

THE LIE STRUCTURE ON THE FIRST HOCHSCHILD COHOMOLOGY GROUPS OF BEILINSON ALGEBRAS OF GRADED DOWN-UP ALGEBRAS WITH WEIGHTS (n, m)

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ABSTRACT. Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $(\deg x, \deg y) = (n, m)$ and $\beta \neq 0$, and ∇A the Beilinson algebra of A . Note that A is a 3-dimensional cubic AS-regular algebra. Assume that $\gcd(n, m) = 1$ and $m \geq n$. In this report, we give the dimensional formula of the Hochschild cohomology group of ∇A for the case that $m \geq 2$ and $n \geq 2$. As an application, when $m > n > 1$, we prove that the derived category of a non-commutative projective scheme associated to A is not equivalent to the derived category of any smooth projective surface. For this case, we give a basis of the first Hochschild cohomology groups of ∇A by constructing the Bar resolution. Moreover, we give the Lie algebra structures on the first Hochschild cohomology groups obtained by the Gerstenhaber brackets for the case that $m \geq n \geq 1$, $m \neq 1$ and $\gcd(n, m) = 1$.

1. INTRODUCTION

This report is based on a part of [8]. Throughout this report, let k be an algebraically closed field with $\text{char } k = 0$. Noncommutative projective schemes associated to AS-regular algebras were formulated by Artin–Zhang [2]. It was defined by the Serre quotient of the category of finite generated right modules by the subcategory of finite dimensional right modules. Let $\text{mod } \nabla A$ denote the category of finitely generated right ∇A -modules, $\text{tails } A$ denote the noncommutative projective scheme. The bounded derived categories of $\text{tails } A$ and $\text{mod } \nabla A$ are denoted by $D^b(\text{tails } A)$ and $D^b(\text{mod } \nabla A)$, respectively. Regarding the relationship between a noetherian AS-regular algebra A and its Beilinson algebra ∇A , an important result was proved by Minamoto–Mori [11]; if A is a coherent AS-regular algebra of dimension d and ∇A is its Beilinson algebra, then ∇A is extremely Fano of the global dimension $d - 1$, and there exists an equivalence of triangulated categories $D^b(\text{tails } A) \cong D^b(\text{mod } \nabla A)$.

The graded algebra $A(\alpha, \beta) := k\langle x, y \rangle / (x^2y - \beta yx^2 - \alpha xyx, xy^2 - \beta y^2x - \alpha yxy)$ is called a graded down-up algebra with weights (n, m) , where $\alpha, \beta \in k$ and $\deg x = n, \deg y = m \in \mathbb{N}^+$. This down-up algebra was introduced by Benkart–Roby [4] for a study to a poset. Kirkman–Musson–Passman [10] showed that a graded down-up algebra $A = A(\alpha, \beta)$ is a 3-dimensional noetherian AS-regular algebra if and only if $\beta \neq 0$ holds.

The aim of our study is to investigate the Hochschild cohomology of ∇A of a down-up algebra $A = A(\alpha, \beta)$ with weights (n, m) and $\beta \neq 0$. Assume that $\gcd(n, m) = 1$ and $m \geq n$. If $n = 1$ and $m = 1$, then the description of the Hochschild cohomology group of ∇A was already given by Belmans [3]. Also, if $n = 1$ and $m \geq 2$, then the dimensional formula for the Hochschild cohomology group of ∇A was given by the first author and Ueyama [9].

The detailed version of this paper will be submitted for publication elsewhere.

In this study, we provide the dimensional formula for the Hochschild cohomology group of ∇A in the case that $n \geq 2$ and $m \geq 2$ (Theorem 7). Using this dimensional formula, we show that, if $m > n > 1$, then $\mathbf{D}^b(\text{tails } A)$ is not equivalent to the derived category of any smooth projective surface (Remark 13). Furthermore, for the case that $n \geq 2$ and $m \geq 2$, we construct a basis of the first Hochschild cohomology group of ∇A via the Bar resolution. Moreover, we give the Lie algebra structures on the first Hochschild cohomology groups induced by the Gerstenhaber bracket for the case that $m \geq n \geq 1$, $m \neq 1$, $\gcd(n, m) = 1$ (Theorem 15).

