathbf{Z}$ (if $m \equiv 1 \pmod 4$). Let p be a prime integer such that $p \mid m$ and $p \neq 2$. It follows from Example in §1.3 that $\hat{H}^*(\Lambda, \Gamma)_{(p)}$ has an invertible element of degree 2. Moreover, it is verified that the element is G-invariant. This says that $\hat{H}^*(\Gamma, \Gamma)_{(p)}$ has an invertible element of degree 2. In particular, in case of $m \equiv 1 \pmod 4$, so does $\hat{H}^*(\Gamma, \Gamma)$ itself. It is, however, derived from [B] that these cohomologies of Γ are periodic of period 2.

2.2. Cohomology of twisted group algebras.

Let R be a commutative ring, G a finite group and θ a normalized factor set; θ : $G \times G \to U(R)$. We consider a twisted group algebra $\Gamma = R_\theta G = \sum_{\sigma \in G} \oplus R w_\sigma$; $w_\sigma w_\tau =$

$\theta(\sigma,\tau)w_{\sigma\tau}, w_\sigma r = rw_\sigma$ for $\sigma,\tau \in G$ and for $r \in R$. Let $\Lambda = R_\theta H$ be a subalgebra of Γ for a subgroup H of G. Then Γ/Λ is a Frobenius extension. $\hat{H}^r(\Lambda, A)$ is a two-sided Γ^Λ-module and $\hat{H}^*(\Lambda,\Gamma)$ is a graded ring under the modified cup product $\cup_t$. A map $\gamma_\sigma : \hat{H}^r(\Lambda, A) \to \hat{H}^r(\Lambda^\sigma, A)$ is defined for left Γ^e-modules A and for $\sigma \in G$ where $\Lambda^\sigma = w_\sigma \Lambda w_\sigma^{-1}$, which coincides with a map $A^\Lambda/N_\Lambda(A) \to A^{\Lambda^\sigma}/N_{\Lambda^\sigma}(A); \bar{a} \mapsto \overline{w_\sigma a w_\sigma^{-1}}$ in case of $r = 0$. This map is called *a conjugation map*. The conjugation map has the properties that $\gamma_\sigma\gamma_\tau = \gamma_{\sigma\tau}, \gamma_\sigma =$ *identity* for $\sigma \in H$, $\gamma_\sigma(\alpha \cup_t \beta) = \gamma_\sigma(\alpha) \cup_t \gamma_\sigma(\beta)$ and so on.

Since $N_\Gamma(\Gamma) = |G| \cdot R$, we have a primary decomposition; $\hat{H}^r(\Gamma, A) = \bigoplus_{p||G|} \hat{H}^r(\Gamma, A)_{(p)}$. We have an isomorphism;

$$Res : \hat{H}^r(\Gamma, A)_{(p)} \xrightarrow{\sim} [\hat{H}^r(\Lambda(P), A)]^G \stackrel{\text{def}}{=} \{\alpha \in \hat{H}^r(\Lambda(P), A) \mid Res^\Lambda_{\Lambda\cap\Lambda^\sigma}(\alpha) = Res^{\Lambda^\sigma}_{\Lambda\cap\Lambda^\sigma} \cdot \gamma_\sigma(\alpha) \text{ for all } \sigma \in G\}$$

for every p-Sylow subgroup P of G, where $\Lambda = \Lambda(P) = R_\theta P$. Hence we obtain a ring isomorphism $Res : \hat{H}^*(\Gamma, \Gamma)_{(p)} \xrightarrow{\sim} [\hat{H}^*(\Lambda(P), \Gamma)]^G$.

Theorem. *Assume that the group of unit elements of $\Gamma^{\Lambda(P)\cap\Lambda(P^g)}/N_{\Lambda(P)\cap\Lambda(P^g)}(\Gamma)$ is a finite group for each Sylow subgroup P and for $g \in G$. Then the graded ring $\hat{H}^*(\Gamma,\Gamma)$ has an invertible element of non-zero degree if and only if so does $\hat{H}^*(\Lambda(P),\Gamma)$ for each Sylow subgroup P of G. Hence, in such a case, $\hat{H}^r(\Gamma, A)$ is periodic.*

Examples. 1. Let Λ be a twisted group algebra of a cyclic group. Then it is verified that $\hat{H}^*(\Lambda,\Lambda)$ has an invertible element of degree 2. So, by the above theorem, $\hat{H}^*(\Gamma,\Gamma)$ has an invertible element of non-zero degree for a twisted group algebra Γ of a finite group whose every Sylow subgroup is cyclic.

2. More precisely, above G is a semi-direct product $N \rtimes H$ with $N = <\tau>, H = <\sigma>$ and $(|N|,|H|) = 1$. We set $\sigma\tau\sigma^{-1} = \tau^r$ for some integer r. Let k be the order of the subgroup $<r>$ of $(\mathbf{Z}/|N|\mathbf{Z})^*$. Then we see that $\hat{H}^*(\Gamma,\Gamma)$ has an invertible element of degree $2k$.

§3. Cohomology of the quaternion algebra over Z.

3.1. Spectral sequence.

Let G be a finite group, N a normal subgroup of G and θ a normalized factor set of G; $\theta : G \times G \to \{\pm 1\}$. We set $\Gamma = \mathbf{Z}_\theta G = \sum_{\sigma\in G} \oplus \mathbf{Z}w_\sigma$. The conjugation map $\gamma_\sigma : H^r(\Lambda, A) \to H^r(\Lambda, A)$ for $r \geq 1$ on the ordinary cohomology groups is defined in §2.2. In case of $r = 0$, we define $\gamma_\sigma : A^\Lambda \to A^\Lambda$ by $(a \mapsto w_\sigma a w_\sigma^{-1})$. Then, since $\gamma_\sigma\gamma_\tau = \gamma_{\sigma\tau}$,

$\gamma_\sigma = \textit{identity}$ for $\sigma \in N$, it follows that $H^r(\Lambda, A)$ for $r \geq 0$ is a G/N-module. We consider a double complex;

$$M_{p,q} = \mathrm{Hom}_{Z(G/N)}(Z_p, \mathrm{Hom}_{\Lambda \otimes \Gamma^\circ}((X_\Gamma)_q, A)).$$

In the above, Z_p denotes a G/N-projective resolution of Z and $(X_\Gamma)_q$ the standard resolution of Γ. Then we obtain the spectral sequence of Hochschild-Serre type;

$$E_2^{p,q} = H^p(G/N, H^q(\Lambda, A)) \underset{p}{\Rightarrow} H^{p+q}(\Gamma, A).$$

3.2. Quaternion algebra.

We set $G = < x > \times < y >$ and $N = < x >$ with $x^2 = y^2 = 1$ in §3.1. We consider a twisted group algebra $\Gamma = Z_\theta G$ with the factor set θ given by the following table, which is *the quaternion algebra* over Z.

\	1	x	y	xy
1	1	1	1	1
x	1	-1	1	-1
y	1	-1	-1	1
xy	1	1	-1	-1

In the above, for example, the (2, 3)-entry represents the value $\theta(x, y)$. Then $M_{p,q}$ is quasi-isomorphic to a certain double complex which is appropriate to calculating indeed the homology of the total complex. In particular, in case of $A = \Gamma$, we have

$$H^n(\Gamma, \Gamma) \simeq \begin{cases} Z & \text{for } n = 0, \\ Z/2Z \oplus \ldots \oplus Z/2Z \quad (2n+1 \text{ times}) & \text{for } n \geq 1. \end{cases}$$

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ON THE INVARIANT SUBRING UNDER THE ACTION OF A FINITE POSET

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0. Introduction. The purpose of this note is to introduce the notion of a P-Galois extension of a ring for a finite poset P which we call a relative sequence of homomorphisms. We can see that a finite group of automorphisms, some type of semigroup of derivations and of higher derivations [cf. [1], p.191] are relative sequences of homomorphisms respectively. Hence, a P-Galois extension is a generalized notion of Galois extensions of separable type [cf. [2]] and of purely inseparable type [cf. [4]].

1. A relative sequence of homomorphisms. Let B be a ring with an identity 1, A a subring of B with common identity 1 of B, and let $P = (P, \leq)$ be a finite poset of $End(B_A)$.

By $P(min)$ (resp. $P(max)$), we denote the set of all $\Omega \in P$ such that Ω is a minimal (resp. maximal) element of P. For an element $\Omega \in P$, $\Lambda \in P(min)$ is said to be a minimal element of Ω if $\Omega > \Lambda$, and $\Delta \in P(max)$ is said to be a maximal element of Ω if $\Delta > \Omega$.

$$\Omega = \Omega_0 > \Omega_1 > \cdots > \Omega_m; \Omega_m \in P(min)$$

is said to be a chian of Ω of the length m + 1 if Ω_{i-1} is a cover of Ω_i, for i = 1,2,...,m, that is, $\Omega_{i-1} > \Omega_i$ and there is no Γ such that $\Omega_{i-1} > \Gamma > \Omega_i$.

Let us consider a finite poset P which satisfies the following conditions (A.1) - (A.4) and (B.1) - (B.4).

The detailed version of this note will be submitted for publication elsewhere.

(A.1) $\Omega \neq 0$ for all $\Omega \in P$ and $P(min) = \{\Omega \in P | \Omega$ is a ring automorphism $\}$

(A.2) For $\Omega \in P$, the length of each chain of Ω is unique. We denote the length by $ht(\Omega)$.

(A.3) For $\Omega, \Gamma \in P, \Omega\Gamma \in P$ if $\Omega\Gamma \neq 0$, and if $\Omega\Gamma = 0$ then $\Gamma\Omega = 0$.

(A.4) Assume $\Omega\Gamma_1 \in P$ and $\Omega\Gamma_2 \in P$.

