SEMIBRICKS IN SUPER CATEGORY O

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Abstract. In this paper we introduce the idea that when it is difficult to understand the structure of a given module strictly in the Jordan–Hölder sense as a successive extension of simple modules, one may gain insight by considering filtrations by larger modules. We explain how this question can make sense in the representation theory of Kac–Moody Lie superalgebras.

1. Semibricks

For simplicity, let the base field k be an algebraically closed field of characteristic 0. In a k-linear abelian category, by Schur's lemma the collection of simple objects has pairwise orthogonal Hom spaces. Conversely, a collection of objects whose Hom spaces are pairwise orthogonal is called a *semibrick*. More precisely, a family $\{M_i\}_{i\in I}$ is a semibrick if

$$\dim \operatorname{Hom}(M_i, M_j) = \delta_{ij} \qquad (\forall\, i, j \in I).$$

The following classical result of Ringel gives, in a sense, a converse to Schur's lemma: a family satisfying Schur's lemma can be realized as the set of simple objects of some Serre subcategory.

Theorem 1 ([Rin76] Theorem 2.1)**.** *Let $\mathcal{A}$ be a k-linear abelian category. The assignments*

$$S \longmapsto \operatorname{Filt} S \quad \textit{(the full subcategory of objects filtered by } S\textit{)}$$

and

$$\mathcal{W} \longmapsto \operatorname{sim} \mathcal{W} \quad \textit{(the set of simple objects of } \mathcal{W}\textit{)}$$

induce a bijection between the following two classes:

1. *Semibricks S in $\mathcal{A}$.*
2. *Extension-closed finite-length abelian subcategories $\mathcal{W}$ of $\mathcal{A}$.*

In the special case where the family of simple objects S is closed under subquotients, the filtration closure is a *Serre subcategory*. Informally, one may view this as regarding some non-simple objects as if they were simple.

The concept of semibricks has been actively studied in recent years in the representation theory of finite-dimensional algebras, particularly in the context of τ-tilting theory (see [Asa20]). The motivating question from the introduction can be rephrased in the language of semibricks (though the flavor is different from τ-tilting theory):

Given a concrete and classical abelian category, can one construct a natural and interesting semibrick? Can one describe its filtration closure?

The detailed version of this paper will be submitted for publication elsewhere.

2. The even Verma theorem

The classical BGG category $\mathcal{O}$ is a well-established abelian category: it is the Serre subcategory generated by simple highest-weight modules in the category of modules over a semisimple Lie algebra, and it has been studied since the 1970s. Highest-weight modules are bricks (i.e., have End $\cong k$) and, by universality, homomorphisms between Verma modules can be controlled to some extent. Thus it is natural to attempt to construct semibricks from highest-weight modules. However, the classical Verma theorem / BGG results [Ver68, BGG76] state that in the semisimple Lie algebra setting, all homomorphisms between Verma modules are either injective or zero, and they are governed by the Bruhat order.

3. Weyl groupoids

Heckenberger–Yamane (2008)[HY08] introduced the Weyl groupoid (generalized root system / Cartan graph), which extends the classical notion of a root system by allowing *essentially different choices of bases*. The correctness of this notion is supported by its motivations and consequences.

The origin lies in the Andruskiewitsch–Schneider program to classify finite-dimensional pointed Hopf algebras via Nichols algebras of diagonal type. Heckenberger (2009) used Weyl groupoids to give a Lie-theoretic classification of Nichols algebras of finite Gelfand–Kirillov dimension; in particular the case of commutative coradicals is essentially resolved.

The nilpotent part of small quantum groups introduced by Nichols (1978) is now understood as a Nichols algebra — a braided analogue of the symmetric algebra in a braided monoidal category. Lusztig (1993) showed that the nilpotent part of Drinfeld–Jimbo type quantum groups is a Nichols algebra, which provides a conceptual algebraic viewpoint on quantum groups distinct from the usual description as q-deformations of universal enveloping algebras of semisimple Lie algebras.

An interesting fact is that rank 2 Weyl groupoids are equivalent to Conway–Coxeter frieze patterns. The classical A_2, B_2, G_2 root systems correspond to frieze patterns arising from triangular dissections, and reflections correspond to shifts of the rows. Frieze patterns arising from other polygon dissections produce rank 2 Weyl groupoids. For example, the frieze pattern coming from a pentagon dissection corresponds to the exceptional-type Nichols algebra ufo(7).

Cuntz–Heckenberger (2015) proved that finite-type Weyl groupoids are classified into frieze-pattern-derived cases, infinite series coming from Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, and 76 exceptional families. See the surveys and data books [AA17, HS20] for more on this topic.

4. Kac–Moody Lie superalgebras and odd reflections

The category of super vector spaces is essentially the category of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces; what matters there is the *braiding* (symmetry), which is independent of the additive and monoidal structures and encodes important intrinsic information such as dimension. In Lie superalgebras the odd part causes generators to square to zero, producing behavior that differs from the classical case.