2. PRELIMINARIES

First, we recall some basic facts of Hochschild cohomology (see [13, Section 2] and [14, Chapter 1], for example). Throughout this section, let Λ be the k -algebra and $\Lambda^e := \Lambda^{\text{op}} \otimes_k \Lambda$ the enveloping algebra of Λ . We consider the following resolution of Λ as right Λ^e -modules which is called the Bar resolution of Λ :

$$\text{Bar}_\Lambda^\bullet : \cdots \xrightarrow{d_\Lambda^{i-1}} \Lambda^{\otimes i} \xrightarrow{d_\Lambda^{i-2}} \cdots \xrightarrow{d_\Lambda^2} \Lambda^{\otimes 3} \xrightarrow{d_\Lambda^1} \Lambda^{\otimes 2} \xrightarrow{\pi_\Lambda} \Lambda \rightarrow 0, \text{ where } \pi_\Lambda \text{ is multiplication map,}$$

$$d_\Lambda^i(b_0 \otimes b_1 \otimes \cdots \otimes b_{i+1}) = \sum_{l=0}^{i+1} (-1)^l b_0 \otimes b_1 \otimes \cdots \otimes b_{l-1} \otimes b_l b_{l+1} \otimes b_{l+2} \otimes \cdots \otimes b_{i+1}.$$

We consider the complex $\bigoplus_{i \geq 0} \text{Hom}_k(\Lambda^{\otimes i}, \Lambda)$ where the differential δ_Λ^i from $\text{Hom}_k(\Lambda^{\otimes(i-1)}, \Lambda)$ to $\text{Hom}_k(\Lambda^{\otimes i}, \Lambda)$ is defined by

$$\begin{aligned} \delta_\Lambda^i(f)(b_1 \otimes b_2 \otimes \cdots \otimes b_i) &= b_1 f(b_2 \otimes \cdots \otimes b_i) + \sum_{l=1}^{i-1} (-1)^l f(b_1 \otimes \cdots \otimes b_l b_{l+1} \otimes \cdots \otimes b_i) \\ &\quad + (-1)^i f(b_1 \otimes \cdots \otimes b_{i-1}) b_i. \end{aligned}$$

For each $i \geq 0$, there exists an isomorphism

$$\varepsilon_\Lambda^i : \text{Hom}_k(\Lambda^{\otimes i}, \Lambda) \rightarrow \text{Hom}_{\Lambda^e}(\Lambda^{\otimes(i+2)}, \Lambda); h \mapsto [b_0 \otimes \cdots \otimes b_{i+1} \mapsto b_0 h(b_1 \cdots b_i) b_{i+1}].$$

Moreover, $\bigoplus_{i \geq 0} \text{Hom}_k(\Lambda^{\otimes i}, \Lambda)$ is isomorphic to the complex $\bigoplus_{i \geq 0} \text{Hom}_{\Lambda^e}(\Lambda^{\otimes(i+2)}, \Lambda)$ as a complex, which is obtained by applying $\text{Hom}_{\Lambda^e}(-, \Lambda)$ to $\text{Bar}_\Lambda^\bullet$. The Hochschild cohomology of Λ is defined by the homology of $\bigoplus_{i \geq 0} \text{Hom}_k(\Lambda^{\otimes i}, \Lambda)$. We identify the elements of $\bigoplus_{i \geq 0} \text{Hom}_{\Lambda^e}(\Lambda^{\otimes(i+2)}, \Lambda)$ that correspond to the basis computed as above. The Hochschild cohomology of Λ is defined by $\text{HH}^i(\Lambda) := \text{Ker } \delta_\Lambda^{i+1} / \text{Im } \delta_\Lambda^i$. When Λ is free k -module, the Bar resolution of Λ is the projective resolution of Λ as right Λ^e -modules. Then $\text{HH}^i(\Lambda) \cong \text{Ext}_{\Lambda^e}^i(\Lambda, \Lambda)$ (see [14, Remarks 1.1.14]).

