SYMMETRIC COHOMOLOGY OF TRIANGULAR BIALGEBRAS

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ABSTRACT. The symmetric cohomology of groups was introduced by Staic [11] as a version of the classical group cohomology, which was extended to cocommutative Hopf algebras over a field by Shiba–Sanada–Itaba [10]. In this report, we explain a procedure to construct the symmetric cohomology of triangular bialgebras, generalizing the previous works to the setting where the symmetry on the category of left modules can be chosen arbitrarily. Our construction makes use of a category $(\mathbf{\Delta S})_+$, which witnesses both the augmented simplicial structure, similar to the case for some other cohomologies, as well as the actions of the symmetric groups, being essential to the symmetric cohomology.

1. INTRODUCTION

Let G be a group, and M a G-module (or equivalently, a $\mathbb{Z}[G]$-module; see Example 4 (1)). The group cohomology $\mathrm{H}^\bullet(G, M)$ of G can be defined to be the cohomology of a chain complex $\mathrm{K}^\bullet(G, M)$ of abelian groups, where for all $n \geq 0$, $\mathrm{K}^n(G, M)$ consists of $\mathbb{Z}[G]$-linear maps $\mathbb{Z}[G]^{\otimes(n+1)} \to M$, and $\mathbb{Z}[G]^{\otimes(n+1)}$ is regarded as a left $\mathbb{Z}[G]$-module via the formula $g(g_0 \otimes \cdots \otimes g_n) := gg_0 \otimes \cdots \otimes gg_n$ for all $g, g_0, \ldots, g_n \in G$, and the tensor product is taken over $\mathbb{Z}$. This complex is isomorphic to the complex $\mathrm{C}^\bullet(G, M)$ of abelian groups, where $\mathrm{C}^n(G, M)$ consists of the (set-theoretic) maps $G^n \to M$ (see Pirashvili [8, page 400]). Motivated from lattice topological field theory, Staic [11] introduced the symmetric cohomology $\mathrm{HS}^\bullet(G, M)$ of G to be the cohomology of a subcomplex $\mathrm{CS}^\bullet(G, M)$ of $\mathrm{C}^\bullet(G, M)$, where $\mathrm{CS}^n(G, M)$ consists of those n-cochains which are invariant under a certain left action of the symmetric group S_{n+1} on $\mathrm{C}^\bullet(G, M)$. If we translate this to the left action on $\mathrm{K}^n(G, M)$, the action is given by the formula

$$(1.1) \quad ((i, i+1) \cdot \phi)(g_0 \otimes \cdots \otimes g_n) := -\phi(g_0 \otimes \cdots \otimes g_{i-2} \otimes g_i \otimes g_{i-1} \otimes g_{i+1} \otimes \cdots \otimes g_n)$$

for all $\phi \in \mathrm{K}^n(G, M)$, $g_0, \ldots, g_n \in G$, and $1 \leq i \leq n$ (when $n > 0$), where, for all $1 \leq i \leq n$, $(i, i+1) \in S_{n+1}$ denotes the adjacent transposition that swaps i and $i+1$.

Staic [12] established a correspondence between the cohomology group $\mathrm{HS}^2(G, M)$ and the set of equivalence classes of extensions of G by M, that is, short exact sequences of groups of the form $0 \to M \to X \to G \to 0$, subject to certain conditions:

Theorem 1 ([12, Theorem 3.2]). *Let G be a group, and M a G-module. Then, there is a one-to-one correspondence between the set $\mathrm{HS}^2(G, M)$ and the set of equivalence classes of extensions $0 \to M \to X \xrightarrow{\pi} G \to 0$ such that π admits a set-theoretic section $t\colon G \to X$ which satisfies the equality $t(g^{-1}) = t(g)^{-1}$ for every $g \in G$.*

On the other hand, Pirashvili [9] established a correspondence between the cohomology group $\mathrm{HS}^3(G, M)$ and the set of crossed extensions of G by M that are subject to certain

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conditions, when G has no elements of order two. We refer to [9, Subsection 2.3] for the definition of crossed extensions, and to [9, Theorem 4.5] for the precise statement. These results are analogous to the standard facts known for the (classical) group cohomology.