Kac classified finite-dimensional Kac–Moody type (basic classical) Lie superalgebras, and this classification can be embedded naturally into the Weyl groupoid classification of Nichols algebras [AYY15]. Regular symmetrizable Kac–Moody Lie superalgebras are also treated within the Weyl groupoid framework (though types P, Q are exceptions).

Classically, reflections split into even reflections and odd reflections; odd reflections change the nature of the base in an essential way. Considering a Weyl groupoid is equivalent to simultaneously handling both even and odd reflections. Azam–Yamane–Yousefzadeh (2015)[AYY15] used the existence of the longest element in a Weyl groupoid to recover Kac's classification of finite-dimensional irreducible representations in a transparent way.

Concretely, for $\mathfrak{gl}(m|n)$ the even subalgebra is isomorphic to $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$, the full set of reflections behaves like the symmetric group S_{m+n}, even reflections form $S_m \times S_n$, and odd reflections are naturally described by an edge-colored graph. In the case $\mathfrak{gl}(2|2)$, Borel subalgebras correspond to Young diagrams fitting in 2×2 , and odd reflections correspond to edges in that lattice.

5. The super category $\mathcal{O}$ and semibricks

For a Borel subalgebra b, the b-Verma module

$$M^b(\lambda) := \mathrm{Ind}_b^{\mathfrak{g}}\, k_\lambda$$

has the usual universality but depends strongly on b. Let $\mathcal{O}^b$ denote the Serre subcategory generated (as the filtration closure) by the simple tops of the modules $M^b(\lambda)$. Then $\mathcal{O}^b$ is a finite-length abelian category containing the b-Verma modules.

$\mathcal{O}^b$ depends only on the even part of b, hence all Borel subalgebras related by odd reflections produce b-Verma modules that lie inside the same category $\mathcal{O}$, although the highest-weight structures they induce differ. Thus the super category $\mathcal{O}$ can be regarded as a typical example of a category admitting multiple highest-weight structures.

Note that the category $\mathcal{O}$ and the category of weight modules are not closed under Ext^1 inside the whole module category, so it is essential that we take our filtration categories inside $\mathcal{O}$. On the other hand, $\mathcal{O}$ itself is closed under Ext^n for every n [CM15].

6. Exchange property of odd reflections

Lemma 2. *For any basic classical Lie superalgebra $\mathfrak{g}$, any weight λ, and any two Borel subalgebras $\mathfrak{b}, \mathfrak{b}'$ sharing the same even part, the following hold. Here $\rho^{\mathfrak{b}}$ denotes the Weyl vector associated to $\mathfrak{b}$:*

- $\mathrm{ch}\, M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}) = \mathrm{ch}\, M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})$.
- $\dim \mathrm{Hom}\big(M^{\mathfrak{b}}(\lambda - \rho^{\mathfrak{b}}),\, M^{\mathfrak{b}'}(\lambda - \rho^{\mathfrak{b}'})\big) = 1$.

For simplicity assume $\lambda = 0$. In this case the following additional property holds:

- The composition of nonzero homomorphisms

  $$M^{\mathfrak{b}}(-\rho^{\mathfrak{b}}) \longrightarrow M^{\mathfrak{b}'}(-\rho^{\mathfrak{b}'}) \longrightarrow M^{\mathfrak{b}}(-\rho^{\mathfrak{b}})$$

  is always zero when $\mathfrak{b} \neq \mathfrak{b}'$.

In particular, homomorphisms between modules attached to adjacent Borel subalgebras via odd reflections are basic in the sense above.

Since hom spaces on the above diagram are unique up to scalars, understanding homomorphisms reduces to deciding precisely when the composition of nonzero homomorphisms is nonzero.

Theorem 3. [Hir25]*[Odd Verma Theorem] The composition of homomorphisms between adjacent modules $M^{\mathfrak{b}}(-\rho^{\mathfrak{b}})$ connected by odd reflections can be naturally identified with a walk w on the above diagram. Under this identification the following are equivalent:*

(1) $w \neq 0$.
(2) *w is* rainbow *(i.e., all edge colors along w are pairwise distinct).*
(3) *w is shortest among all walks with the same start and end points.*

The equivalence of (2) and (3) is the exchange property for odd reflections noted by Gorelik–Hinich–Serganova [GHS22]; the Odd Verma Theorem may be seen as a representation-theoretic consequence of this property. This is analogous to the well-known Coxeter-group exchange property for even reflections: the equality of length and number of inversions corresponds to the statement that a walk is shortest if and only if it is rainbow in the Cayley graph.

One emphasized point of this paper is that Heckenberger–Yamane [HY08] generalized the Coxeter case to Weyl groupoids and that by removing even reflections from the Cayley graph of a Weyl groupoid one obtains the exchange property for odd reflections. This approach is, in our view, considerably simpler than the explanation in [GHS22]. If one restricts attention to $\mathfrak{gl}(m|n)$ alone, the exchange property is straightforward and the explicit machinery of Weyl groupoids might not be strictly necessary. However, the Weyl groupoid viewpoint allows extension of the Odd Verma Theorem to regular symmetrizable Kac–Moody Lie superalgebras and to diagonal-type Nichols algebras, and it explains the naturality of the result.