We recall the definition of the Gerstenhaber bracket of Hochschild cohomology (see [14]).

Definition 1. For each $[f] \in \text{HH}^i(\Lambda)$ and $[g] \in \text{HH}^{\otimes j}(\Lambda)$ represented by $f \in \text{Hom}_k(\Lambda^{\otimes i}, \Lambda)$ and $g \in \text{Hom}_k(\Lambda^{\otimes j}, \Lambda)$, the Gerstenhaber bracket is the residue class of the k -linear

homomorphism $[f, g]_\Lambda : \Lambda^{\otimes(i+j-1)} \rightarrow \Lambda$. The bracket $[f, g]_\Lambda$ is defined by $[f, g]_\Lambda = f \circ g - (-1)^{(i-1)(j-1)}g \circ f$, where

$$(f \circ g)(b_1 \otimes \cdots \otimes b_{i+j-1}) = \sum_{l=1}^i (-1)^{(n-1)(l-1)} f(b_1 \otimes \cdots \otimes b_{l-1} \otimes g(b_l \otimes \cdots \otimes b_{i+j-1}) \otimes b_{l+j} \otimes \cdots \otimes b_{i+j-1}).$$

In particular, $f \circ g$ is a standard composition of homomorphism when $i = j = 1$.

The following theorem is one of our motivations in this report for studying a down-up algebra.

Theorem 2 ([10]). *Let $A(\alpha, \beta)$ be the graded down-up algebra with weights $\deg x = n$ and $\deg y = m$, $\beta \neq 0$. Then, $A(\alpha, \beta)$ is an AS-regular algebra of the dimension 3 and the Gorenstein parameter $\ell = 2(n + m)$.*

We consider assumptions that we can assume without loss of generality when computing the Hochschild cohomology of the Beilinson algebra of a graded down-up algebra.

Proposition 3 ([9]). *Let $A := A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = n$ and $\deg y = m$. We write $A' := A'(\alpha, \beta)$ as a graded down-up algebra with the generators x', y' has weights $\deg x = n/r$ and $\deg y = m/r$ where $r := \gcd(n, m)$. Assume that $\beta \neq 0$. Then the Hochschild cohomology groups of Beilinson algebras of the above down-up algebras satisfy $\mathrm{HH}^i(\nabla A) \cong (\mathrm{HH}^i(\nabla A'))^r$ for all $i \geq 0$.*

By Proposition 3, we can assume that $\gcd(\deg x, \deg y) = 1$ without loss of generality.

Lemma 4 ([6, 4.1. Lemma (i)]). *Let $A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = n$ and $\deg y = m$. If $\beta \neq 0$, then $A(\alpha, \beta) \cong A(-\alpha\beta^{-1}, \beta^{-1})$ via the map interchanging x and y .*

Hence, we can assume that $\deg x \leq \deg y$ without loss of generality.

To introduce these results of [3] and [9], we define a sequence $\lambda_{i+2} = \alpha\lambda_{i+1} + \beta\lambda_i$ with $\lambda_0 = 0$ and $\lambda_{-1} = \beta^{-1}$. Here, we name the following cases:

Case I	$n + m$ is even and $\alpha = 0$	Case 1	$\lambda_{m+1} = 0$
Case II	otherwise	Case 2	$\lambda_{m+1} \neq 0$ and $\alpha^2 + 4\beta = 0$
		Case 3	$\lambda_{m+1} \neq 0$ and $\alpha^2 + 4\beta \neq 0$

Theorem 5 ([3, Table 2]). *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = 1$, $\deg y = 1$, $\beta \neq 0$, and ∇A the Beilinson algebra of A . Then we obtain the dimensional formula of the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A as follows:*

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 6 & \text{for Case I (and Case 1),} \\ 3 & \text{for Case II and Case 2,} \\ 1 & \text{for Case II and Case 3;} \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} 9 & \text{for Case I (and Case 1),} \\ 6 & \text{for Case II and Case 2,} \\ 4 & \text{for Case II and Case 3;} \end{cases}$

- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ for $i \geq 3$.