(i) $\Omega\Gamma_1 \geq \Omega\Gamma_2$ (resp. $\Gamma_1\Omega \geq \Gamma_2\Omega$) if and only if $\Gamma_1 \geq \Gamma_2$.

(ii) $\Omega\Gamma \geq \Lambda$ if and only if $\Lambda = \Omega_0\Gamma_0$ for some $\Omega_0 \leq \Omega$ and $\Gamma_0 \leq \Gamma$.

(B.1) $\Omega(1) = 0$ for all $\Omega \in P - P(min)$.

(B.2) For $\Omega \in P$, there exist $g(\Omega, \Gamma) \in End(B_A)$ for all $\Gamma \leq \Omega$ such that

$\Omega(xy) = \Sigma_{\Gamma \leq \Omega} g(\Omega, \Gamma)(x)\Gamma(y)$

for $x, y \in B$, where the sum $\Sigma_{\Gamma \leq \Omega}$ means the sum of all Γ such that $\Gamma \leq \Omega$.

(B.3) (i) $g(\Omega, \Lambda)(xy) = \Sigma_{\Lambda \leq \Gamma \leq \Omega} g(\Omega, \Gamma)(x) g(\Gamma, \Lambda)(y)$ for $x, y \in B$ where the sum $\Sigma_{\Lambda \leq \Gamma \leq \Omega}$ means the sum of all Γ such that $\Lambda \leq \Gamma \leq \Omega$.

(ii) For $\Omega, \Lambda \in P$ such that $\Gamma = \Omega_0\Lambda_0$ for some $\Omega_0 \leq \Omega$ and $\Lambda_0 \leq \Lambda$

$\Sigma_{\Omega', \Lambda'}(\Sigma_{\Omega''} g(\Omega, \Omega'')(x) g(\Omega'', \Omega')(g(\Lambda, \Lambda')(y))$

$= \Sigma_{\Omega', \Lambda'}(\Sigma_{\Omega''} g(\Omega, \Omega'')(x) g(\Omega''\Lambda, \Omega'\Lambda')(y))$

for $x, y \in B$, where $\Sigma_{\Omega', \Lambda'}$ means the sum of all Ω', Λ' such that $\Omega' \leq \Omega, \Lambda' \leq \Lambda$ and $\Omega'\Lambda' = \Gamma$, and $\Sigma_{\Omega''}$ means the sum of all Ω'' such that $\Omega' \leq \Omega'' \leq \Omega$ for each Ω'.

(B.4) (i) $g(\Omega, \Omega)$ is a ring automorphism for each $\Omega \in P$.

(ii) $g(\Omega, \Lambda) = \Omega$ for all $\Lambda \in P(min)$.

(iii) $g(\Omega, \Gamma)(1) = 0$ for $\Gamma < \Omega$.

A finite poset P which satisfies conditions (A.1) - (A.4) and (B.1) - (B.4) is called **a relative sequence of homomorphisms** (abbreviate r.s.h) of B. In this case $P(min)$ forms a multiplicative group of $End(B_A)$, and hence P contains the identity map 1.

In what follows, we assume that P is a r.s.h, and use the following notations:

$T := \Sigma_{\Lambda \in P(min)} \Lambda$.

Let $P(max) = \{\Delta_1, \Delta_2, ..., \Delta_k\}$ and $\Delta_1 > 1$.

$\Delta := \Sigma_{i=1}^{k} \Delta_i$

If we define the relation $\Delta_i \sim \Delta_j \Leftrightarrow \Lambda\Delta_i = \Delta_j$ for some $\Lambda \in P(min)$, then the relation $\sim$ is an equivalence relation of $P(max)$ and we may understand that $P(max)/\sim$, the classification of $P(max)$, is $\{[\Delta_1], [\Delta_2], ..., [\Delta_h], h \leq k\}$ where $[\Delta_i]$ is the coset of Δ_i.

For a system of representatives $\Delta_1, \Delta_2, ..., \Delta_h$, we have $T\Delta_i \neq T\Delta_j$ if $i \neq j$ and

$\Delta = \Sigma_{i=1}^{h} T\Delta_i$.

In the rest, we assume that $\Delta = \Sigma_{i=1}^{h} T\Delta_i$ for $P(max) = \{\Delta_1, \Delta_2, ..., \Delta_k\}$.

For $\Lambda\Delta_i$, we put $g(\Lambda\Delta_i, \Gamma) = 0$, if $\Lambda\Delta_i$ is not a maximal element of Γ and we put

$g(T\Delta_i, \Gamma) = \Sigma_{\Lambda \in P(min)} g(\Lambda\Delta_i, \Gamma)$, and

$g(\Delta, \Gamma) = \Sigma_{i=1}^{h} g(T\Delta_i, \Gamma)$.

Let $B^P = B_1 \cap B_0$ where $B_1 = B^{P(min)} = \{b \in B | \Lambda(b) = b$ for all $\Lambda \in P(min)\}$ and $B_0 = \{b \in B | \Omega(b) = 0$ for all $\Omega \in P - P(min)\}$. Then we can easily see that B^P is a subring of B with an identity 1 of B, and we call that B^P is a P-invariant subring of B.

2. Trivial crossed product of P over B. Let $D = D(B, P) = \Sigma_{\Omega \in P} \oplus Bu_\Omega$ be a free left B-module with a B-basis $\{u_\Omega : \Omega \in P\}$. Then D becomes a right B-module via

$u_\Omega \cdot b := \Sigma_{\Gamma \leq \Omega} g(\Omega, \Gamma)(b) u_\Gamma$

Then we have the following

Theorem 2. 1. (1) $D(B, P)$ is a free right B-module with a B-basis $\{u_\Omega | \Omega \in P\}$.

(2) $D(B,P)$ forms a ring under the multiplication defined by
$(au_\Omega)(bu_\Lambda) = \Sigma_{\Gamma\leq\Omega} ag(\Omega,\Gamma)(b)u_{\Gamma\Lambda}$ where $u_{\Gamma\Lambda} = 0$ if $\Gamma\Lambda = 0$.
(3) The map j of $D(B,P)$ to $End(B_A)$ defined by
$j(u_\Omega b)(x) := \Omega(bx)$
is a ring homomorphism.

In the rest of this note, we assume following two conditions on P.
(1) P is a pure poset, that is, $ht(\Delta_i)$ is same for all $\Delta_i \in P(max)$.
(A.6) If Δ_i is a maximal element of Γ, then there exist Γ_i and Γ_i' in P such that $\Delta_i = \Gamma_i\Gamma = \Gamma\Gamma_i'$.

Theorem 2. 2. Let $B^P = A$ and j is an isomorphism. Then
$j(\Sigma_{i=1}^h (T\Delta_i \cdot B)) = Hom(B_A, A_A) = B^*$.
Moreover, if this is the case,
$B_A \oplus > A_A$ if and only if there exist $x_1, x_2, ..., x_h \in B$ such that $\Sigma_{i=1}^h T\Delta_i(x_i) = 1$.

3. P-Galois extensions and P-Galois systems.

Definition 3. 1. B/A is said to be a P-Galois extension if
(1) $B^P = A$
(2) B_A is a f.g. projective module and j is an isomorphism.

Theorem 3. 1. If B/A is a P-Galois extension, then there exists a system $\{x_i, y_{it}; i = 1,2,...,s \text{ and } t = 1,2,...,h\} \subseteq B$ such that
$\Sigma_{i=1}^s x_i(\Sigma_{t=1}^h g(T\Delta_t, \Gamma)(y_{it})) = \delta_{1,\Gamma}$
for all $\Gamma \in P$.
Moreover if this is the case
$\Sigma_{i=1}^s \Omega(x_i)(\Sigma_{t=1}^h g(T\Delta_t, \Gamma)(y_{it})) = \delta_{\Omega,\Gamma}$
for all $\Omega \in P$.

Definition 3. 2. For an element $\Omega \in P$, a system $\{x_i, y_{it}; i = 1, 2, ..., s \text{ and } t = 1, 2, .., h\} \subseteq B$ such that

$\Sigma_{i=1}^{s} x_i(\Sigma_{t=1}^{h} g(T\Delta_t, \Gamma)(y_{it})) = \delta_{\Omega,\Gamma}$

is called a (P, Ω)-system. In particular, the system is called a P-Galois system when $\Omega = 1$.

Theorem 3. 2. Let $B^P = A$. Then B/A is a P-Galois extension if and only if there exists a P-Galois system.

4. A r.s.h satisfying (A.5). In this section we assume that a r.s.h P satisfies the following condition

(A.5) $|P(min)| = |P(max)|$.

In this case $P(max)$ is obtained by $\{\Lambda\Delta_i; \Lambda \in P(min)\}$, and hence,

$\Delta = T\Delta_1$.

Corollary 4. 1. Let $B^P = A$. Then B/A is a P-Galois extension if and only if there exists a P-Galois system $\{x_i, y_i; i = 1, 2, ..., s\}$ such that $\Sigma_{i=1}^{s} g(\Delta, \Gamma)(y_i) = \delta_{1,\Gamma}$.

B/A is called a **projective Frobenius extension** if B_A is a f.g. projective module and ${}_AB_B \cong_A B_B^*$ [cf. [3], p.121].

If B/A is a P-Galois extension, then $b \longrightarrow u_\Delta b \longrightarrow j(u_\Delta b)$ gives an isomorphism ${}_AB_B \cong_A u_\Delta B_B \cong_A B_B^*$. Thus we have

Theorem 4. 2. A P-Galois extension is a projective Frobenius extension.