Let A be a cocommutative Hopf algebra over a field k and M a left A-module (see Section 2 for the relevant definitions). Shiba–Sanada–Itaba [10] defined the symmetric cohomology $\mathrm{HS}^\bullet(A, M)$ of A, which generalizes the symmetric cohomology of groups in the sense that group algebras are cocommutative Hopf algebras. They also defined the symmetric Hochschild cohomology $\mathrm{HHS}^\bullet(A, X)$ of A. Here, X appearing in the coefficient is an A-bimodule such that $\lambda x = x\lambda$ for all $\lambda \in k$ and $x \in X$, or equivalently, X is a left module over the enveloping algebra $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{op}}$ of A. The following result by [10] is a symmetric version of the result due to Ginzburg–Kumar [4, page 197, Proposition].

Theorem 2 ([10, Theorem 4.2])**.** *Let A be a cocommutative Hopf algebra over a field k that is equipped with the antipode S, and X an A-bimodule. Then, there is an isomorphism*

$$\mathrm{HHS}^n(A, X) \cong \mathrm{HS}^n(A, {}^{\mathrm{ad}}X)$$

of k-vector spaces for every $n \geq 0$, where ${}^{\mathrm{ad}}X$ is the k-vector space X whose left A action is given by the formula $a \cdot x := ((A \otimes S) \circ \Delta)(a)x$ for all $a \in A$ and $x \in X$.

In this report, we explain a procedure to construct the symmetric cohomology of triangular bialgebras. This notion is a generalization of a cocommutative bialgebra, where the cocommutativity condition (see Example 8) is replaced by the existence of a symmetry on the category of left modules on a given bialgebra. The key step to our generalization is the observation that the group action (1.1) comes from the fact that the representations of the group G forms a symmetric monoidal category.

2. Triangular bialgebras and symmetric monoidal categories

Throughout this report, we fix a nonzero commutative base ring k, and the unadorned tensor product $\otimes$ means $\otimes_k$, except in Definitions 5 and 11 and in Remark 14. Algebras and their homomorphisms are defined over k, and modules are left modules over an algebra which are compatible with the underlying k-module structures. Given an algebra Λ, we let $\Lambda\,\mathrm{Mod}$ denote the category of Λ-modules. We refer to Etingof–Gelaki–Nikshych–Ostrik [2] for the basic concepts of bialgebras and of monoidal categories.

Definition 3 (cf. [2, Definition 5.2.2])**.** A *bialgebra* is a triple (A, Δ, ε) consisting of an algebra A and algebra homomorphisms $\Delta\colon A \to A \otimes A$, $\varepsilon\colon A \to k$ such that the diagrams

$$
\begin{array}{ccc}
& A & \\
{}^{\Delta}\swarrow & & \searrow^{\Delta} \\
A \otimes A & & A \otimes A \\
{}_{\Delta \otimes A}\downarrow & & \downarrow^{A \otimes \Delta} \\
(A \otimes A) \otimes A & \xrightarrow{\ \sim\ } & A \otimes (A \otimes A);
\end{array}
$$

$$\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \otimes A \\
{\scriptstyle \mathrm{id}} \downarrow & & \downarrow {\scriptstyle \varepsilon \otimes A} \\
A & \xrightarrow[\sim]{\ \ } & k \otimes A;
\end{array}
\qquad\qquad
\begin{array}{ccc}
A \otimes A & \xleftarrow{\ \Delta\ } & A \\
{\scriptstyle A \otimes \varepsilon} \downarrow & & \downarrow {\scriptstyle \mathrm{id}} \\
A \otimes k & \xleftarrow[\sim]{\ \ } & A,
\end{array}$$

commute, where the unlabeled morphisms $\xrightarrow{\sim}$ are the canonical isomorphisms.

Example 4. (1) Let G be a monoid. The monoid algebra $k[G]$ of G becomes a bialgebra $(k[G], \Delta, \varepsilon)$, where $\Delta(g) := g \otimes g$ and $\varepsilon(g) := 1$ for all $g \in G$. When G is a group, the (bi)algebra $k[G]$ is called the *group algebra* of G.