Theorem 6 ([9, Theorem 1.4]). *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = 1$, $\deg y > 1$, $\beta \neq 0$, and ∇A the Beilinson algebra of A . Then we obtain the dimensional formula of the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A as follows:*

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 4 & \text{for Case I (and Case 1),} \\ 3 & \text{for Case II and Case 1,} \\ 2 & \text{for Case II and Case 2,} \\ 1 & \text{for Case II and Case 3;} \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} 8 & \text{for } m = 2, \text{ Case II, and Case 1,} \\ 7 & \text{for } m = 2, \text{ Case II, and Case 2,} \\ 6 & \text{for } m = 2, \text{ Case II, and Case 3,} \\ m + 5 & \text{for } m \geq 3 \text{ and Case I hold (and Case 1),} \\ m + 4 & \text{for } m \geq 3, \text{ Case II, and Case 1,} \\ m + 3 & \text{for } m \geq 3, \text{ Case II, and Case 2,} \\ m + 2 & \text{for } m \geq 3, \text{ Case II, and Case 3;} \end{cases}$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ for $i \geq 3$.

3. MAIN RESULTS

3.1. The dimensional formula of the Hochschild cohomology groups. In this subsection, we show the dimensional formula of the Hochschild cohomology group of the Beilinson algebras of graded down-up algebras with $\beta \neq 0$, $\deg x = n$ and $\deg y = m$ where $m \geq n > 1$ and $\gcd(n, m) = 1$ in a similar way to Green–Snashall [7] and [9].

Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = n$ and $\deg y = m$, $\beta \neq 0$, where $m \geq n \geq 1$ and $\gcd(n, m) = 1$, and ∇A the Beilinson algebra of A . Then, ∇A is represented by the following quiver $Q_{\nabla A} = ((Q_{\nabla A})_0, (Q_{\nabla A})_1, s, t)$ and the admissible ideal \mathcal{I} of $kQ_{\nabla A}$, that is, the quiver $Q_{\nabla A} = ((Q_{\nabla A})_0, (Q_{\nabla A})_1, s, t)$ is defined by

$$\begin{cases} (Q_{\nabla A})_0 := \{e_1, e_2, \dots, e_{2(n+m)}\}, \\ (Q_{\nabla A})_1 := \{x_1, x_2, \dots, x_{n+2m}, y_1, y_2, \dots, y_{2n+m}\}, \\ s(x_i) := e_i, t(x_i) = e_{i+n} \text{ for } 1 \leq i \leq n + 2m, \\ s(y_j) := e_j, t(y_j) = e_{j+m} \text{ for } 1 \leq j \leq 2n + m; \end{cases}$$

and the admissible ideal \mathcal{I} generated by the relations

$$\begin{cases} f_i := x_i x_{i+n} y_{i+2n} - \alpha x_i y_{i+n} x_{i+n+m} - \beta y_i x_{i+m} x_{i+n+m} & \text{for } 1 \leq i \leq m, \\ g_j := x_j y_{j+n} y_{j+n+m} - \alpha y_j x_{j+m} y_{j+n+m} - \beta y_j y_{j+m} x_{j+2m} & \text{for } 1 \leq j \leq n. \end{cases}$$

By the similar calculations of [7] and [9], we obtain the dimensions of $\mathrm{HH}^i(\nabla A)$ for all $i \geq 0$.