Let $P_1 = \{\Omega \in P | \Omega \leq \Delta_1\}$. Since any element Ω of P is obtained by $\Lambda\Omega_1$ for some $\Lambda \in P(min)$ and $\Omega_1 \in P_1$, we can see that

$\{b \in B | \Omega(b) = 0 \text{ for all } \Omega \in P_1 - P(min)\} = B_0$.

Further, if the number m_{Δ_1} of minimal elements of Δ_1 is 1, then P_1 becomes a r. s. h, and $B^{P_1} = B_0$.

Theorem 4. 3. Assume B/A is a P-Galois extension and $m_{\Delta_1} = 1$. Then

(1) B/B_0 is a $P(min)$-Galois extension.

Moreover if we added the assumption that $g(\Delta_1, \Delta_1) = 1$, then

(2) B/B_1 is a $P(min)$-Galois extension

(3) $B = B_0[B_1]$.

Theorem 4. 4. Assume $B^P = A, m_{\Delta_1} = 1$ and $g(\Delta_1, \Delta_1) = 1$. If B_0/A is a $P(min)$-Galois extension and B_1/A is a P_1-Galois extension, then B/A is a P-Galois extension.

Theorem 4. 5. Assume $B^P = A, m_{\Delta_1} = 1, g(\Delta_1, \Delta_1) = 1$ and $P(min) \subsetneq$ the center of P. Then B/A is a P-Galois extension with $B_A \oplus > A_A$ if and only if B_0/A is a $P(min)$-Galois extension with ${B_0}_A \oplus > A_A$ and B_1/A is a P_1-Galois extension with ${B_1}_A \oplus > A_A$.

Remark. If B/A is a commutative P-Galois extension, then $B_A \oplus > A_A$.

5. Examples. Let A be a commutative ring of characteristic 3, and let $B \cong A[X]/(X^3 - X - a)$. Then we may write $B = A[x] = A \oplus xA \oplus x^2 A$ with $x^3 = x + a$.

Then the map τ of B defined by

$$\tau : xa \longrightarrow (x+1)a$$

is an A-ring automorphism of B.

(1) Let $P_1 = \{1, \tau, \tau^2\}$ be a clutter (P_1 is a poset such that distinct elements are imcomparable). Then P_1 is a r.s.h with $P_1 = P_1(min) = P_1(max)$ and $g(\tau^i, \tau^i) = \tau^i$ for $i = 0, 1, 2$.

(2) Let $D = \tau - 1$. Then D is a $(\tau, 1)$-derivation with $D^3 = 0$. Then

$$P_2 = \{1, D, D^2\}$$

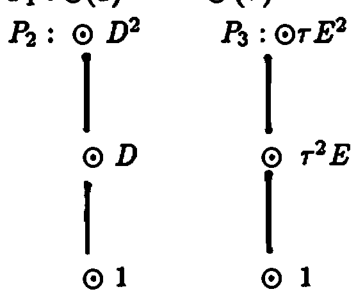
becomes a r.s.h by the ordering $D^2 > D > 1$, and $g(D^2, D^2) = \tau^2, g(D^2, D) = 2\tau D, g(D^2, 1) = D^2, g(D, D) = \tau, g(D, 1) = D$ and $g(1, 1) = 1$, and hence $P_2(min) = \{1\}$ and $P_2(max) = \{D^2\}$.

(3) Let $E = 1 - \tau$. Then E is a $(1, \tau)$-derivation with $E^3 = 0$. Then

$$P_3 = \{1, \tau^2 E, \tau E^2\}$$

becomes a r.s.h by the ordering $\tau E^2 > \tau^2 E > 1$ and $g(\tau E^2, \tau E^2) = \tau, g(\tau E^2, \tau^2 E) = 2\tau E, g(\tau E^2, 1) = \tau E^2, g(\tau^2 E, \tau^2 E) = \tau^2, g(\tau^2 E, 1) = \tau^2 E$ and $g(1, 1) = 1$. The Hasse diagram of P_i are followings:

$P_1 : \odot(1) \cdots\cdots \odot(\tau) \cdots\cdots \odot(\tau^2)$



We can easily see that $B^{P_1} = B^{P_2} = B^{P_3} = A$.

Theorem 5. 1. B/A is a P_i-Galois extension (of separable type) for each $i = 1, 2, 3$.

Proof. It suffices to show the existence of a P_i-Galois system.

(i) The case P_1: Since $\tau(x) = x + 1$, we have

$-1 \cdot \tau^i(x^2) - x\tau^i(x) + (1 - x^2)\tau^i(1) = \delta_{1,\tau^i}$.

This shows that B has a P_1-Galois system.

(ii) The case P_2: Since $D^2(-x^2) = -(\tau(x)D(x) + D(x)x) = -D(2x + 1) = 1$,

$-1g(D^2, \Omega)(x^2) - xg(D^2, \Omega)(x) - (x^2 - 1)g(D^2, \Omega)(1) = \delta_{1,\Omega}$

This shows that B has a P_2-Galois system.

(iii) The case P_3: Since $\tau E^2(-x^2) = -\tau E(2x + 1) = 1$, we have

$-1g(\tau E^2, \Omega)(x^2) + xg(\tau E^2, \Omega)(x) + xg(\tau E^2, \Omega)(1) = \delta_{1,\Omega}$

This shows that B has a P_3-Galois system.

Let $B_1 \cong A[X]/(X^3 - a)$. Then we may write $B_1 = A[x] = A \oplus xA \oplus x^2A$ with $x^3 = a$. Then the map D defined by

$D : \Sigma_{i=0}^2 x^i a_i \longrightarrow \Sigma_{i=1}^2 i x^{i-1} a_i$

is a derivation of B such that $D^3 = 0$.

Then $P_1 = \{1, D, D^2\}$ becomes a r.s.h by the ordering $D^2 > D > 1$ and $g(D^2, D^2) = 1, g(D^2, D) = 2D, g(D^2, 1) = D^2, g(D, D) = 1, g(D, 1) = D$ and $g(1, 1) = 1$. Thus $P_1(min) = \{1\}$ and $P_1(max) = \{\Delta = D^2\}$ and $B^{P_1} = A$. Since

$-1 \cdot g(D^2, D^i)(x^2) - x \cdot g(D^2, D^i)(x) - x^2 \cdot g(D^2, D^i)(1) = \delta_{1,D^i}$

B_1/A is a P_1-Galois extension.

Next, let $B_2 \cong A[Y]/(Y^2 - u)$ for some unit $u \in A$. Then we may write $B_2 = A[y] = A \oplus yA$ with $y^2 = u$. Then the map defined by

$\sigma : ya_1 + a_0 \mapsto (-y)a_1 + a_0$

is an automorphism such that $\sigma^2 = 1$. Then the clutter $P_2 = \{1, \sigma\}$ is a r.s.h and $B_2^{P_2} = A$.

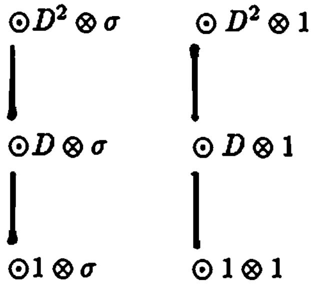
Since $\Delta = 1 + \sigma$, $-yg(\Delta, \sigma^i)(y^{-1}) - 1g(\Delta, \sigma^i)(1) = \delta_{1,\sigma^i}$, and hence B_2/A is a P_2-Galois extension.

Let $B = B_1 \otimes_A B_2$ and $P = P_1 \otimes P_2 = \{D^i \otimes \sigma^j; i = 0, 1, 2$ and $j = 0, 1\}$. Then $P \in End(B_A)$ by $(D^i \otimes \sigma^j)(x^s \otimes y^t) = D^i(x^s) \otimes \sigma^j(y^t)$

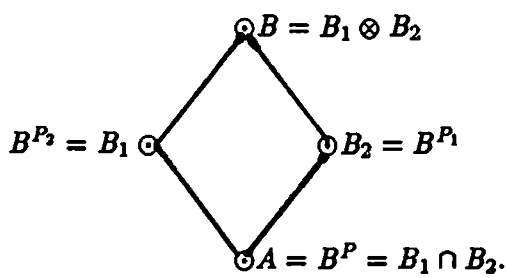
and P becomes a r.s.h with $P(min) = \{1 \otimes 1, 1 \otimes \sigma\}$ and $P(max) = \{D^2 \otimes 1, D^2 \otimes \sigma\}$ by the ordering

$D^i \otimes \sigma^j \geq D^k \otimes \sigma^h$ if and only if $D^i \geq D^k$ and $\sigma^j = \sigma^h$ and $g(D^i \otimes \sigma^j, D^k \otimes \sigma^h) = g(D^i, D^k) \otimes \sigma^j$.

The Hasse diagram of P is the following:



Further



Theorem 5. 2. (1) B/A is a P-Galois extension.

(2) B/B_1 is a P_2-Galois extension and B_1/A is a P_1-Galois extension.

(3) B/B_2 is a P_1-Galois extension and B_2/A is a P_2-Galois extension.

Proof. (1) $(-1 \otimes y)g(\Delta,\Omega)(x^2 \otimes y^{-1}) + (x \otimes y)g(\Delta,\Omega)(x \otimes y^{-1})$
$+(x^2 \otimes y)g(\Delta,\Omega)(1 \otimes y^{-1}) + (1 \otimes 1)g(\Delta,\Omega)(x^2 \otimes 1)$
$+(x \otimes 1)g(\Delta,\Omega)(x \otimes 1) + (x^2 \otimes 1)g(\Delta,\Omega)(1 \otimes 1) = \delta_{1,\Omega}$
shows that B/A is a P-Galois extension.

(2) A P_2-Galois system for B_2/A is a P_2-Galois system for B/B_1.

(3) A P_1-Galois system for B_2/A is a P_1-Galois system for B/B_2.