(2) Let $\mathfrak{g} = (\mathfrak{g}, [-,-])$ be a Lie algebra. The *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the algebra

$$U(\mathfrak{g}) := T(\mathfrak{g})/\langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g}\rangle,$$

where $T(\mathfrak{g})$ is the tensor algebra of (the underlying k-module of) $\mathfrak{g}$. This becomes a bialgebra $(U(\mathfrak{g}), \Delta, \varepsilon)$, where $\Delta(\overline{x}) := \overline{x} \otimes 1 + 1 \otimes \overline{x}$ and $\varepsilon(\overline{x}) := 0$ for all $x \in \mathfrak{g}$. Here, we identify $\mathfrak{g}$ with the k-submodule $T_1(\mathfrak{g})$ of $T(\mathfrak{g})$, and we write $\overline{x}$ for the image of $x \in \mathfrak{g}$ under the quotient morphism $T(\mathfrak{g}) \to U(\mathfrak{g})$.

Let (A, Δ, ε) be a bialgebra. The algebra homomorphisms Δ and ε induce functors

$$\Delta^*\colon (A \otimes A)\,\mathrm{Mod} \longrightarrow A\,\mathrm{Mod}, \quad \varepsilon^*\colon k\,\mathrm{Mod} \longrightarrow A\,\mathrm{Mod}.$$

Composing Δ^* with the evident functor $A\,\mathrm{Mod} \times A\,\mathrm{Mod} \to (A \otimes A)\,\mathrm{Mod}$ defines a functor

$$\otimes\colon A\,\mathrm{Mod} \times A\,\mathrm{Mod} \longrightarrow A\,\mathrm{Mod}; \ (L, M) \longmapsto L \otimes M.$$

On the other hand, since k can be regarded as a k-module in the obvious way, we obtain an A-module $\varepsilon^*(k)$, which we still denote by k. Concretely, for every pair of A-modules L and M, we regard $L \otimes M$ as an A-module via $a(\ell \otimes m) := \Delta(a)(\ell \otimes m)$ for all $a \in A$, $\ell \in L$, and $m \in M$, and we regard k as an A-module via $a\lambda := \varepsilon(a)\lambda$ for all $a \in A$ and $\lambda \in k$, where juxtaposition on the right hand side is the multiplication in k. This enables us to deal with tensor products of finitely many A-modules, which are associative and unital up to isomorphism in the following sense: For every triple of A-modules L, M, and N, we have an isomorphism $(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$ of A-modules, and for every A-module M, we have an isomorphism $k \otimes M \cong M \cong M \otimes k$ of A-modules, and these (fixed) isomorphisms are natural and satisfy some coherence conditions. Therefore, a bialgebra (A, Δ, ε) gives rise to a structure of a (*k-linear*) *monoidal category* on $A\,\mathrm{Mod}$; see [2, Definition 2.1.1] for a precise definition.

The monoidal category $k\,\mathrm{Mod}$ of k-modules is equipped with a natural isomorphism $V \otimes W \to W \otimes V; v \otimes w \mapsto w \otimes v$ of k-modules for every pair of k-modules V and W. The notion of a symmetry axiomatizes the properties of these flip morphisms:

Definition 5 (cf. [2, Definitions 8.1.1 and 8.1.12]). Let $\mathcal{C}$ be a monoidal category. A *symmetry* β on $\mathcal{C}$ is a collection of isomorphisms $\beta_{L,M}\colon L \otimes M \to M \otimes L$ which is natural in $L, M \in \mathcal{C}$ and satisfies the following conditions:

(1) For every $L, M, N \in \mathcal{C}$, the following diagram commutes:

$$
\begin{array}{ccccc}
(L \otimes M) \otimes N & \xrightarrow{\ \sim\ } & L \otimes (M \otimes N) & \xrightarrow{\ \beta_{L, M \otimes N}\ } & (M \otimes N) \otimes L \\
\downarrow{\scriptstyle \beta_{L,M} \otimes \mathrm{id}} & & & & \downarrow{\scriptstyle \wr} \\
(M \otimes L) \otimes N & \xrightarrow[\ \sim\ ]{} & M \otimes (L \otimes N) & \xrightarrow[\ \mathrm{id} \otimes \beta_{L, N}\ ]{} & M \otimes (N \otimes L),
\end{array}
$$

where the unlabeled morphisms $\xrightarrow{\ \sim\ }$ are canonical isomorphisms which witnesses the associativity of the tensor product.