Theorem 7. *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = n$, $\deg y = m$, $\beta \neq 0$, where $m \geq n > 1$ and $\gcd(n, m) = 1$, and ∇A the Beilinson algebra*

of A . Then we obtain the dimensional formula of the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A as follows:

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 2 & \text{for Case I,} \\ 1 & \text{for Case II;} \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} m + 5 & \text{for } n = 2 \text{ and Case I,} \\ m + 4 & \text{for } n = 2 \text{ and Case II,} \\ n + m + 1 & \text{for } n \geq 3 \text{ and Case I,} \\ n + m & \text{for } n \geq 3 \text{ and Case II;} \end{cases}$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ for $i \geq 3$.

3.2. Applications of Theorem 7. In this subsection, we give the similar result in [9, Section 3] for the case that $m > n > 1$. Regarding the bounded derived category of coherent sheaves on a smooth projective variety, Bondal–Polishchuk [5] proved the following theorem:

Theorem 8 ([5, Lemma 3.1]). *Let $\mathrm{Db}(\mathrm{coh} X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X and κ the automorphism on $K_0(\mathrm{D}^b(\mathrm{coh} X))$ induced by the Serre functor K of $\mathrm{D}^b(\mathrm{coh} X)$. The action of $(-1)^{\dim X} \kappa$ on $K_0(\mathrm{D}^b(\mathrm{coh} X))$ is unipotent.*

To use Theorem 8 to check the correspondence between down-up algebras and some geometric objects, we recall a useful matrix which is called the Gram matrix.

Definition 9 (see [5]). Let \mathbf{T} be a triangulated category and $K_0(\mathbf{T})$ the Grothendieck group of \mathbf{T} . We denote $[X] \in K_0(\mathbf{T})$ as the isomorphic class of $X \in \mathbf{T}$. Suppose that χ is the bilinear form on $K_0(\mathbf{T})$ defined by $\chi([X], [Y]) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}^i(X, Y)$. If the category \mathbf{T} is generated by an exceptional collection $\{E_i\}_{1 \leq i \leq r}$ of length r , then the Gram matrix M of χ for this collection of vectors is the $r \times r$ matrix defined as $M = (\chi([E_i], [E_j]))_{1 \leq i, j \leq r}$.

Definition 10 (see [1]). Let Λ be a basic finite dimensional k -algebra with a complete set $\{e_1, \dots, e_r\}$ of primitive orthogonal idempotents of Λ . The Cartan matrix of Λ is the $r \times r$ matrix $C_\Lambda = (\dim_k e_i \Lambda e_j)_{1 \leq i, j \leq r}$.

Let $K_0(\mathrm{D}^b(\mathrm{tails} A))$ be the Grothendieck group of the bounded derived category $\mathrm{D}^b(\mathrm{tails} A)$ and \mathfrak{s} the automorphism on $K_0(\mathrm{D}^b(\mathrm{tails} A))$ induced by the Serre functor S of $\mathrm{D}^b(\mathrm{tails} A)$. (In this case, the existence of Serre functor \mathfrak{s} on $\mathrm{D}^b(\mathrm{tails} A)$ was proved by de Naeghel–Van den Bergh [12, Appendix A]). We recall a useful proposition as follows:

Proposition 11 (see [9, Section 3]). *Let $A = A(\alpha, \beta)$ be a down up algebra with weights $\deg x = n$, $\deg y = m$ and $\beta \neq 0$. Let $\mathrm{D}^b(\mathrm{tails} A)$ be the bounded derived category of tails A and ∇A the Beilinson algebra of A . Then $\{A(i) \mid 0 \leq i \leq 2(n + m) - 1\}$ is a full strong exceptional collection in $\mathrm{D}^b(\mathrm{tails} A)$ and the Gram matrix M of χ on $\mathrm{D}^b(\mathrm{tails} A)$ for this collection of vectors is equal to the Cartan matrix of ∇A .*