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Totally real algebras

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1. Introduction. Let K be a commutative ring with identity 1. We consider an ordering P on K, i.e. (K, P) is a partially ordered ring which has a positive cone $P^{\cdot}$ satisfying the conditions $1 \in P^{\cdot}$, $P^{\cdot} \cdot P^{\cdot} \subseteq P^{\cdot}$ and $P^{\cdot} \cap -P^{\cdot} = \phi$. As a maximal one of orderings under some condition, the following notion was defined in [K₃]:

Let F be a commutative multiplicative semi-group which has zero element 0, unit element 1 and a unique element -1 of order 2, and suppose that $F^{\cdot} = \{\alpha \in F \mid \alpha \neq 0\}$ becomes a group. For a subset K_{σ} of K, let $\sigma : K_{\sigma} \to F$ be a map satisfying the following conditions:

(1) $-1 \in K_{\sigma}$ and $\sigma(-1) = -1$,

(2) for any $a, b \in K_{\sigma}$, $ab \in K_{\sigma}$ and $\sigma(ab) = \sigma(a)\sigma(b)$,

(3) for $a, b \in K_{\sigma}$, either $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ implies $a + b \in K_{\sigma}$ and $\sigma(a+b) = \sigma(b)$,

(4) if $a \in K$ and $a \notin K_{\sigma}$, there exists $b \in K_{\sigma}$ with $\sigma(b) = 0$ and $\sigma(ab) = 1$.

Then, $P_{\sigma} = \{x \in K \mid \sigma(x) = 0 \text{ or } 1\}$ is an ordering of K with $P_{\sigma}^{\cdot} = \{x \in K \mid \sigma(x) = 1\}$. If K is a field, $(K_{\sigma}, \mathfrak{p}_{\sigma})$ is a valuation ring of K, where $\mathfrak{p}_{\sigma} = P_{\sigma} \cap -P_{\sigma}$ is the prime ideal of K_{σ}.

From now, we consider a simple case; $F = GF(3) = \{0, 1, -1\}$ and $K = K_{\sigma}$, then the map $\sigma : K \to F = \{0, 1, -1\}$ satisfies the following conditions: (1) $\sigma(-1) = -1$, (2) for any $a, b \in K$, $\sigma(ab) = \sigma(a)\sigma(b)$, and (3) for any $a, b \in K$, if either $\sigma(a) = 0$ or $\sigma(a) = \sigma(b)$ then $\sigma(a+b) = \sigma(b)$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Such a subset $P=\{x\in K \mid \sigma(x)=0 \text{ or } 1\}$ of K satisfies conditions:
(1°) $P: P\subseteq P$, (2°) $P\cup -P=K$, (3°) $P\cap -P=\mathfrak{p}_P$ is a prime ideal of K.
If P is a subset of K satisfying (1°), (2°) and (3°), then $(K/\mathfrak{p}_P, P/\mathfrak{p}_P)$ is a totally ordered domain, and the quotient field of $(K/\mathfrak{p}_P, P/\mathfrak{p}_P)$ is a totally ordered field $(\kappa_P, \bar{P})$. We denote by R_P a real closure of $(\kappa_P, \bar{P})$. Then, $R_P(\sqrt{-1})$ is an algebraic closure of κ_P. The canonical map $K\to K/\mathfrak{p}$ and inclusion maps $K/\mathfrak{p}_P \hookrightarrow \kappa_P$ and $\kappa_P \hookrightarrow R_P$, induce ring homomorphisms of Witt rings (signatures) $\psi_P: W(K)\longrightarrow W(R_P)=\mathbb{Z}$ and $\overline{\psi}_P: W(\kappa_P)\longrightarrow W(R_P)=\mathbb{Z}$.

(1.1) ([D], [K, K]) There are bijections
$\{\sigma: K\to F \mid (1), (2) \text{ and } (3) \text{ hold}\} \rightarrowtail\!\!\!\twoheadrightarrow \{P\subseteq K \mid (1^\circ), (2^\circ), (3^\circ) \text{ hold}\}$; $\sigma \rightsquigarrow P$, and
$\{P\subseteq K \mid (1^\circ), (2^\circ), (3^\circ) \text{ hold}\} \rightarrowtail\!\!\!\twoheadrightarrow \{\psi: W(K)\to \mathbb{Z} \mid \text{ring homomorphism}\}$; $P \rightsquigarrow \psi_P$.

Definition. Put <u>$\mathrm{Sig}(K)$</u>$:=\{P\subseteq K \mid (1^\circ), (2^\circ) \text{ and } (3^\circ) \text{ hold}\}$ and $Q(K):=\bigcap_{P\in \mathrm{Sig}(K)} P$, and elements P of $\mathrm{Sig}(K)$ are called <u>orderings of K</u>.

(1.2) ([K_3]; Theorem 13) $Q(K)$ is characterized as follows:
Put $T_0=\Sigma\{x^2 \mid x\in K\}$, and inductively, for $i=0, 1, 2, \cdots$,
$T_{2i+1}=\Sigma\{x\in K \mid \exists n\geqq 0,\ \exists t\in T_{2i}: x(x^{2n}+t)\in T_{2i}\}$ and
$T_{2(i+1)}=\Sigma\{x\in K \mid \exists n\geqq 0,\ \exists t\in T_{2i+1}: xt-x^{2n}\in T_{2i+1}\}$, where $\Sigma\{\cdots\}$ denotes the set of all finite sums of elements in $\{\cdots\}$. Then, $T_0\subseteq T_1\subseteq T_2\subseteq \cdots$ and $Q(K)=\bigcup_{i=0}^{\infty} T_i$ hold.

Let A be a K-algebra such that A is a finitely generated projective K-module with a projective dual basis $b_1, b_2, \ldots, b_n (\in A)$ and $f_1, f_2, \ldots, f_n$ $(\in \mathrm{Hom}_K(A, K))$ satisfying $x=\Sigma_{i=1}^{n} f_i(x)b_i$ for all $x\in A$. The projective dual basis $\{b_i, f_i; i=1, 2, \ldots, n\}$ defines a map $t_A: A\to K; x\rightsquigarrow \Sigma_{i=1}^{n} f_i(xb_i)$ which is called the <u>trace map of A.</u>

(1.3) For any projective dual bases $\{b_i, f_i; i=1, 2, \ldots, n\}$ and $\{b'_i, f'_i; i=1, 2, \ldots, m\}$ of A, $\Sigma_{i=1}^{n} f_i(xb_i)=\Sigma_{i=1}^{m} f'_i(xb'_i)$ and $t_A(xy)=t_A(yx)$ hold for all $x, y\in A$.

Proof. Let $\{b_i, f_i; i=1, 2, \ldots, n\}$ and $\{b'_i, f'_i; i=1, 2, \ldots, m\}$ be projective dual bases of A. Since $x=\Sigma_{i=1}^{n} f_i(x)b_i=\Sigma_{i=1}^{m} f'_i(x)b'_i$ for all $x\in A$, it is easily checked that $\Sigma_{i=1}^{n} f_i(xb_i)=\Sigma_{i=1}^{n} f_i(x\Sigma_{j=1}^{m} f'_j(b_i)b'_j)=\Sigma_{i=1}^{n}\Sigma_{j=1}^{m} f'_j(b_i)f_i(xb'_j)=\Sigma_{i=1}^{n}\Sigma_{j=1}^{m} f'_j(f_i(xb'_j)b_i)=\Sigma_{j=1}^{m} f'_j(xb'_j)$, and $t_A(xy)=\Sigma_{i=1}^{n} f_i(xyb_i)=\Sigma_{i=1}^{n} f_i(x\Sigma_{j=1}^{n} f_j(yb_i)b_j)=\Sigma_{i,j=1}^{n} f_j(yb_i)f_i(xb_j)=\Sigma_{i,j=1}^{n} f_j(yf_i(xb_j)b_i)=\Sigma_{j=1}^{n} f_j(yxb_j)=t_A(yx)$.

Definition. The trace map t_A of A defines a symmetric bilinear form

$A\times A\to K;(x,y)\mapsto t_r(xy)$, which is called the trace form of A and is denoted by $\langle A\rangle$. The trace form $\langle A\rangle$ induce a quadratic form $\rho: A\to K$; $x\mapsto t_r(x^2)$.

(1.4) ([K_5]; Proposition 1) $\langle A\rangle$ is nondegenerate if and only if A is a strongly separable K-algebra, where the "strongly separable K-algebra A" means that there exists a $\sum_{i=1}^{n} a_i\otimes b_i\in A\otimes_K A$ such that $\sum_{i=1}^{n} a_i\otimes b_i=\sum_{i=1}^{n} b_i\otimes a_i$, $\sum_{i=1}^{n} a_i b_i=1$ and $\sum_{i=1}^{n} xa_i\otimes b_i=\sum_{i=1}^{n} a_i\otimes b_i x$ for all $x\in A$ hold, (cf. [K_1], [K_2]).

(1.5) ([K_1], [K_2]) If either A is commutative or K is a field of characteristic 0, then "strongly separable" coincides with "separable".

(1.6) ([K_5]; Lemma 1) For a $P\in Sig(K)$, $\mathfrak{a}_P=\{x\in A\mid t_r(xA)\subseteq \mathfrak{p}_P\}$ is a two sided ideal of A, and $(A/\mathfrak{a}_P)\otimes_K \bar{k}_P(\cong A\otimes_K \bar{k}_P/J(A\otimes_K \bar{k}_P))$ is a separable $\bar{k}_P$-algebra, where $J(A\otimes_K \bar{k}_P)$ denotes the Jacobson-radical of $A\otimes_K \bar{k}_P$.

