ON RELATIVE PROJECTIVE COVERS FOR THE PRINCIPAL BLOCKS OF FINITE GROUPS WITH METACYCLIC SYLOW SUBGROUPS

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Abstract. Let p be an odd prime and G a finite group with non-abelian split metacyclic Sylow p-subgroup $P \rtimes Q$, where P is a cyclic group of order p^n for $n \geq 2$ and Q is a cyclic group of order p. Under some assumptions, we give the relative Q-projective covers of the simple modules in the principal p-block of G.

1. Introduction

In this paper, we study the relative projective covers of simple modules in the principal p-block of a finite group with non-abelian split metacyclic Sylow p-subgroup, where p is an odd prime.

Our motivation for this study comes from Rouquier's conjecture [7, A.2], which can be viewed as a non-abelian analogue to Broué's abelian defect group conjecture [1, 2]. Roughly speaking, Rouquier's conjecture predicts that if a p-block of a finite group has a defect group whose hyperfocal p-subgroup is abelian, then the block and its Brauer correspondent in the normalizer of the hyperfocal subgroup are derived equivalent. In this paper, we focus on the following special case. Let k be an algebraically closed field of characteristic p where p is an odd prime. Let $\widetilde{G}$ be a finite group with non-abelian split metacyclic Sylow p-subgroup $P \rtimes Q$ where P is a cyclic group of order p^a for $a \geq 2$ and Q is a cyclic group of order p. Then there is a normal subgroup G of $\widetilde{G}$ such that $\widetilde{G}$ is the semidirect product $G \rtimes Q$ and a Sylow p-subgroup of G is P. In this case, the principal block $B_0(k\widetilde{G})$ of $\widetilde{G}$ has $P \rtimes Q$ as a defect group and its hyperfocal p-subgroup is P, which is abelian; thus Rouquier's conjecture predicts that $B_0(k\widetilde{G})$ is derived equivalent to its Brauer correspondent $B_0(kN_{\widetilde{G}}(P))$. Remarkably, the principal block $B_0(kG)$ of G has a cyclic defect group P, a case where Broué's conjecture is known to hold; hence $B_0(kG)$ is derived equivalent to its Brauer correspondent $B_0(kN_G(P))$. Moreover, $N_{\widetilde{G}}(P)$ is isomorphic to the semidirect product $N_G(P) \rtimes Q$. Hence, one may expect that $B_0(k\widetilde{G})$ is derived equivalent to $B_0(kN_{\widetilde{G}}(P))$, that is, Rouquier's conjecture holds for this case.

To establish derived equivalences between blocks of finite groups, we often employ Okuyama's method [4], which lifts stable equivalences of Morita type to derived equivalences (see [6, 6.1]). The following theorem is a combination of Okuyama's method and simple-minded collections:

Theorem 1 ([6, Theorem 6.1]). *Let A and B be finite-dimensional symmetric algebras over a field k, let F : mod-$A \to$ mod-B be an exact functor inducing a stable equivalence*

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of Morita type, and let $\{S_i \mid 1 \leq i \leq n\}$ *be a complete set of non-isomorphic simple* A*-modules.*

If there is a simple-minded collection $X_1, \ldots, X_n$ *in the bounded derived category* $D^b(\text{mod-}B)$ *such that* X_i *is isomorphic to* $F(S_i)$ *in the stable category* $\underline{\text{mod}}\text{-}B$ *for each* $1 \leq i \leq n$*, then* A *and* B *are derived equivalent.*

This approach requires constructing a stable equivalence of Morita type between $B_0(k\widetilde{G})$ and $B_0(kN_{\widetilde{G}}(P))$ and analyzing the images of simple modules under this equivalence. While Okuyama [5] constructed such a stable equivalence of Morita type, understanding the images of simple modules remains a challenge. By [5], the images of simple modules under the stable equivalence of Morita type can be described via their Green correspondents and relative projective covers under certain conditions. These conditions require us to determine the Brauer construction of simple modules with respect to Q, which necessitates determining relative projective covers of simple modules. Therefore, as a first step towards establishing the derived equivalence between $B_0(k\widetilde{G})$ and $B_0(kN_{\widetilde{G}}(P))$, we focus on determining the relative Q-projective covers of the simple modules in $B_0(k\widetilde{G})$.