(2) For every $L, M \in \mathcal{C}$, the equality $\beta_{M,L} \circ \beta_{L,M} = \mathrm{id}_{L \otimes M}$ holds.

A *symmetric monoidal category* is a pair $(\mathcal{C}, \beta)$ consisting of a monoidal category $\mathcal{C}$ and a symmetry β on $\mathcal{C}$.

The notion of an R-matrix was introduced by Drinfel'd [1, page 811] for a study of quantum groups. It will be sufficient to define only the unitary R-matrices for our purposes.

Definition 6 (cf. [2, Definitions 8.3.1 and 8.3.3]). Let $A := (A, \Delta, \varepsilon)$ be a bialgebra. An element $R = \sum_u (R_u^1 \otimes R_u^2) \in A \otimes A$ is called a *unitary R-matrix* of A if the pair $(A\,\mathrm{Mod}, \beta^R)$ is a symmetric monoidal category, where the symmetry $\beta^R = \{\beta_{L,M}^R\}_{L,M}$ on the monoidal category $A\,\mathrm{Mod}$ is defined by the formula

$$
(2.1) \qquad \beta_{L,M}^R(\ell \otimes m) := \sum_u (R_u^2 m) \otimes (R_u^1 \ell) \qquad (\forall \ell \in L, \forall m \in M)
$$

for every pair of A-modules L and M. A *triangular bialgebra* is a pair (A, R) consisting of a bialgebra A and a unitary R-matrix R of A.

Remark 7. Let $A := (A, \Delta, \varepsilon)$ be a bialgebra. In general, an element $R \in A \otimes A$ defines a natural transformation $\beta^R = \{\beta_{L,M}^R\}_{L,M}$ via the equality (2.1) if and only if the equality

$$
(\tau \circ \Delta)(a) R = R \Delta(a)
$$

holds for every $a \in A$, where $\tau \colon A \otimes A \to A \otimes A; a \otimes a' \mapsto a' \otimes a$ denotes the flip morphism. In this case, β^R is a symmetry on $A\,\mathrm{Mod}$ if and only if the following equalities hold:

$$
(2.2) \qquad\qquad\qquad (A \otimes \Delta)(R) = R^{13} R^{12};
$$

$$
(2.3) \qquad\qquad\qquad R^{21} R = 1 \otimes 1,
$$

which correspond to conditions (1) and (2) of Definition 5, respectively. Here, we used the abbreviations

$$
R^{13} := \sum_u (R_u^1 \otimes 1 \otimes R_u^2); \quad R^{12} := \sum_u (R_u^1 \otimes R_u^2 \otimes 1); \quad R^{21} := \sum_u (R_u^2 \otimes R_u^1),
$$

for the given element $R =: \sum_u (R_u^1 \otimes R_u^2)$ in $A \otimes A$.

Example 8. Let $A := (A, \Delta, \varepsilon)$ be a bialgebra. The element $1 \otimes 1 \in A \otimes A$ satisfies the equalities (2.2) and (2.3) automatically since $\Delta(1) = 1 \otimes 1$. Therefore, the pair $(A, 1 \otimes 1)$ is a triangular bialgebra if and only if the equality $\tau \circ \Delta = \Delta$ holds. A bialgebra with this property is said to be *cocommutative*. In Example 4, the monoid algebras and the universal enveloping algebras are cocommutative.

We end this section with some remarks on Hopf algebras.