(1.7) For $P\in Sig(K)$, $(A/\mathfrak{a}_P)\otimes_K R_P$ is isomorphic to a product of matrix rings over R_P, H_P and $R_P(\sqrt{-1})$; $(R_P)_{m_1}\times\cdots\times(R_P)_{m_r}\times(H_P)_{m_1}\times\cdots\times(H_P)_{m_s}\times(R_P(\sqrt{-1}))_{k_1}\times\cdots\times(R_P(\sqrt{-1}))_{k_t}$, where $H_P=R_P\oplus R_P i\oplus R_P j\oplus R_P k$ is a quaternion R_P-algebra with $ij=-ji=k$, $i^2=j^2=k^2=-1$.

Definition. $\deg(A/\mathfrak{a}_P)$ or $\deg((A/\mathfrak{a}_P)\otimes_K \bar{k}_P)$ denote the number $n_1+\cdots+n_t$, provided $(A/\mathfrak{a}_P)\otimes_K R_P(\sqrt{-1})\cong(R_P(\sqrt{-1}))_{n_1}\times\cdots\times(R_P(\sqrt{-1}))_{n_t}$. We call it the degree of $A/\mathfrak{a}_P$ or $(A/\mathfrak{a}_P)\otimes_K \bar{k}_P$. If A is commutative, then $\deg(A/\mathfrak{a}_P)=[(A/\mathfrak{a}_P)\otimes_K \bar{k}_P:\bar{k}_P]$.

2. The commutative case. Let A be a commutative K-algebra such that A is a finitely generated and projective K-module, and let $P\in Sig(K)$. We put $\underline{Sig_P(A/K)}:=\{Q\in Sig(A)\mid Q\cap K=P\}$: the set of extensions on A of the ordering P, and $\underline{Hom_{K\text{-}alg}(A, R_P)}:=\{f: A\to R_P\mid K\text{-algebra homomorphisms}\}$.

(2.1)([K_4];Lemma 1) Maps Q: $Hom_{K\text{-}alg}(A, R_P)\longrightarrow Sig_P(A/K)$; $f\mapsto Q_f$, defined by $Q_f=\{x\in A\mid f(x)\geqq 0$ in $R_P\}$,

λ: $Hom_{K\text{-}alg}(A, R_P)\longrightarrow Hom_{\bar{k}_P\text{-}alg}(A\otimes_K \bar{k}_P, R_P)$; $f\mapsto f\otimes 1$ and

μ: $Hom_{K\text{-}alg}(A, R_P)\longrightarrow Hom_{R_P\text{-}alg}(A\otimes_K R_P, R_P)$; $f\mapsto f\otimes 1$, defined by $f\otimes 1(x\otimes r)=f(x)r$ for $x\otimes r\in A\otimes_K \bar{k}_P$ or $x\otimes r\in A\otimes_K R_P$, are bijections.

(2.2) The canonical homomorphism $A \to A/\mathfrak{a}_P$ induces a bijection
$\nu : \mathrm{Hom}_{k_P\text{-alg}}((A/\mathfrak{a}_P)\otimes_K k_P,\ R_P) \twoheadrightarrow \mathrm{Hom}_{k_P\text{-alg}}(A\otimes_K k_P,\ R_P)$.

Because, $(A/\mathfrak{a}_P)\otimes_K k_P \cong A\otimes_K k_P / J(A\otimes_K k_P)$ and $g(J(A\otimes_K k_P))=(0)$ for all $g\in \mathrm{Hom}_{k_P\text{-alg}}(A\otimes_K k_P,\ R_P)$, since $J(A\otimes_K k_P)$ is nilpotent.

The trace map $t_P : A \to K$ induces a map $\bar{t}_P : A/\mathfrak{a}_P \to (K/\mathfrak{p}_P) \hookrightarrow k_P$ and a trace map $\bar{t}_P \otimes 1 : A/\mathfrak{a}_P\otimes_K k_P \to k_P$ of $A/\mathfrak{a}_P\otimes_K k_P$. $(A/\mathfrak{a}_P\otimes_K k_P,\ \bar{t}_P\otimes 1) = \langle A/\mathfrak{a}_P\otimes_K k_P\rangle$ is a nondegenerate quadratic k_P-module. We denotes the Witt class of $\langle A/\mathfrak{a}_P\otimes_K k_P\rangle$ in the Witt ring $W(k_P)$ by $[A/\mathfrak{a}_P\otimes_K k_P]$. Then, the image $\overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P])$ by the ring homomorphism $\overline{\Psi}_P : W(k_P) \to \mathbb{Z}$ means the signature of $\langle A/\mathfrak{a}_P\otimes_K k_P\rangle$ on the ordered field $(k_P, \bar{P})$.

(2.3) Lemma. For the K-algebra A, there is a separable polynomial $f(X)\in k_P[X]$ of degree n, i.e. $f(X)=0$ has mutually distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $R_P(\sqrt{-1})$, such that $A/\mathfrak{a}_P\otimes_K k_P \cong k_P[X]/(f(X))$. Then, the number r of roots of $f(X)=0$ contained in R_P is equal to the signature of the trace form $\langle k_P[X]/(f(X))\rangle$, that is, $r=\overline{\Psi}_P([k_P[X]/(f(X))]) = \overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P])$.

Proof. Since $A/\mathfrak{a}_P\otimes_K k_P$ is a separable k_P-algebra, it follows that $A/\mathfrak{a}_P\otimes_K k_P \cong L_1\times L_2\times\cdots\times L_m$, where L_i is a finite separable extension field. There are irreducible polynomials $f_1(X), f_2(X), \ldots, f_m(X)$ in $k_P[X]$ which are mutually prime and $k_P[X]/(f_i(X)) \cong L_i$ for $i=1,2,\ldots,m$. Then, for $f(X)=f_1(X)f_2(X)\cdots f_m(X)$, one has $k_P[X]/(f(X)) \cong L_1\times L_2\times\cdots\times L_m$. If $f(X)$ decomposes to a product $f(X)=(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_r)g_1(X)g_2(X)\cdots g_s(X)$ of irreducible polynomials in $R_P[X]$, then $R_P[X]/(f(X))$ is isomorphic to a direct product of r-copies of R_P and some copies of $R_P(\sqrt{-1})$. Then, the trace form $\langle R_P[X]/(f(X))\rangle$ decomposes a orthogonal sum of r-copies of $\langle 1\rangle$ and some copies of $\langle 1,-1\rangle$. Accordingly, we get $\overline{\Psi}_P([k_P[X]/(f(X))]) = r$.

(2.4)([K4];Lemma 5) $\overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P]) = |\mathrm{Sig}_P(A/K)| = |\mathrm{Hom}_{K\text{-alg}}(A,\ R_P)|$, where $|M|$ denotes the number of elements in M.

From the above statements, the following corollaries are derived:

(2.5) Corollary. For any $P\in \mathrm{Sig}(K)$, the following conditions are equivalent:

① $(A/\mathfrak{a}_P)\otimes_K R_P \cong R_P\times\cdots\times R_P$.

② $|\mathrm{Sig}_P(A/K)| = [A/\mathfrak{a}_P\otimes_K k_P : k_P]$.

③ there exists a separable polynomial $f(X)\in k_P[X]$ such that $f(X)=(X-\alpha_1)(X-\alpha_2)\cdots(X-\alpha_n)$ for $\alpha_1,\alpha_2,\dots,\alpha_n\in R_P$ and $A/\mathfrak{a}_P\otimes_K k_P\cong k_P[X]/(f(X))$.

④ $\Psi_P([A/\mathfrak{a}_P\otimes_K k_P])=[(A/\mathfrak{a}_P)\otimes_K k_P : k_P]$.

⑤ $\langle A/\mathfrak{a}_P\otimes_K k_P\rangle$ is positive definite.

⑥ for $\langle A\rangle=(A,\rho)$, $\rho(A)\subseteq P$.

(2.6) Corollary. For every $P\in Sig(K)$, $(A/\mathfrak{a}_P)\otimes_K R_P\cong R_P\times\cdots\times R_P$ holds if and only if $\rho(A)\subseteq Q(K)$ holds.

(2.7) Corollary. If A is a separable K-algebra, then $\langle A\rangle$ is nondegenerate, and $\psi_P([A])=|Sig_P(A/K)|$ ($[A]\in W(K)$).

3. The case of commutative G-Galois extension. Let $A\supseteq K$ be a commutative Galois extension with Galois group G in the sense of [C,H,R].

(3.1)([K₄];Theorem) For any $P\in Sig(K)$, $\psi_P([A])=|Sig_P(A/K)|$ is equal to $|G|$ or 0. $\psi_P([A])=|G|$ if and only if for every subgroup H of G with $|H|=2$, $A\otimes_K R_P\cong(A^H\otimes_K R_P)\times(A^H\otimes_K R_P)$ holds. If $|G|$ is odd, then $\rho(A)\subseteq Q(K)$ holds.

(3.2) Lemma If $G=\langle\sigma\rangle$ is a cyclic group and $|G|$ is divisible by 4, then $\rho(A)\subseteq Q(K)$ holds.

