COHOMOLOGY ANNIHILATORS AND STRONG GENERATION OF SYZYGY SUBCATEGORIES

KAITO KIMURA

Abstract. Let R be a commutative Noetherian ring of dimension d. The category $\operatorname{mod} R$ of finitely generated R-modules is said to be strongly generated if there exist a finitely generated module G and nonnegative integers s and n such that the s-th syzygy of any finitely generated module can be built, up to direct summands and finite direct sums, from G by at most n extensions. The aim of this paper is to study the cohomology annihilator and, from this perspective, to address the stronger question of whether all d-th syzygies can be controlled by a single module.

1. Introduction

This paper is based on [4]. Throughout the present paper, let R be a commutative Noetherian ring of dimension d, and $\operatorname{mod} R$ the category of finitely generated R-modules. All subcategories are assumed to be full and strict. In the category of modules, the middle term of a short exact sequence is often called an extension of the modules on both sides. First, we introduce "generation" by extensions, which is the most important concept in this paper.

Definition 1. (1) Let $\mathcal{X}$ and $\mathcal{Y}$ be subcategories of $\operatorname{mod} R$. Denote by $|\mathcal{X}|$ the smallest subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$ that is closed under direct summands and finite direct sums. We denote by $\mathcal{X} \circ \mathcal{Y}$ the subcategory of $\operatorname{mod} R$ consisting of objects Z such that there exists an exact sequence $0 \to X \to Z \to Y \to 0$ in $\operatorname{mod} R$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We set $|\mathcal{X}|_0 = \{0\}$, $|\mathcal{X}|_1 = |\mathcal{X}|$, and $|\mathcal{X}|_n = ||\mathcal{X}|_{n-1} \circ |\mathcal{X}||$ for any $n \geq 2$. Following Dao and Takahashi [1], we define the *size* of $\mathcal{X}$, denoted by $\mathsf{size}(\mathcal{X})$, as the infimum of n such that $\mathcal{X} \subseteq |G|_{n+1}$ for some object $G \in \operatorname{mod} R$.

(2) A finitely generated R-module M is called *maximal Cohen–Macaulay* if $\operatorname{depth} M_{\mathfrak{p}} \geq \dim R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R. The subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules is denoted by $\mathsf{CM}(R)$. We set $\Omega^0 \operatorname{mod} R = \operatorname{mod} R$, and for $n \geq 1$, we denote by $\Omega^n \operatorname{mod} R$ the subcategory of $\operatorname{mod} R$ consisting of modules M such that there is an exact seuquence $0 \to M \to P_{n-1} \to \cdots \to P_1 \to P_0$ of finitely generated R-modules, where $P_0, \dots, P_{n-1}$ are projective. Following Iyengar and Takahashi [3], we call a finitely generated R-module G a *strong generator* for $\operatorname{mod} R$ if there exist integers s and n such that $\Omega^s \operatorname{mod} R$ is contained in $|\mathcal{X}|_n$.

While it is a natural trend to lift the representation theory of finite-dimensional (or Artinian) algebras to higher dimensions, studying the category of finitely generated modules over rings of positive dimension is difficult. What we focus on, therefore, are $\mathsf{CM}(R)$ and $\Omega^d \operatorname{mod} R$, and one reason for this is remarked below.

The detailed version of this paper will be submitted for publication elsewhere.

Remark 2. (1) If $\operatorname{mod} R$ has a strong generator G, then the global finiteness of a certain homological dimension is characterized by the finiteness of that of G. For example, the residue field of a local ring often satisfies such a property; what distinguishes G, however, is that it retains the same property even after localization of the ring. This allows the"locus" to be characterized by G, and in particular, the singular locus of a ring with a strong generator is closed. Conversely, it follows from this that there exist rings that do not admit a strong generator.

(2) Note that $\operatorname{mod} R$ has a strong generator if and only if $\mathsf{size}(\Omega^s \operatorname{mod} R) < \infty$ for some s. Since $\Omega^s \operatorname{mod} R$ forms a descending chain with respect to inclusion in s, we would like to consider generability for the smallest possible s. There do not exist $G \in \operatorname{mod} R$ and intege n such that $\Omega^{d-1} \operatorname{mod} R$ is contained in $|\mathcal{X}|_n$; see the proof of [6, Proposition 3.2] for instance. On the other hand, $\Omega^d \operatorname{mod} R$ is very close to $\mathsf{CM}(R)$. In fact, if R is Cohen–Macaulay, then $\Omega^d \operatorname{mod} R$ is contained in $\mathsf{CM}(R)$, and moreover, if R is locally Gorenstein on the punctured spectrum, they coincide.

In view of these facts, this paper considers the following question.

Question. When does $\mathsf{size}(\Omega^d \operatorname{mod} R) < \infty$ or $\mathsf{size}(\mathsf{CM}(R)) < \infty$?

2. Key facts and previous results

Note that $\mathsf{CM}(R)$ is basically considered only when R is Cohen–Macaulay. We present the key facts that are important for addressing this question and the previous research. Using the following method based on the proof of [1, Corollary 5.9], the problem is often reduced to the finiteness of the size of $\Omega^d \operatorname{mod} R$. Dao and Takahashi [1] also used the fact below to prove that the above question holds when R is a complete local ring with a perfect coefficient field.