For a general weight λ, it may happen that $M^{\mathfrak{b}}(\lambda-\rho^{\mathfrak{b}}) \cong M^{\mathfrak{b}'}(\lambda-\rho^{\mathfrak{b}'})$ even when $\mathfrak{b} \neq \mathfrak{b}'$. To ensure that modules attached to adjacent odd reflections are non-isomorphic in a suitable sense, we introduce a quotient graph by identifying colors appropriately. The exchange property descends to this quotient graph, and the Odd Verma Theorem continues to hold in the same form.

7. Applications

As an application of the Odd Verma Theorem, we construct a natural semibrick for $\mathfrak{g} = \mathfrak{gl}(2|1)$. For a nonzero homomorphism

$$M^{\mathfrak{b}}(-\rho^{\mathfrak{b}}) \longrightarrow M^{\emptyset}(-\rho^{\emptyset})$$

denote its image by $\operatorname{Im} M^{\mathfrak{b}}(-\rho^{\mathfrak{b}})$. Then by the Odd Verma Theorem the following family

$$
\begin{aligned}
H^{\emptyset} &:= \{B^{\emptyset},\, B^{(1)},\, B^{(1^2)}\},\\
B^{\emptyset} &:= M^{\emptyset}(-\rho^{\emptyset}) \big/ \operatorname{Im} M^{(1)}(-\rho^{(1)}),\\
B^{(1)} &:= \operatorname{Im} M^{(1)}(-\rho^{(1)}) \big/ \operatorname{Im} M^{(1^2)}(-\rho^{(1^2)}),\\
B^{(1^2)} &:= \operatorname{Im} M^{(1^2)}(-\rho^{(1^2)}).
\end{aligned}
$$

is a semibrick, and moreover the projective covers of the $\mathfrak{b}$-highest-weight modules $B^{\mathfrak{b}}$ in $\operatorname{Filt} H^{\emptyset}$ are given by the Verma modules $M^{\mathfrak{b}}(-\rho^{\mathfrak{b}})$. Vanishing of certain Ext^1 between Verma modules inside the subcategory is easy to verify using universality. The main difficulty lies in showing that Verma modules indeed belong to the filtration closure subcategory.

Using the Odd Verma Theorem together with Morita theory, the category $\operatorname{Filt}(H^{\emptyset})$ is equivalent to the category of finite-dimensional modules over the finite-dimensional algebra given by the double quiver of the Young lattice $L(2,1)$ modulo the relations:

- Every non-rainbow walk is zero.
- Any two paths with the same start and end points are equal.

Consequently,

$$
B^{\mathfrak{b}} \cong \frac{M^{\mathfrak{b}}(-\rho^{\mathfrak{b}})}{\sum_{\mathfrak{b}' \neq \mathfrak{b}} \operatorname{Im} M^{\mathfrak{b}'}(-\rho^{\mathfrak{b}'})}.
$$

This can be interpreted as a decomposition of the difficulties in describing Verma modules: separating the even part into modules $B^{\mathfrak{b}}$ absorbs the even complications, leaving the odd part describable in a combinatorial (quiver) manner.

Moreover, we obtain the decomposition

$$
\operatorname{Soc} M^{\mathfrak{b}}(-\rho^{\mathfrak{b}}) = \bigoplus_{\left\{ \mathfrak{b}' \in \mathrm{OR}(\mathfrak{g}) \;\middle|\; \begin{array}{l} \exists\, \mathfrak{b}'' \neq \mathfrak{b}, \mathfrak{b}' \text{ such that} \\ \text{there is a rainbow path from } \mathfrak{b}'' \\ \text{to } \mathfrak{b} \text{ passing through } \mathfrak{b}' \end{array} \right\}} \operatorname{Soc} B^{\mathfrak{b}'}.
$$

For example, one has:

- $\operatorname{Soc} M^{\emptyset}(-\rho^{\emptyset}) \cong \operatorname{Soc} B^{(2^2)}$.
- $\operatorname{Soc} M^{(1)}(-\rho^{(1)}) \cong \operatorname{Soc} B^{\emptyset} \oplus \operatorname{Soc} B^{(2^2)}$.
- $\operatorname{Soc} M^{(1^2)}(-\rho^{(1^2)}) \cong \operatorname{Soc} B^{\emptyset} \oplus \operatorname{Soc} B^{(2)} \oplus \operatorname{Soc} B^{(2^2)}$.

Although $\operatorname{Soc} B^{\mathfrak{b}}$ is generally nontrivial, this seems a plausible first step toward the subtle unresolved problem of determining the socles of Verma supermodules. In the special case $\mathfrak{b} = \emptyset$, Chen–Coulembier–Mazorchuk [CCM21] showed that the socle of the Verma module is simple.

Analogous results hold for $\mathfrak{gl}(m|1)$ and for appropriate partial semibricks. In the general case, a canonical semibrick exists for basic classical Lie superalgebras and for finite-dimensional Nichols algebras as a direct consequence of the Odd Verma Theorem, but one encounters technical difficulties in describing the filtration closure.

We note that for other settings, such as some affine cases, similar coarse-grained considerations about semibricks hold more broadly.

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