Proof. Let $A\supseteq K$ be G-Galois extension with $G=\langle\sigma\rangle$ and $|G|=4m$. Suppose for a $P\in Sig(K)$, $Sig_P(A/K)=\phi$, that is, $A\otimes_K R_P=R_P(\sqrt{-1})e_1\oplus R_P(\sqrt{-1})e_2\oplus\cdots\oplus R_P(\sqrt{-1})e_{2m}$, where $e_1,\dots,e_{2m}$ are orthogonal idempotents in $A\otimes_K R_P$ with $\Sigma_{i=1}^{2m}e_i=1\otimes1$. Since $\{\sigma\otimes1(e_1),\sigma\otimes1(e_2),\dots,\sigma\otimes1(e_{2m})\}=\{e_1,e_2,\dots,e_{2m}\}$, we may assume $\sigma\otimes1(e_1)=e_2$, $\sigma\otimes1(e_2)=e_3,\dots,$ $\sigma\otimes1(e_{2m})=e_1$. $A\otimes_K R_P\supseteq R_P$ is also G-Galois extension, hence the G-fixed ring is $(A\otimes_K R_P)^G=R_P$, so $\sigma\otimes1(\sqrt{-1})=-\sqrt{-1}$. For any $x=\Sigma_{i=1}^{2m}(a_i+b_i\sqrt{-1})e_i$ in $A\otimes_K R_P$ with $a_i,b_i\in R_P$, $\sigma\otimes1(x)=x$ if and only if $a_1=a_2=\cdots=a_{2m}$ and $b_1=-b_2=b_3=-b_4=b_5=\cdots=(-1)^{2m-1}b_{2m}$. Hence, we get $(A\otimes_K R_P)^G=\{\Sigma_{i=1}^{2m}(a+(-1)^{i-1}b\sqrt{-1})e_i\mid a,b\in R_P\}\neq R_P$, this is a contradiction.

4. The non-commutative case.

(4.1) ([K,W]) For any $P\in Sig(K)$, signatures of trace forms $\langle(R_P)_n\rangle$, $\langle(H_P)_m\rangle$ and $\langle(R_P(\sqrt{-1}))_k\rangle$ are equal to n, $-2m$ and 0, respectively.

(4.2) ([K₅]) For any $P\in Sig(K)$,

(a) $(A/\mathfrak{a}_P)\otimes_K R_P \cong (R_P)_{n_1}\times\cdots\times(R_P)_{n_t}$ holds if and only if $\overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P])=\deg(A/\mathfrak{a}_P)$ holds.

(b) $(A/\mathfrak{a}_P)\otimes_K R_P \cong (H_P)_{m_1}\times\cdots\times(H_P)_{m_s}$ holds if and only if $\overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P])=-\deg(A/\mathfrak{a}_P)$ holds.

(c) $(A/\mathfrak{a}_P)\otimes_K R_P \cong (R(\sqrt{-1}))_{k_1}\times\cdots\times(R(\sqrt{-1}))_{k_t}$ implies $\overline{\Psi}_P([A/\mathfrak{a}_P\otimes_K k_P])=0$.

Example 1. Let (M,q) be a nondegenerate, finitely generated and projective quadratic K-module, and $P\in \mathrm{Sig}(K)$. We denote the Witt class of $(M,q)\otimes_K k_P$ in $W(k_P)$ by $[(M,q)\otimes_K k_P]$. For the Clifford algebra $C(M,q)$ of (M,q), the trace form of k_P-algebra $C(M,q)\otimes_K k_P$ is denoted by $\langle C(M,q)\otimes_K k_P\rangle$, and its Witt class by $[C(M,q)\otimes_K k_P](\in W(k_P))$. As is well known, $(M,q)\otimes_K k_P$ is expressed $\langle a_1,a_2,\dots,a_n\rangle$ for $a_i\neq 0\in k_P$; $i=1,2,\dots,n$, using an orthogonal basis of $M\otimes_K k_P$.

(4.3)(cf.[W1]) If $n(=2r)$ is even, $\deg(C(M,q))=2^r$. If $n(=2r+1)$ is odd, $\deg(C(M,q))=2^{r+1}$.

(4.4)([K5];Corollary 2)

(a) If $\psi_P([M,q])\equiv 0,1,2 \pmod 8$, then $\overline{\Psi}_P([C(M,q)\otimes_K k_P])=\deg(C(M,q))$, and $C(M,q)\otimes_K R_P$ is isomorphic to $(R_P)_{2^r}$ or $(R_P)_{2^r}\oplus(R_P)_{2^r}$.

(b) If $\psi_P([M,q])\equiv 4,5,6 \pmod 8$, then $\overline{\Psi}_P([C(M,q)\otimes_K k_P])=-\deg(C(M,q))$, and $C(M,q)\otimes_K R_P$ is isomorphic to $(H_P)_{2^{r-1}}$ or $(H_P)_{2^{r-1}}\oplus(H_P)_{2^{r-1}}$.

(c) If $\psi_P([M,q])\equiv 3,7 \pmod 8$, then $\overline{\Psi}_P([C(M,q)\otimes_K k_P])=0$, and $C(M,q)\otimes_K R_P\cong(R_P(\sqrt{-1}))_{2^r}$.

Example 2. Let G be a finite group with order n.
For any $P\in\mathrm{Sig}(K)$, we denote by $\xi_1,\xi_2,\dots,\xi_t$ the irreducible representations of G on $R_P(\sqrt{-1})$, and by $I(G)$ the set of all involutions in G. Using the number $k=(|G|-|I(G)|-1)/2$, the trace form $\langle k_P G\rangle$ of the group algebra $k_P G$ is expressed by $\langle|G|\rangle\cdot\{(\perp^{|I(G)|+1}\langle 1\rangle)\perp(\perp^{k}\langle 1,-1\rangle)\}$, where $\perp^{|I(G)|+1}\langle 1\rangle$ and $\perp^{k}\langle 1,-1\rangle$ mean orthogonal sums $\langle 1\rangle\perp\cdots\perp\langle 1\rangle$ of $(|I(G)|+1)$-copies of $\langle 1\rangle$, and $\langle 1,-1\rangle\perp\cdots\perp\langle 1,-1\rangle$ of k-copies of $\langle 1,-1\rangle$. Furthermore, we know that $\deg(k_P G)=\Sigma_{i=1}^{t}\xi_i(e)$ for the unit element e of G and $\overline{\Psi}_P([k_P G])=1+|I(G)|$.

(4.5) (cf. [I]; (4.6) and (4.19)) For the Witt class $[k_r G]$ ($\in W(k_r)$) of trace form $\langle k_r G\rangle$, $R_r G \cong (R_r)_{*_1} \times \cdots \times (R_r)_{*_r}$ if and only if $\overline{\varphi}_r([k_r G]) = \deg(k_r G)$, i.e. $1 + |\, I(G)\,| = \Sigma_{i \in I}\, \xi_i(e)$.

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Modular representations of Iwahori-Hecke algebras

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0. Introduction

Let (W, S) be a finite Coxeter system (see [3]), q an indeterminate, and $H_q(W)$ a free $\mathbb{C}[\sqrt{q}]$-module with a basis $\{T_w\}_{w\in W}$ parametrized by the elements of W. Here $\mathbb{C}$ denotes the field of complex numbers. Then $H_q(W)$ has an associative $\mathbb{C}[\sqrt{q}]$-algebra structure characterized by the conditions

$$(T_s + 1)(T_s - q) = 0, \text{ if } s \in S \text{ and}$$
$$T_w T_{w'} = T_{ww'}, \text{ if } \ell(w) + \ell(w') = \ell(ww'),$$

where ℓ is the length function. This is called an (Iwahori)-Hecke algebra. See [3, Chap.4, §2, Ex.23],[4] and [11]. The Poincaré polynomial $P_W(q)$ of (W, S) is defined by $P_W(q) = \sum_{w\in W} q^{\ell(w)}$.

Now we recall modular representation theory of finite groups. Let G be a finite group, and R a suitable discrete valuation ring, and let K and k be the quotient and the residue fields of R, respectively. We assume that Char$K = 0$ and Char$k = p \neq 0$ and take R, K and k as coefficient rings. Relationship between representations over K and k are studied via those over R. For example, we can define so called decomposition numbers as follows. Let V be a simple (irreducible) KG-module. Take an RG-lattice V' such that $K \otimes_R V' = V$ and consider a composition

The detailed version of this note is cited as [12] in References below.

series of $k \otimes_R V'$. The multiplicities of simple kG-modules in $k \otimes V'$ depend only on V and not on the choice of V', and they are called decomposition numbers.

Return to Hecke algebras. Fix a complex number α, and, as coefficient rings, take $\mathbf{C}(\sqrt{q})$, $\mathbf{C}[\sqrt{q}]_\alpha$ and $\mathbf{C}[\sqrt{\alpha}]$ instead of K, R and k, respectively, where $\mathbf{C}[\sqrt{q}]_\alpha$ is the localization of $\mathbf{C}[\sqrt{q}]$ by the ideal $(\sqrt{q} - \sqrt{\alpha})$. Then we can consider 'modular representations' of Hecke algebras which is analogous to those of finite groups. Notice that $\mathbf{C}[\sqrt{q}]_\alpha \otimes H_q(W)$, which we denote by $H_\alpha(W)$ thereafter, is obtained by formally letting q be α (see p.637 of [5]). The C-algebra $H_\alpha(W)$ is called a specialized algebra of $H_q(W)$ under the specialization $q \mapsto \alpha$. (Note : We actually consider the specialization $\sqrt{q} \mapsto \sqrt{\alpha}$ choosing $\sqrt{\alpha}$. However, we indicate it by $q \mapsto \alpha$ for convenience.) In this note, we mention some results in the modular representation theory of Hecke algebras. Remark that the specialization $q \mapsto 1$ of $H_q(W)$ gives the group algebra $\mathbf{C}W$ and $P_W(1) = |W|$. By those facts, a Hecke algebra is called a q-analogue of the group algebra $\mathbf{C}W$.

1. Some known facts

Here we list several properties of Hecke algebras. (There are many others which we omit. See, for example, [14].) These should be compared with the corresponding properties of the group algebra kW. For convenience, we assume that $\alpha \neq 0$. (The phenomenon when $\alpha \neq 0$ is quite different from others.) For the proof, see [5],[6],[7],[9] and [12].

