

ON THE ROUQUIER DIMENSIONS OF SINGULARITY CATEGORIES AND THEIR ANNIHILATORS

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ABSTRACT. Let R be a commutative noetherian ring. In this article, we investigate upper bounds for the Rouquier dimension of the singularity category of R . As an application, we establish a formula that gives an upper bound for the Rouquier dimension of the singularity category in terms of invariants associated with a certain ideal of R . This extends a theorem of Liu to the case where R is neither an isolated singularity nor a local ring.

Let R be a commutative noetherian ring. Denote by $\text{mod } R$ the category of finitely generated R -modules, by $D^b(R)$ the bounded derived category of $\text{mod } R$. The singularity category of R , introduced by Buchweitz [4], is defined as the Verdier quotient of $D^b(R)$ by the category of perfect complexes over R ; that is,

$$D_{\text{sg}}(R) = D^b(R) / \text{thick } R.$$

This category reflects the singularity of R in the sense that $D_{\text{sg}}(R)$ is trivial if and only if R is regular. For an essentially small triangulated category \mathcal{T} , we can define the Rouquier dimension of \mathcal{T} , denoted by $\dim \mathcal{T}$ [15]. This invariant measures how many mapping cones are needed to generate the entire category \mathcal{T} from a single object. The Rouquier dimension of the singularity category of R is closely related to the representation dimension of R ; see [2, 3, 13, 14, 15] for instance. In this article, we study upper bounds for the Rouquier dimension of the singularity category $D_{\text{sg}}(R)$.

Throughout this article, R denotes a commutative noetherian ring. All triangulated categories are assumed to be essentially small, and all subcategories are assumed to be strictly full.

We begin by recalling the definition of the Rouquier dimension and reviewing related previous results. We then state our main theorem and give an application example.

Definition 1. Let (\mathcal{T}, Σ) be a triangulated category, and \mathcal{X}, \mathcal{Y} subcategories of \mathcal{T} .

- (1) We denote by $\text{add}_{\mathcal{T}} \mathcal{X}$ the *additive closure* of \mathcal{X} in \mathcal{T} .
- (2) We define $\langle \mathcal{X} \rangle^{\mathcal{T}}$ as the additive closure of objects of the form $\Sigma^i X$, where $i \in \mathbb{Z}$, and $X \in \mathcal{X}$.
- (3) We define $\mathcal{X} * \mathcal{Y}$ as the subcategory of \mathcal{T} consisting of objects Z such that there exists an exact triangle $X \rightarrow Z \rightarrow Y \rightarrow \Sigma X$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

The detailed version [12] of this article will be submitted for publication elsewhere.

(4) For a nonnegative integer r , we define

$$\langle \mathcal{X} \rangle_r^{\mathcal{T}} = \begin{cases} 0 & \text{if } r = 0, \\ \langle \mathcal{X} \rangle^{\mathcal{T}} & \text{if } r = 1, \\ \langle \langle \mathcal{X} \rangle_{r-1}^{\mathcal{T}} * \langle \mathcal{X} \rangle^{\mathcal{T}} \rangle^{\mathcal{T}} & \text{if } r > 1. \end{cases}$$

(5) The *Rouquier dimension* of \mathcal{T} , denoted by $\dim \mathcal{T}$, is defined as the infimum of nonnegative integers n such that there exists an object G in \mathcal{T} with $\mathcal{T} = \langle G \rangle_{n+1}^{\mathcal{T}}$. That is,

$$\dim \mathcal{T} = \inf \{ n \geq 0 \mid \mathcal{T} = \langle G \rangle_{n+1}^{\mathcal{T}} \text{ for some } G \in \mathcal{T} \}.$$

Thus, the Rouquier dimension of a triangulated category measures how many mapping cones are needed to generate the entire category from a single object, up to direct summands, finite direct sums, and shifts.

Remark 2. Let \mathcal{T} be a triangulated category. The Rouquier dimension of \mathcal{T} being zero means that every object in \mathcal{T} is, up to shifts, a direct summand of a finite direct sum of a single object in \mathcal{T} . The property that the singularity category has dimension zero is related to local finiteness and finite Cohen–Macaulay representation type; see [1, 5] for instance.

Example 3. Let R be a commutative noetherian ring.