Definition 9 (cf. [2, Proposition 5.3.1]). A *Hopf algebra* is a bialgebra (A, Δ, ε) such that there exists a k-linear map $S \colon A \to A$ which makes the following diagram commutative:

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\ S \otimes \mathrm{id}\ } & A \otimes A \\
{\scriptstyle \Delta} \nearrow & & \searrow {\scriptstyle \pi} \\
A \xrightarrow{\ \varepsilon\ } k \xrightarrow{\ \eta\ } A, \\
{\scriptstyle \Delta} \searrow & & \nearrow {\scriptstyle \pi} \\
A \otimes A & \xrightarrow[\ \mathrm{id} \otimes S\ ]{} & A \otimes A
\end{array}
$$

where $\pi \colon A \otimes A \to A; a \otimes a' \mapsto aa'$ and $\eta \colon k \to A; \lambda \mapsto \lambda 1$ are the multiplication and the unit of A. In this case, S uniquely exists, and is called the *antipode* of (A, Δ, ε).

Example 10.   (1) The monoid algebra $k[G]$ of a monoid G is a Hopf algebra if and only if G is a group. In this case, the antipode S of $k[G]$ is given by the formula $S(g) := g^{-1}$ for all $g \in G$.
    (2) The universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra, and the antipode S of $U(\mathfrak{g})$ is given by the formula $S(\overline{x}) := -\overline{x}$ for all $x \in \mathfrak{g}$.

3. MAIN RESULT

Finally, in this section, we explain the way to construct the symmetric cohomology of triangular bialgebras. When we associate to geometric or algebraic inputs certain cohomologies, such as the singular cohomology of topological spaces or the Hochschild cohomology of algebras, the construction often factors through an augmented simplicial object, which is defined to be a contravariant functor from the augmented simplex category $\mathbf{\Delta}_+$ (see Remark 12). However, as we will explain in the same remark, the category $\mathbf{\Delta}_+$ does not a priori contain information about interesting actions of the symmetric groups. Therefore, in order to encode this data on the associated functor, it is necessary and reasonable to deal with a certain enlargement $(\mathbf{\Delta S})_+$ of $\mathbf{\Delta}_+$.

Definition 11. The category $(\mathbf{\Delta S})_+$ is defined as follows:
- The objects of $(\mathbf{\Delta S})_+$ are written as $[n]$, indexed by the integers $n \geq -1$.
- The morphisms of $(\mathbf{\Delta S})_+$ are generated by the face maps $\delta_j^n \colon [n-1] \to [n]$, the degeneracy maps $\sigma_j^n \colon [n+1] \to [n]$, where $n \geq 0$ and $0 \leq j \leq n$, as well as the "adjacent transpositions" $\tau_i^n \colon [n] \to [n]$ where $n > 0$ and $1 \leq i \leq n$, subject to
  - the cosimplicial relations for δ_j^n and σ_j^n (see Goerss–Jardine [5, I.(1.2)]);
  - the Moore relations for τ_i^n (same as those in the Coxeter presentation of S_{n+1} which appear in [10, page 7]);

– the relations

$$(3.1) \qquad \tau_i^n \circ \delta_j^n = \begin{cases} \delta_j^n \circ \tau_{i-1}^{n-1} & \text{if } j < i - 1; \\ \delta_i^n & \text{if } j = i - 1; \\ \delta_{i-1}^n & \text{if } j = i; \\ \delta_j^n \circ \tau_i^{n-1} & \text{if } j > i, \end{cases}$$

and

$$(3.2) \qquad \tau_i^n \circ \sigma_j^n = \begin{cases} \sigma_j^n \circ \tau_{i+1}^{n+1} & \text{if } j < i - 1; \\ \sigma_i^n \circ \tau_i^{n+1} \circ \tau_{i+1}^{n+1} & \text{if } j = i - 1; \\ \sigma_{i-1}^n \circ \tau_{i+1}^{n+1} \circ \tau_i^{n+1} & \text{if } j = i; \\ \sigma_j^n \circ \tau_i^{n+1} & \text{if } j > i. \end{cases}$$