1. $H_\alpha(W)$ is a symmetric algebra.
2. $\mathbf{C}(\sqrt{q}) \otimes H_q(W)$ is a symmetric semisimple algebra over $\mathbf{C}(\sqrt{q})$.
3. $H_\alpha(W)$ is semisimple if and only if $P_W(\alpha) \neq 0$ ([9]).
4. If $H_\alpha(W)$ is semisimple, then it is isomorphic to $\mathbf{C}W$ ([5, §68]).
5. Let S' be a subset of S and $W' = < S' >$. Then a trace map can be defined from the center of $H_\alpha(W')$ into that of $H_\alpha(W)$. Also, the Frobenius reciprocity laws

hold between modules over $H_\alpha(W')$ and $H_\alpha(W)$.

6. Let $G(\mathbb{C}(\sqrt{q})\otimes H_q(W))$ and $G(H_\alpha(W))$ be the Grothendieck groups of the categories of modules over $\mathbb{C}(\sqrt{q})\otimes H_q(W)$ and $H_\alpha(W)$, respectively. Then the decomposition map d from $G(\mathbb{C}(\sqrt{q})\otimes H_q(W))$ to $G(H_\alpha(W))$ can be defined. (See Introduction.)

2. Hecke algebras of type A_l

In this section, we assume that $\alpha \neq 0$ and that (W, S) is of type A_l. Then W is isomorphic to the symmetric group S_{l+1} on $l+1$ letters, and we have

$$P_W(q) = (1+q)(1+q+q^2)\dots(1+q+q^2+\dots+q^l).$$

Thus, $H_\alpha(W)$ is not semisimple if and only if α is a primitive r-th root of unity with $2 \leq r \leq l+1$. (See § 1, 3.) Moreover, there is a one-to-one correspondence between the set of isomorphism classes of simple $\mathbb{C}(\sqrt{q})\otimes H_q(W)$-modules and the set of Young diagrams with $l+1$ nods. Also, this correspondence gives the natural well known correspondence between the set of isomorphism classes of simple $\mathbb{C}S_{l+1}$-modules and the set of Young diagrams under the specialization $q \mapsto 1$. Moreover, concerning $H_\alpha(W)$, the 'Nakayama conjecture' holds. Namely, if α is a primitive r-th root of unity, then two simple modules over $\mathbb{C}(\sqrt{q})\otimes H_q(W)$ lie in the same block under $q \mapsto \alpha$ if and only if the corresponding Young diagrams have the same r-core. Furthermore, the number of isomorphism classes of simple $H_\alpha(W)$-modules is equal to that of r-regular Young diagrams (with $l+1$ nods). Here a Young diagram is r-regular if the corresponding partition $(p_1, p_2, \dots p_m)$ with $p_1 \geq p_2 \geq \dots \geq p_m > 0$ satisfies $|\{i | p_i = n\}| < r$ for all n. It is well known that similar hold if we replace $H_\alpha(W)$ and r by kS_{l+1} and p, respectively, in the above. For the proofs, see [6] and [7]. Many other analogous results can also be found loc. cit. If α is a primitive $(l+1)$-th root of unity, then simple $H_\alpha(W)$-modules are obtained explicitly in [13].

As is seen so far, there are many resemblance between $H_\alpha(W)$ and kS_{l+1}. Next, we consider indecomposable modules. We ask how many isomorphism classes there are. For kW (or more generally, for an arbitrary finite group algebra) a result of Higman [10] asserts that :

Theorem. *kW is of finite representation type if and only if a p-Sylow subgroup of W is cyclic.*

Here, we say that an algebra is of finite representation type if there are only finitely many isomorphism classes of indecomposable modules. It is easily seen that the symmetric group S_{l+1} has a nontrivial cyclic p-Sylow subgroup if and only if $(l+1)/2 < p \leq l+1$, namely, p is the highest power of p that divides the group order $(l+1)!$. So, a statement analogous to Higman's may be the following.

Question. *Let α be a non zero complex number. Then is it true (for a general finite Coxeter system (W,S)) that $H_\alpha(W)$ is of finite representation type if and only if α is a simple root of $P_W(q) = 0$?*

The above is partially answered as follows.

Theorem. *If (W,S) is of type A_l or $|S| \leq 2$, then the above question is affirmatively answered.*

The proof can be found in [12]. However, some words may be in order. Higman's theorem is proved by using the notion of relative projectivity and Higman's criterion. Hecke algebras have also those notion and criterion. However, there is certain difficulty in using these devices in order to prove the above theorem, because trace maps, which are important to Higman's criterion, seem to have nothing to do with the Poincaré polynomial. Thus, instead, we use Auslander-Reiten and Gabriel theories ([2], [8]). In those theories, Auslander-Reiten quivers and separated graphs give several conditions on finiteness of isomorphism classes of indecomposable modules ([1] ,[8]). Since simple $H_\alpha(S_{l+1})$-modules are relatively well known, we can compute those graphs for $H_\alpha(S_{l+1})$ in critical cases.

3. Problems

Since 'modular representation theory' of Hecke algebras seems to be in the beginning stage, there are many problems. Here, we mention two of them.

1. Is there any criterion for tameness of the module category of $H_\alpha(W)$?
2. Describe the decomposition matrix.

Remark. In previous sections, we insist on similarities between $H_\alpha(W)$ and kW. However, there are also some differences between them:

Decomposition matrices of the principal blocks of kS_5 when Char$k = 2$ and $H_{-1}(S_5)$ are the transposes of the following, respectively.

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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ON THE COHOMOLOGY OF FINITE GROUPS AND THE MODULAR REPRESENTATIONS

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1. Introduction. Let G be a finite group and k be a field of prime characteristic p. We also denote the trivial kG-module by k. All kG-modules considered here are assumed to be finite dimensional over k. For a kG-module M, we set $\mathrm{Ext}_{kG}^{*}(M,M)=\sum_{n\geq 0}\mathrm{Ext}_{kG}^{n}(M,M)$. The theory of module varieties is based on the following theorem.

Theorem 1.1 (Evens). The cohomology ring, with the cup product, $\mathrm{Ext}_{kG}^{*}(k,k)$ is finitely generated as a k-algebra. Moreover, for any kG-modules M and N, $\mathrm{Ext}_{kG}^{*}(M,N)$ is a finitely generated $\mathrm{Ext}_{kG}^{*}(k,k)$-module.

Here, we hope to study some modules and exact sequences corresponding to homogeneous elements of $\mathrm{Ext}_{kG}^{*}(k,k)$, that is, these are concerned with the argument of homological algebra.

Definition 1.2 (Carlson). Let ζ be an element in $\mathrm{Ext}_{kG}^{n}(k,k)\cong\mathrm{Hom}_{kG}(\Omega^{n}(k),k)$. we let L_{ζ} be the kernel of $\hat{\zeta}:\Omega^{n}(k)\to k$ for $\zeta\neq 0$. If $\zeta=0$, let $L_{\zeta}=\Omega^{n}(k)\oplus\Omega(k)$.

The following is the fundamental properties of the above module L_{ζ}, needed in later arguments.

The final version of this paper will be submitted for publication elsewhere.

Proposition 1.3. (1) Let $0 \neq \zeta_1 \in \mathrm{Ext}^n_{kG}(k,k)$ and $0 \neq \zeta_2 \in \mathrm{Ext}^m_{kG}(k,k)$. Then we have an exact sequence

$$0 \to \Omega^n(L_{\zeta_2}) \to L_{\zeta_1 \cdot \zeta_2} \oplus (\text{projective } kG\text{-module}) \to L_{\zeta_1} \to 0 .$$

(2) Let $\zeta \in \mathrm{Ext}^n_{kG}(k,k)$ and H be a subgroup of G. Then $(L_\zeta)_H \cong L_{\mathrm{res}_{G,H}(\zeta)} \oplus$ (projective kH-module).

Next, we state the exact sequence with an interesting diagram ([8]). Let H be a normal subgroup of a finite group G of index p. Then there exists an element β in $\mathrm{Ext}^2_{kG}(k,k)$ such that $\mathrm{inf}_{G/H,G}(\mathrm{Ext}^2_{k(G/H)}(k,k)) = k\cdot\beta$.

Note that β is unique up to scalar multiples. Following [8], we call β a Bockstein element corresponding to H. Then it is known that β can be represented by $0 \to k \to k_H{}^G \to k_H{}^G \to k \to 0$ as a sequence of kG-modules. Moreover, this sequence have the following commutative diagram :

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L_\beta & \to & \mathrm{Ker}\,\lambda_1 & \to & \mathrm{Ker}\,\lambda_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^2(k) & \to & P_1 & \to & P_0 & \to & k & \to & 0 \\
 & & \downarrow\sigma & & \downarrow\lambda_1 & & \downarrow\lambda_0 & & \| & & \\
0 & \to & k & \to & k_H{}^G & \to & k_H{}^G & \to & k & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where P_0 and P_1 are the projective covers of k and $\Omega(k)$, the rows and columns are exact. Immediately, this diagram gives the following lemma.

Lemma 1.4. Let H be a normal subgroup of G of index p and β be a non zero Bockstein element corresponding to H. If a kG-module M is projective as a kH-module, then $L_\beta \otimes M$ is a projective kG-module.

2. Main results. Carlson introduced the following number in [2].

Definition 2.1. Let E be a maximal elementary abelian p-subgroup of G. Let A_E be an abelian p-subgroup of G which contains E and which has maximal order among such subgroups. Define $n(E)=|G:A_E|$ and $n(G)=L.C.M_{E\in\Gamma}\{n(E)\}$, where Γ is the set of all maximal elementary abelian p-subgroups of G.

The next theorem is the main result.

Theorem 2.2. Let G be a finite group and k be a field of characteristic $p>0$. Then there exist $\zeta_1,\ldots,\zeta_t$ in $\mathrm{Ext}_{kG}^{2n(G)}(k,k)$ such that $L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_t}$ is a projective kG-module.