(1) For every object X in $D_{\text{sg}}(R)$, there exist an integer n and a module $M \in \text{mod } R$ such that X is isomorphic to $M[n]$ in $D_{\text{sg}}(R)$; see [6, Lemma 2.4] for instance. Hence, one has

$$D_{\text{sg}}(R) = \langle \text{mod } R \rangle^{D_{\text{sg}}(R)}.$$

(2) If (R, \mathfrak{m}, k) is an artinian local ring, then we have

$$D_{\text{sg}}(R) = \langle k \rangle_{\ell\ell(R)}^{D_{\text{sg}}(R)},$$

where $\ell\ell(R) = \inf \{ n \geq 0 \mid \mathfrak{m}^n = 0 \}$ is the Loewy length of R . Indeed, let $n = \ell\ell(R)$. Then, for any $M \in \text{mod } R$, there exists a filtration

$$0 = \mathfrak{m}^n M \subseteq \mathfrak{m}^{n-1} M \subseteq \dots \subseteq \mathfrak{m} M \subseteq M.$$

This filtration induces the following exact triangle:

$$\mathfrak{m}^i M \longrightarrow \mathfrak{m}^{i-1} M \longrightarrow k^{\oplus} \longrightarrow (\mathfrak{m}^i M)[1]$$

in $D_{\text{sg}}(R)$ for each $i = 1, \dots, n$. It follows that $M \in \langle k \rangle_{\ell\ell(R)}^{D_{\text{sg}}(R)}$.

Next, we state previous results concerning upper bounds for the Rouquier dimension of $D_{\text{sg}}(R)$. The annihilator ideal of $D_{\text{sg}}(R)$ is defined by

$$\text{ann}_R D_{\text{sg}}(R) = \bigcap_{X \in D_{\text{sg}}(R)} \text{ann}_R \text{End}_{D_{\text{sg}}(R)}(X).$$

By using this notion, one can compute upper bounds for the Rouquier dimension of $D_{\text{sg}}(R)$. The following result was recently established by Liu [11], while in the Cohen–Macaulay case, it had previously been shown by Dao and Takahashi [6].

Theorem 4 (Liu [11]). *Let (R, \mathfrak{m}, k) be a noetherian local ring with an isolated singularity and I an \mathfrak{m} -primary ideal of R contained in $\text{ann}_R D_{\text{sg}}(R)$. Then one has*

$$D_{\text{sg}}(R) = \langle k \rangle_{\ell(R/I)(\mu(I) - \text{depth } R + 1)}^{D_{\text{sg}}(R)}.$$

In particular, we have

$$\dim D_{\text{sg}}(R) \leq \ell(R/I)(\mu(I) - \text{depth } R + 1) - 1.$$

Remark 5. If R is a quasi-excellent ring of finite Krull dimension, then the ideal $\text{ann}_R D_{\text{sg}}(R)$ defines the singular locus of R ; see [7, 8, 9] for instance. Hence, if (R, \mathfrak{m}, k) is a quasi-excellent noetherian local ring with an isolated singularity, then the ideal $\text{ann}_R D_{\text{sg}}(R)$ is \mathfrak{m} -primary.

We would like to take a nontrivial and computable ideal I in the preceding theorem. Suppose that k is a field and

$$R = k[[x_1, \dots, x_n]]/\mathfrak{a} \quad (\text{or } k[x_1, \dots, x_n]/\mathfrak{a}),$$

where \mathfrak{a} is an ideal of the formal power series (or polynomial) ring over k . If $\mathfrak{a} = (f_1, \dots, f_m)$ and $h = \text{ht}(\mathfrak{a})$, then the *Jacobian ideal* of R is defined by

$$\text{Jac}(R) = I_h \left(\left\{ \frac{\partial f_i}{\partial x_j} \right\}_{i,j} \right).$$

The following theorem suggests that when the ring is close to being Cohen–Macaulay, one can take the ideal I in the preceding theorem to be $\text{Jac}(R)$.

Theorem 6 (Iyengar–Takahashi [10] and Liu [11]). *Let R be either a finitely generated algebra over a field or an equicharacteristic complete noetherian local ring of dimension d . Assume that R is equidimensional and half-Cohen–Macaulay; that is,*

$$\dim R/\mathfrak{p} = \dim R \quad \text{for all } \mathfrak{p} \in \text{Min } R,$$

and

$$2 \text{ depth } R_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}} \quad \text{for all } \mathfrak{p} \in \text{Spec } R.$$

Then one has

$$\text{Jac}(R) \subseteq \text{ann}_R D_{\text{sg}}(R).$$

We now give the main result of this article, which extends a theorem of Liu to the case where R is neither an isolated singularity nor a local ring.