2. Relative projective covers

In this section, we recall the definition and basic properties of relative projective covers. Throughout this section, let k be a field and let G and H be finite groups. All modules are assumed to be finitely generated.

Definition 2. A kG-module M is said to be *relatively* H*-projective* if M is a direct summand of an induced module $\operatorname{Ind}_H^G(N)$ for some kH-module N.

If $|H| = 1$, then relatively H-projective modules are just projective modules. We have the following characterization of relative projective modules.

Lemma 3. *If* M *be a* kG*-module then the following are equivalent:*

1. M *is relatively* H*-projective.*
2. *If* $\phi\colon L \to N$ *is a surjective homomorphism of* kG*-modules which splits as a homomorphism of* kH*-modules, then* $\operatorname{Hom}_{kG}(M, \phi)$ *is surjective.*

In this paper, we define relative projective covers as follows.

Definition 4. Let M be a kG-module. A *relative* H*-projective cover* of M is a surjection homomorphism $\phi : P \to M$ of kG-modules such that

1. P is relatively H-projective,
2. ϕ is split as a homomorphism of kH-modules, and
3. any homomorphism $\psi\colon P \to P$ satisfying $\phi \circ \psi = \phi$ is an automorphism.

By Lemma 3, relative projective covers are unique up to isomorphism if they exist. Unlike projective covers, relative projective covers cannot be determined by comparing radical quotients. However, we have the following property.

Lemma 5. *Let* M *be a* kG*-module. Then the canonical surjection*

$$\epsilon_M : \operatorname{Ind}_H^G \operatorname{Res}_H^G(M) \to M \tag{2.1}$$

given the adjunction between induction and restriction functors satisfies (1) and (2) of Definiton 4. Moreover, there is a direct summand P of $\mathrm{Ind}_H^G \mathrm{Res}_H^G(M)$ such that the restriction of $\epsilon_M\,|_P\colon P \to M$ is the relative H-projective cover of M.

Thus, relative H-projective covers always exist.

3. The main result

We retain the notation from Section 1. Let $\{S_1, \ldots, S_n\}$ be a complete set of non-isomorphic simple $B_0(k\widetilde{G})$-modules and $\{S'_1, \ldots, S'_m\}$ a complete set of non-isomorphic simple $B_0(kC_G(Q))$-modules. Then we have $n = m$ automatically. Since $C_{\widetilde{G}}(Q) = C_G(Q) \times Q$, each projective cover of S'_i can be lifted to a $kC_{\widetilde{G}}(Q)$-module by inflation. Let N_i denote the Green correspondent of this lifted module in $B_0(k\widetilde{G})$ with respect to the triple $(\widetilde{G}, Q, C_{\widetilde{G}}(Q))$ for $1 \leq i \leq n$.

We make the following assumptions:

(1) The shape of the Brauer tree of $B_0(kG)$ is a line with the exceptional vertex at the right end. Each edge is labeled by $S_1, \ldots, S_n$ from left to right order.
(2) The shape of the Brauer tree of $B_0(kC_G(Q))$ is a line with the exceptional vertex at the right end. Each edge is labeled by $S'_1, \ldots, S'_n$ from left to right order.
(3) For $1 \leq i \leq n-1$, the restriction $\mathrm{Res}_G^{\widetilde{G}}(N_i)$ is indecomposable.

Our main result is the following:

Theorem 6 ([3])**.** *Under the above assumptions, N_i is the relative Q-projective cover of S_i for $1 \leq i \leq n$.*

For examples satisfying the assumptions, G can be taken as the group $GL(n, q^p)$ where q is a prime power having order n in $(\mathbb{Z}/p\mathbb{Z})^\times$. Since the Galois group $\mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q)$ is cyclic of order p, we have the semidirect product $\widetilde{G} = G \rtimes Q$ where Q is isomorphic to $\mathrm{Gal}(\mathbb{F}_{q^p}/\mathbb{F}_q)$ and $\widetilde{G}$ has a non-abelian split metacyclic Sylow p-subgroup. In this case, the assumptions are satisfied.

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