Remark 12. Let us write $\mathcal{S}$ for the subcategory of $(\mathbf{\Delta S})_+$, whose objects are same as those of $(\mathbf{\Delta S})_+$, and whose morphisms are those generated by δ_j^n and σ_j^n. Then, $\mathcal{S}$ is isomorphic to the *augmented simplex category* $\mathbf{\Delta}_+$, whose objects are finite ordinals i.e., linearly ordered sets of the form $n = \{0, \ldots, n-1\}$ for all $n \geq 0$, and whose morphisms are nondecreasing maps. The isomorphism $\mathbf{\Delta}_+ \cong \mathcal{S}$ is given on objects by sending $[n]$ to $n + 1 = \{0, \ldots, n\}$ for all $n \geq -1$. We thus have an embedding $\mathbf{\Delta}_+ \to \mathcal{S} \subset (\mathbf{\Delta S})_+$, which is far from being full, since for every $n \geq -1$, the group $\mathrm{Aut}_{\mathbf{\Delta}_+}(n + 1)$ is trivial, whereas the group $\mathrm{Aut}_{(\mathbf{\Delta S})_+}([n])$ is isomorphic to the symmetric group S_{n+1} (see Remark 14).

Remark 13. Let us write $\mathbf{\Delta S}$ for the full subcategory of $(\mathbf{\Delta S})_+$ obtained by deleting the object $[-1]$. This category is constructed from the collection of symmetric groups $\{S_{n+1}\}_{n \geq 0}$ using the structure of a crossed simplicial group, introduced independently by Krasauskas [6] and Fiedorowicz–Loday [3]. The relations (3.1) and (3.2) are derived using the crossed simplicial group structure on $\{S_{n+1}\}_{n \geq 0}$; we omit the details.

Remark 14. The category $(\mathbf{\Delta S})_+$ is a strict symmetric monoidal category, where for $n, m \geq -1$, the tensor product $[n] \otimes [m]$ is defined to be $[n + m + 1]$, and $[-1]$ is the unit object. It follows from a classical fact (see MacLane [7, page 254] for example) that for every $n \geq -1$, there exists a group homomorphism

$$(3.3) \qquad S_{n+1} \longrightarrow \mathrm{Aut}_{(\mathbf{\Delta S})_+}([0]^{\otimes(n+1)}) = \mathrm{Aut}_{(\mathbf{\Delta S})_+}([n]),$$

which in fact is an isomorphism in this case. In contrast to the countably many morphisms given in Definition 11, as a strict monoidal category, $(\mathbf{\Delta S})_+$ is generated by an object $[0]$ and morphisms $\delta_0^0 \colon [-1] \to [0]$, $\sigma_0^0 \colon [1] \to [0]$, and $\tau_1^1 \colon [1] \to [1]$.

Recall that a *monoidal functor* between monoidal categories is a functor which is compatible with the tensor products and the unit objects up to some natural isomorphisms; see [2, Definition 2.4.1] for a precise definition.

We can give the construction of the symmetric cohomology of triangular bialgebras.

Construction 15. Let (A, R) be a triangular bialgebra, and M an A-module. The *symmetric cohomology* $\mathrm{HS}^\bullet(A, R, M)$ of (A, R) is defined as follows:

(1) In view of Remark 14, the triangular bialgebra (A, R) gives rise to a monoidal functor $C := \widetilde{\mathrm{T}}(A, R) \colon (\mathbf{\Delta S})_+^{\mathrm{op}} \to A\mathrm{Mod}$ such that $C_n := C([n]) := A^{\otimes(n+1)}$ for every $n \geq -1$, $C(\delta_0^0) := \varepsilon$, $C(\sigma_0^0) := \Delta$, and $C(\tau_1^1) := \beta_{A,A}^R$.

(2) The functor $\widetilde{\mathrm{T}}(A, R)$ gives rise to an augmented chain complex

$$(3.4) \qquad \cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \longrightarrow C_{-1} \longrightarrow 0 \longrightarrow \cdots ,$$

of A-modules, where for all $n \geq 0$, the differential $C_n \to C_{n-1}$ is defined to be the alternating sum $\sum_{j=0}^{n} (-1)^j C(\delta_j^n)$. For every $n \geq -1$, by composing the group homomorphism (3.3) with the group homomorphism $\mathrm{Aut}_{(\mathbf{\Delta S})_+}([n]) \to (\mathrm{Aut}_A(C_n))^{\mathrm{op}}$ induced from the functor $\widetilde{\mathrm{T}}(A, R)$, we obtain a right action of S_{n+1} on C_n.