Corollary 12. Let $A = A(\alpha, \beta)$ be a down up algebra with weights $\deg x = n$, $\deg y = m$, $\beta \neq 0$, where $\gcd(n, m) = 1$ and $m > n > 1$ and the bounded derived category $D^b(\text{tails } A)$ of $\text{tails } A$. Then

$$\chi_{\text{HH}}(\text{HH}^\bullet(\nabla A)) = \begin{cases} m + 4 & \text{if } n = 2, \\ n + m & \text{if } n \geq 3, \end{cases} \neq 2(n + m) = \text{rk} K_0(D^b(\text{tails } A)),$$

where $\chi_{\text{HH}}(\text{HH}^\bullet(\nabla A)) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \text{HH}^i(\nabla A)$. Also, \mathfrak{s} does not act unipotently on $K_0(D^b(\text{tails } A))$.

Remark 13. By Corollary 12, we obtain the same result in [9]: When $m > n > 1$, $D^b(\text{tails } A)$ is not equivalent to the derived category of any smooth projective surface.

3.3. The Lie algebra structures on the first Hochschild cohomology groups.

Finally, in this subsection, we describe the Lie algebra structures on the first Hochschild cohomology groups $\text{HH}^1(\nabla A)$ obtained by the Gerstenhaber bracket for the case that $m \geq n \geq 1$, $m \neq 1$ and $\gcd(n, m) = 1$.

Remark 14. For $m = n = 1$, the Lie algebra structures of the first Hochschild cohomology groups were calculated by [3, Table 7].

We define some elements of $\text{Hom}_k(\nabla A, \nabla A)$ under some cases as follows:

- (1) When $m \geq n \geq 1$ and $\gcd(n, m) = 1$, we define k -linear mappings $T_1, T_2 : \nabla A \rightarrow \nabla A$ as follows:

- For each vertex $a \in (Q_{\nabla A})_0$, $T_r(a) = 0$ for $r = 1, 2$.
- For each arrow $a \in (Q_{\nabla A})_1$,

$$T_1(a) = \begin{cases} x_r & \text{if } a = x_\gamma \text{ for } 1 \leq r \leq 2(n + m), \\ 0 & \text{otherwise;} \end{cases}$$

$$T_2(a) = \begin{cases} (-1)^{i-(n+1)} x_i & \text{if } a = x_i \text{ for } n + 1 \leq i \leq n + m, \\ 0 & \text{otherwise.} \end{cases}$$

- For each path $a_1 \cdots a_r$ of $Q_{\nabla A}$, $T_r(a_1 \cdots a_r) = \sum_{i=1}^r a_1 \cdots a_{i-1} T_r(a_i) a_{i+1} \cdots a_r$ for $r = 1, 2$.

- (2) When $m \geq n = 1$ and $\gcd(n, m) = 1$, we define k -linear mappings $T_3, T_4, T_5 : \nabla A \rightarrow \nabla A$ as follows:

For each vertex $a \in (Q_{\nabla A})_0$, $T_r(a) = 0$ for $r = 3, 4, 5$.

- For each arrow $a \in (Q_{\nabla A})_1$,

$$T_3(a) = \begin{cases} \lambda_{i-1} x_i \cdots x_{i+m-1} & \text{if } a = y_i \text{ for } 1 \leq i \leq m + 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$T_4(a) = \begin{cases} \beta \lambda_{i-2} x_i \cdots x_{i+m-1} & \text{if } a = y_i \text{ for } 1 \leq i \leq m + 2, \\ 0 & \text{otherwise;} \end{cases}$$

$$T_5(a) = \begin{cases} \left(\frac{\alpha}{2}\right)^{i-1} x_i \cdots x_{i+m-1} & \text{if } a = y_i \text{ for } 1 \leq i \leq m + 2, \\ 0 & \text{otherwise.} \end{cases}$$