For the proof, it suffices to show that given an elementary abelian p-subgroup F of G, there exist $\zeta_1,\ldots,\zeta_r$ in $\mathrm{Ext}_{kG}^{2n(G)}(k,k)$ such that $(L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_r})_F$ is a projective kF-module. For, if this is shown, then consider all those $\zeta_1,\ldots,\zeta_t\in\mathrm{Ext}_{kG}^{2n(G)}(k,k)$ taken over the elementary abelian p-subgroups of G . Then by Chouinard's theorem, we have that $L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_t}$ is a projective kG-module.

We prove this assertion by the induction on $|F|$, using Proposition 1.3, Lemma 1.4 and the next lemma which is an analogue of a result of Quillen. (for the detail of the proof, see the final version)

Lemma 2.3. Let A be an abelian p-subgroup of G and F be an elementary abelian subgroup of A . Then there exists an element ζ in $\mathrm{Ext}_{kG}^{2|G:A|}(k,k)$ such that $\mathrm{res}_{G,F}(\zeta)$ is a product of Bockstein elements.

Next, we give the equivalent conditions to the one of our main theorem.

Let K be an algebraically closed field of characteristic $p>0$. Let $H^*(G,K)=\Sigma_{n\geq 0}\mathrm{Ext}_{KG}^n(K,K)$ if $p=2$ and $H^*(G,K)=$

$\sum_{n\geq 0} \mathrm{Ext}_{KG}^{2n}(K,K)$ if $p>2$. From Theorem 1.1, we know that $H^*(G,K)$ has an associated affine variety $V_G(K)=\mathrm{Max}(H^*(G,K))$, which is the set of all maximal ideals of $H^*(G,K)$. Let M be a KG-module and $J_G(M)$ be the annihilator in $H^*(G,K)$ of $\mathrm{Ext}_{KG}^*(M,M)$. The variety $V_G(M)$ of M is defined as the subvariety of $V_G(K)$ associated to $J_G(M)$.

Proposition 2.1. Let n be a positive integer and $\zeta_1,\ldots,\zeta_t$ be elements in $\mathrm{Ext}_{KG}^{2n}(K,K)$. Then the following are equivalent.

(1) For every elementary abelian p-subgroup $F=\langle x_1,\ldots,x_r\rangle$ of G and for its every cyclic shifted subgroup $\langle u_\alpha\rangle$, that is, $u_\alpha=1+\sum_{j=1}^{r}\alpha_j(x_j-1)$, $\alpha=(\alpha_j)\neq 0\in K$, there exists ζ_i $(1\leq i\leq t)$ such that $\mathrm{res}_{G,\langle u_\alpha\rangle}(\zeta_i)\neq 0$.

(2) $L_{\zeta_1}\otimes\cdots\otimes L_{\zeta_t}$ is projective.

(3) $\bigcap_{i=1}^{t} V_G(L_{\zeta_i})=\{0\}$.

(4) $\sqrt{(\zeta_1,\ldots,\zeta_t)}=\sum_{n>0}\mathrm{Ext}_{KG}^{n}(K,K)$, where $\sqrt{(\zeta_1,\ldots,\zeta_t)}=\{\zeta\in \mathrm{Ext}_{KG}^*(K,K)\mid \zeta^c\in\sum_{i=1}^{t}\mathrm{Ext}_{KG}^*(K,K)\zeta_i$ for some $c>0\}$.

3. Applications. Let G be a finite group and k be an arbitrary field of characteristic $p>0$. Using the main result, we can extend Carlson's result in [2]. As another application, we also give a homological criterion for a kG-module to be projective.

Corollary 3.1 (Periodicity of periodic modules). The period of a periodic kG-module divides $2n(G)$.

Corollary 3.2 (Criterion for a module to be projective). A kG-module M is projective if and only if $\mathrm{Ext}_{kG}^{2n(G)}(M,M)=\{0\}$.

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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA [1]

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We should observe that elementary abelian groups play important roles to compute Loewy series or Loewy lengths of group algebras. We here consider a good basis of the radical N of the modular group algebra KU of an elementary abelian group U over a field K (see [4]). We may assume that a field K contains a finite field F of order $p^r = |U|$ and an elementary abelian group U is the permutation group of F defined by

$$U = \{u_a : x \to x + a \; ; \; a \in F\}.$$

We usually use $\{u_a - 1; a \in F\}$ as a basis of the radical N of KU but it is not so useful for the products of basis elements and actions on N. It is the purpose of this paper to give a much better basis B of N and to state some results obtained by using B. The basis B is defined as in the following. Let λ be an element of the character group $\hat{F^*} =$ Hom (F^*, F^*) of the multiplicative group F^* of F. Then we set

$$R_\lambda = \textstyle\sum_{a \in F} \lambda(a) u_a \text{ where } \lambda(0) = \begin{cases} 0 & \text{if } \lambda \neq 1 \\ 1 & \text{if } \lambda = 1. \end{cases}$$

It is easy to see $\{R_\lambda; \lambda \in \hat{F^*}\}$ is a basis of the radical N of the group algebra KU. The next will show that this basis is useful.

1. $R_\lambda R_\mu = J(\lambda, \mu) R_{\lambda\mu}$ where $J(\lambda, \mu) = \sum_{a \in F} \lambda(a) \mu(1-a)$.
2. $R_\lambda^\sigma = R_{\lambda^{\sigma^{-1}}}$ for an automorphism σ of F.

We can find a much better basis B of N as in the following.
Let η be a generator of F^*. Then $\phi : \eta \to \eta^{-1}$ is a generator of $\hat{F^*}$.
We set $\Phi_k = R_{\phi^{p^k}}$. It is evident $\Phi_k^p = 0$. We obtain the next

[1] The final version of results III,IV,V in this paper will be submitted for publication elsewhere. Another parts of this paper were already published.

$$\Phi_0^{p-1}\Phi_1^{p-1}\cdots\Phi_{r-1}^{p-1}\neq 0.$$

We can identify an element

$$\Phi_0^{i_0}\Phi_1^{i_1}\cdots\Phi_{r-1}^{i_{r-1}}\qquad(0\leq i_k<p)$$

with a natural number

$$i_0+i_1p+\cdots+i_{r-1}p^{r-1}.$$

For two natural numbers $a=i_0+i_1p+\cdots+i_{r-1}p^{r-1}$ and $b=j_0+j_1p+\cdots+j_{r-1}p^{r-1}$, we shall define

$$a\#b=\begin{cases} a+b & \text{if } i_k+j_k<p \text{ for all } k\\ 0 & \text{otherwise.}\end{cases}$$

It follows from our observations that the set

$$B=\{1,2,\ldots,p^r-1\}$$

is a basis of N.

Moreover we shall use some notations. We set $a^* = \sum_{k=0}^{r-1} i_k$ for $a = i_0+i_1p+\cdots+i_{r-1}p^{r-1}$. Let $t(G)$ be the nilpotency index of the radical of KG. Let F be a finite field of order p^{pt} and let S be a subgroup of F^*. Then we consider the next permutation group $M_{p,t,S}$ on F.

$$M_{p,t,S}=\{x\to ax^{p^{tk}}+b\ ;\ a\in S, b\in F, k=0,1,\ldots,p-1\}.$$

If the order of S is $h_0=(p^{pt}-1)/(p^t-1)$, we set simply $M_{p,t}=M_{p,t,S}$. Using the basis B, we can obtain the next results I,...,V.

I. The set $\{b\in B\ ;\ b^*=k\}$ is a basis of N^k/N^{k+1} (see [4]).

II. If the order of S is a multiple of h_0, then we can obtain $t(M_{p,t,S}) = (pt+1)(p-1)+1$ (see [2]) and the formula to compute Loewy series of the group algebra of $M_{p,t,S}$ over a field K (see [4,5]).

III. Recently, in his paper H. Fukushima obtained the group structure of the next groups G satisfying the following conditions (see [1]).

1. $G = UH, U \lhd G$ and $U \cap H = 1$.
2. $H = VW$ is a Frobenius group with kernel V and complement W.
3. U is a p-subgroup, V is an abelian p'-group, and W is a p-group.
4. $t(G) = s(p-1)+1$ where p^s is the order of a p-Sylow subgroup of G.

The most difficult part in his proof is to prove that $C_G(x)$ contains a p-Sylow subgroup of G for every $x \in U$. However, it is easy to prove this part by using our basis B.

IV. If G is a group of the minimal order satisfying the next conditions, then G is isomorphic to $M_{p,1}$ (see [3]).

1. G is a p-solvable group with a p-Sylow subgroup P of order p^s.
2. P is not elementary abelian.
3. $t(G) = s(p-1) + 1$.
4. $O_{p'}(G/O_p(G))$ is abelian.

V. If the order h of S is a proper divisor of h_0, then it is not so easy to compute $t(M_{p,t,S})$. We use the notation $\alpha^{1-\sigma} = \alpha - \alpha^\sigma$ for $\alpha \in KU$ where $a^\sigma \equiv ap^t \bmod p^{pt} - 1$ for $a \in B$ such that $a^\sigma \in B$. We set

$$d = \mathrm{Max}\ \{\textstyle\sum k_i\ ;\ b_1^{(1-\sigma)^{k_1}} \# b_2^{(1-\sigma)^{k_2}} \# \cdots \# b_m^{(1-\sigma)^{k_m}} \neq 0\}$$

where $k_s < p$ for all s and $b_1, b_2, \ldots, b_m$ run through the set $\{b \in B\ ;\ h|b,\ h_0 \not| b\}$. Then we have $t(M_{p,t,S}) = d + (pt+1)(p-1) + 1$.

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