Proposition 3. [1, See the proof of Corollary 5.9] *Let R be a Cohen–Macaulay ring with a canonical module. Then $\mathsf{size}(\Omega^d \operatorname{mod} R) < \infty$ if and only if $\mathsf{size}(\mathsf{CM}(R)) < \infty$.*

Research by Iyengar and Takahashi [3] shows that the existence of a strong generator is closely related to the properties of the annihilator ideals of Ext modules. In particular, the concept below known as the cohomology annihilator and their results concerning it play an essential role in the proof of the main result of this paper.

Definition 4. For each integer $n \geq 0$, we denote by $\mathsf{ca}^n(R)$ the ideal consisting of elements a that annihilate $\operatorname{Ext}_R^n(M, N)$ for all finitely generated R-modules M, N. The union $\bigcup_{n \geq 0} \mathsf{ca}^n(R)$ is called *cohomology annihilator* of R, which is denoted by $\mathsf{ca}(R)$.

Theorem 5. [3, Theorems 1.2 and 5.2] *The following statements hold.*
(1) *If for each prime ideal $\mathfrak{p}$ in R, $\mathsf{ca}(R/\mathfrak{p}) \neq 0$, then $\operatorname{mod} R$ has a strong generator.*
(2) *If for each prime ideal $\mathfrak{p}$ in R, $\mathsf{ca}^{\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$, then $\mathsf{size}(\Omega^d \operatorname{mod} R) < \infty$.*

Applying this method for determining the finiteness of the size of $\Omega^d \operatorname{mod} R$, the following results are known. The latter statement follows from Proposition 3.

Theorem 6. [3, Theorems 5.3 and 5.4] *In each of the following cases, $\mathsf{size}(\Omega^d \operatorname{mod} R) < \infty$ holds. Moreover, if R is Cohen–Macaulay, then $\mathsf{size}(\mathsf{CM}(R)) < \infty$.*
(1) *R is an equicharacteristic comlete local ring.*

(2) *R is a localization of a finitely generated algebra over a field.*

Another important point in the work of Iyengar and Takahashi is the relationship between the cohomology annihilator and the singular locus. When considering whether the assumptions of Theorem 5 are satisfied, this perspective is highly relevant. Indeed, in the context of Theorem 6, one can see that the assumptions of Theorem 5 are satisfied by considering the Jacobian ideal. We present the key previous work that is crucial for obtaining our main result without assuming equicharacteristic.

Theorem 7. [2, Corollary 3.12] and [3, Theorem 1.1] *If R is quasi-excellent, then the cohomology annihilator of R is a defining ideal of the singular locus of R. In particular, $\mathsf{ca}(R/\mathfrak{p}) \neq 0$ for any prime ideal $\mathfrak{p}$ in R.*

Note that the latter immediately follows from the fact that $R/\mathfrak{p}$ is a quasi-excellent integral domain.

3. Main results

The following is the main result of this paper. The proof is conducted via homological methods, by using the existence of a dualizing complex.

Theorem 8. [4, Theorem 1.1] *If R is either a homomorphic image of a Gorenstein ring or a local ring, then $\sqrt{\mathsf{ca}^{d+1}(R)} = \sqrt{\mathsf{ca}(R)}$ holds.*

The reason this theorem constitutes the main result is that, when combined with the facts from the previous section, it provides an affirmative answer to the question posed in the introduction.

Corollary 9. *Suppose that R is either a homomorphic image of a Gorenstein ring or a local ring. If R is quasi-excellent, then $\mathsf{ca}^{d+1}(R)$ is a defining ideal of the singular locus of R. In particular, $\mathsf{ca}^{\dim R/\mathfrak{p}+1}(R/\mathfrak{p}) \neq 0$ for any prime ideal $\mathfrak{p}$ in R, and hence $\mathsf{size}(\Omega^d \operatorname{mod} R) < \infty$.*

Proof. The statement is a direct consequence of Theorems 5, 7, and 8. The only point to note is that an ideal in an integral domain is nonzero if its radical is nonzero.

Finally, we conclude this paper by noting a few remarks about the main result.

Remark 10. (1) A Cohen–Macaulay ring that is a homomorphic image of a Gorenstein ring admits a canonical module. If R is either an equicharacteristic comlete local ring or a localization of a finitely generated algebra over a field, then it is excellent and a homomorphic image of a Gorenstein ring. From these facts and Proposition 3, Corollary 9 generalizes Theorem 6.

(2) When $(R, \mathfrak{m})$ is a quasi-excellent local ring, $\mathsf{ca}^{d+1}(R)$ is a defining ideal of the singular locus of R. In particular, if R has an isolated singularity, $\mathsf{ca}^{d+1}(R)$ is an $\mathfrak{m}$-primary ideal, so we can choose a system of parameters contained in $\mathsf{ca}^{d+1}(R)$. Such a system of parameters is called (or corresponds to) an efficient or faithful system of parameters, and it also plays an important role in Cohen–Macaulay representation theory; see [7, Chapter 6] and [5, Section 2 of Chapter 15] for instance.

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Graduate School of Mathematics
Nagoya University
Nagoya, Aichi 464-8602 JAPAN

Email address: `m21018b@math.nagoya-u.ac.jp`