- For each path $a_1 \cdots a_r$ of $Q_{\nabla A}$, $T_r(a_1 \cdots a_r) = \sum_{i=1}^r a_1 \cdots a_{i-1} T_3(a_i) a_{i+1} \cdots a_r$
for $r = 3, 4, 5$.
- (3) When $m = n = 1$ and $\gcd(n, m) = 1$, we define k -linear mappings $T_{3'}, T_{4'}, T_{5'} : \nabla A \rightarrow \nabla A$ as follows:
- For each vertex $a \in (Q_{\nabla A})_0$, $T_r(a) = 0$ for $r = 3', 4', 5'$.
 - For each arrow $a \in (Q_{\nabla A})_1$,
- $$T_{3'}(a) = \begin{cases} y_2 & \text{if } a = x_2 \\ 0 & \text{otherwise;} \end{cases} \quad T_{4'}(a) = \begin{cases} \beta y_1 & \text{if } a = x_1 \\ y_3 & \text{if } a = x_3 \\ 0 & \text{otherwise;} \end{cases}$$
- $$T_{5'}(a) = \begin{cases} \left(\frac{\alpha}{2}\right)^2 y_1 & \text{if } a = x_1 \\ \left(\frac{\alpha}{2}\right) y_2 & \text{if } a = x_2 \\ y_3 & \text{if } a = x_3 \\ 0 & \text{otherwise.} \end{cases}$$
- For each path $a_1 \cdots a_r$ of $Q_{\nabla A}$, $T_r(a_1 \cdots a_r) = \sum_{i=1}^r a_1 \cdots a_{i-1} T_r(a_i) a_{i+1} \cdots a_r$
for $r = 3', 4', 5'$.

Theorem 15. *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with weights $\deg x = n$, $\deg y = m$, $\beta \neq 0$ where $\gcd(n, m) = 1$, $m \geq n \geq 1$, and $(n, m) \neq (1, 1)$. Then the Lie structures of $\text{HH}^1(\nabla A)$ are as follows:*

(1) When $m > n = 1$.

(a) If Case I (and Case 1) hold, then we have

$$\begin{aligned} \text{HH}^1(\nabla A) &= \langle [T_1], [T_2], [T_3], [T_4] \rangle_k, \\ [[T_1], [T_2]]_{\nabla A} &= 0, \quad [[T_1], [T_3]]_{\nabla A} = m[T_3], \quad [[T_1], [T_4]]_{\nabla A} = m[T_4], \\ [[T_2], [T_3]]_{\nabla A} &= [T_3], \quad [[T_2], [T_4]]_{\nabla A} = 0, \quad [[T_3], [T_4]]_{\nabla A} = 0. \end{aligned}$$

(b) If Case II and Case 1 hold, then we have

$$\begin{aligned} \text{HH}^1(\nabla A) &= \langle [T_1], [T_3], [T_4] \rangle_k, \\ [[T_1], [T_3]]_{\nabla A} &= m[T_3], \quad [[T_1], [T_4]]_{\nabla A} = m[T_4], \quad [[T_3], [T_4]]_{\nabla A} = 0. \end{aligned}$$

(c) If Case II and Case 2 hold, then we have

$$\begin{aligned} \text{HH}^1(\nabla A) &= \langle [T_1], [T_5] \rangle_k, \\ [[T_1], [T_5]]_{\nabla A} &= m[T_5]. \end{aligned}$$

(d) If Case II and Case 3 hold, then we have $\text{HH}^1(\nabla A) = \langle [T_1] \rangle_k$.

(2) When $m > n > 1$.

(a) If Case I holds, then we have

$$\begin{aligned} \text{HH}^1(\nabla A) &= \langle [T_1], [T_2] \rangle_k, \\ [[T_1], [T_2]]_{\nabla A} &= 0. \end{aligned}$$

(b) If Case II holds, then we have $\text{HH}^1(\nabla A) = \langle [T_1] \rangle_k$.

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