Theorem 7. *Let R be a commutative noetherian ring and I an ideal of R contained in $\text{ann}_R D_{\text{sg}}(R)$. Then one has*

$$D_{\text{sg}}(R) = \langle \text{mod } R/I \rangle_{\mu(I) - \text{grade } I + 1}^{D_{\text{sg}}(R)}.$$

Sketch of the proof of Theorem 7.

Set $I = (\mathbf{x})$, where $\mathbf{x} = x_1, \dots, x_{\mu(I)}$. Let $X \in D_{\text{sg}}(R)$. We may assume that

$$X = \Omega_R^n M \quad \text{for some } M \in \text{mod } R \text{ and sufficiently large } n.$$

Since I is contained in

$$\text{ann}_R D_{\text{sg}}(R) \subseteq \text{ann}_R \text{End}_{D_{\text{sg}}(R)}(X),$$

the object X is a direct summand of $K(\mathbf{x}, X)$ in $D_{\text{sg}}(R)$. On the other hand, one can see that

$$K(\mathbf{x}, X) \in \langle H(\mathbf{x}, X) \rangle_{\mu(I) - \text{grade}(I, X) + 1}^{\text{D}^b(R)}$$

by using the grade sensitivity and the truncation of $K(\mathbf{x}, X)$. Since n has been taken sufficiently large, it follows that

$$\text{grade}(I, X) \geq \text{grade } I.$$

Combining these observations, we obtain

$$X \in \langle H(\mathbf{x}, X) \rangle_{\mu(I) - \text{grade}(I, X) + 1}^{\text{D}_{\text{sg}}(R)} \subseteq \langle H(\mathbf{x}, X) \rangle_{\mu(I) - \text{grade } I + 1}^{\text{D}_{\text{sg}}(R)} \subseteq \langle \text{mod } R/I \rangle_{\mu(I) - \text{grade } I + 1}^{\text{D}_{\text{sg}}(R)}. \quad \square$$

Finally, we present an application of the above theorem.

Example 8. Consider the ring

$$R = \mathbb{C}[[x, y, z]]/(x^3 - y^4).$$

This is a two-dimensional complete hypersurface domain. Note that R has neither countable Cohen–Macaulay type nor an isolated singularity.

Since $\text{Jac}(R) = (\bar{x}^2, \bar{y}^3)$, we have the following:

- $S = R/\text{Jac}(R) \cong \mathbb{C}[[x, y, z]]/(x^2, y^3)$.
- $\mu(\text{Jac}(R)) = 2$.
- $\text{grade}(\text{Jac}(R)) = \text{ht}(\text{Jac}(R)) = 1$.

Set $K = \text{nil}(S) = (\bar{x}, \bar{y})$. Then $K^4 = 0$. Hence, for every $M \in \text{mod } S$, there exists a filtration

$$0 = K^4 M \subseteq K^3 M \subseteq K^2 M \subseteq KM \subseteq M.$$

It follows that

$$\text{mod } S \subseteq \langle \text{mod } S/K \rangle_4^{\text{D}_{\text{sg}}(R)}.$$

On the other hand, since $S/K \cong \mathbb{C}[[z]]$, which is a discrete valuation ring, we have

$$\text{mod}(S/K) \subseteq \langle S/K \rangle_2^{\text{D}_{\text{sg}}(R)} = \langle R/(\bar{x}, \bar{y}) \rangle_2^{\text{D}_{\text{sg}}(R)}.$$

Consequently,

$$\text{mod } S \subseteq \langle R/(\bar{x}, \bar{y}) \rangle_8^{\text{D}_{\text{sg}}(R)}.$$

By applying Theorem 7, we obtain

$$D_{\text{sg}}(R) = \langle \text{mod } S \rangle_{\mu(\text{Jac}(R)) - \text{grade}(\text{Jac}(R)) + 1}^{\text{D}_{\text{sg}}(R)} = \langle R/(\bar{x}, \bar{y}) \rangle_{16}^{\text{D}_{\text{sg}}(R)}.$$

Therefore,

$$\dim D_{\text{sg}}(R) \leq 15.$$

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