(3) We define a cochain complex

$$\mathrm{K}^\bullet(A, M)\colon \ \cdots \longrightarrow 0 \longrightarrow \mathrm{K}^0(A, M) \longrightarrow \mathrm{K}^1(A, M) \longrightarrow \mathrm{K}^2(A, M) \longrightarrow \cdots$$

of k-modules by sending (the stupid truncation of) the chain complex (3.4) via the additive functor $\mathrm{Hom}_A(-, M)\colon (A\,\mathrm{Mod})^{\mathrm{op}} \to k\,\mathrm{Mod}$. It follows from Step (2) that for every $n \geq 0$, we have a left action of S_{n+1} on $\mathrm{K}^n(A, M) = \mathrm{Hom}_A(C_n, M)$.

(4) For $n \geq 0$, we define a k-submodule $\mathrm{KS}^n(A, M)$ of $\mathrm{K}^n(A, M)$ by

$$\mathrm{KS}^n(A, M) := \{\phi \in \mathrm{K}^n(A, M) \mid \forall \tau \in S_{n+1}, \tau \cdot \phi = (\mathrm{sgn}\,\tau)\phi\}.$$

Using the relations (3.1), it follows that $\mathrm{KS}^\bullet(A, M)$ is a subcomplex of $\mathrm{K}^\bullet(A, M)$.

(5) We define $\mathrm{HS}^\bullet(A, R, M)$ by $\mathrm{HS}^n(A, R, M) := H^n(\mathrm{KS}^\bullet(A, M))$ for all $n \geq 0$.

Remark 16. Let (A, R) be a triangular bialgebra, M an A-module, and $n > 0$ a positive integer. For all $\phi \in \mathrm{K}^n(A, M)$ and $1 \leq i \leq n$, the A-linear map $(i, i+1) \cdot \phi$ is equal to $\phi \circ (A^{\otimes(i-1)} \otimes \beta_{A,A}^R \otimes A^{\otimes(n-i)})$, which concretely is defined as

$$((i, i+1)\phi)(a_0 \otimes \cdots \otimes a_n) := \phi(a_0 \otimes \cdots a_{i-2} \otimes \beta_{A,A}^R(a_{i-1} \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_n)$$

$$= \sum_u \phi(a_0 \otimes \cdots a_{i-2} \otimes (R_u^2 a_i) \otimes (R_u^1 a_{i-1}) \otimes a_{i+1} \otimes \cdots \otimes a_n)$$

for all $a_0, \ldots, a_n \in A$. In the case that A is cocommutative and $R = 1 \otimes 1$, the above formula coincides with the formula [10, (3.2)] up to sign, which contains the formula (1.1) in the introduction as a more special case that $A = k[G]$ for some monoid G.

References

[1] V. G. Drinfel'd, *Quantum groups*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 798–820.

[2] P. I. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, AMS Mathematical Surveys and Monographs **205**, 2015.

[3] Z. Fiedorowicz and J.-L. Loday, *Crossed simplicial groups and their associated homology*, Trans. Amer. Math. Soc. **326** (1991), no. 1, 57–87.

[4] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no. 1, 179–198.

[5] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009.

[6] R. Krasauskas, *Skew-simplicial groups*, Litovsk. Mat. Sb. **27** (1987), no. 1, 89–99.

[7] S. MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics **5**, Springer, New York-Berlin, 1971.

[8] M. Pirashvili, *Symmetric cohomology of groups*, J. Algebra **509** (2018), 397–418.

[9] _____, *Crossed modules and symmetric cohomology of groups*, Homology Homotopy Appl. **22** (2020), no. 2, 123–134.

[10] Y. Shiba, K. Sanada and A. Itaba, *Symmetric cohomology and symmetric Hochschild cohomology of cocommutative Hopf algebras*, J. Algebra Appl. **23** (2024), no. 13, 2450223, 25pp.

[11] M. D. Staic, *From 3-algebras to Δ-groups and symmetric cohomology*, J. Algebra **322** (2009), no. 4, 1360–1378.

[12] _____, *Symmetric cohomology of groups in low dimension*, Arch. Math. (Basel) **93** (2009), no. 3, 205–211.

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