AN EXTENSION THEOREM OF SEMIBRICKS

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Abstract. Bricks and semibricks are generalizations of (semi)simple modules, and they are fundamental notions in the representation theory of algebras. By definition, any subset of a semibrick is a semibrick again. Thus it is natural to consider whether a given semibrick can be extended to another semibrick or not. We give a useful sufficient condition for a semibrick to be extendable in terms of representation schemes, which can be called an extension theorem of semibricks.

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1. Preliminaries

We let Λ be a finite-dimensional algebra over an algebraically closed field K. We write $\mathsf{mod}\,\Lambda$ for the category of finite-dimensional right Λ-modules.

Bricks and semibricks in $\mathsf{mod}\,\Lambda$ are defined as modules satisfying Schur's Lemma as follows.

Definition 1. We define the following notions.

(1) Let $B \in \mathsf{mod}\,\Lambda$. Then B is called a *brick* if $B \neq 0$ and every nonzero endomorphism $f \colon B \to B$ is isomorphic. We write $\mathsf{brick}\,\Lambda$ for the set of isoclasses of bricks in $\mathsf{mod}\,\Lambda$.

(2) Let $\mathcal{S} \subset \mathsf{brick}\,\Lambda$ be a subset. Then $\mathcal{S}$ is called a *semibrick* if any distinct two elements $B_1 \neq B_2$ in $\mathcal{S}$ satisfy $\mathsf{Hom}_\Lambda(B_1, B_2) = 0$. We write $\mathsf{sbrick}\,\Lambda$ for the set of semibricks in $\mathsf{mod}\,\Lambda$.

Since K is algebraically closed, $B \in \mathsf{mod}\,\Lambda$ is a brick if and only if $\mathsf{End}_\Lambda(B) \simeq K$.

It is clear that, if $\mathcal{S}$ is a semibrick and $\mathcal{S}' \subset \mathcal{S}$, then $\mathcal{S}'$ is also a semibrick. Moreover $\mathsf{sbrick}\,\Lambda$ has a poset structure by inclusions. Thus we study whether a given semibrick is properly contained in another semibrick or not. We say that a semibrick $\mathcal{S} \in \mathsf{sbrick}\,\Lambda$ is *maximal* if there exists no $\mathcal{S}' \in \mathsf{sbrick}\,\Lambda$ such that $\mathcal{S} \subsetneq \mathcal{S}'$.

Note that we allow an *infinite* semibrick; that is, a semibrick consisting of infinitely many isoclasses of bricks. Many algebras including the Kronecker quiver algebra are known to have infinite semibricks. If Λ has an infinite semibrick, then $\mathsf{sbrick}\,\Lambda$ is obviously an infinite set. It is conjectured that the converse also holds [7, Conjecture 2.2].

The detailed version of this paper will be submitted for publication elsewhere.

On the other hand, any algebra Λ admits a maximal finite semibrick, because the semibrick consisting of all isoclasses of simple modules is maximal and finite. Moreover any left finite semibrick $\mathcal{S}$ can be extended to some left finite semibrick $\mathcal{S}'$ which is maximal finite, and we have $\#\mathcal{S} \leq \#\mathcal{S}' \leq |\Lambda|$ [1, Corollary 2.10, Theorem 2.21], where $|\Lambda|$ is the number of isoclasses of simple Λ-modules.

The main question of this proceeding is:

> when a finite semibrick can be extended to another semibrick (or equivalently, not maximal)?

We have obtained a sufficient condition (Theorem 3(2)) in our study. To explain this, we use *representation schemes*. We may assume that Λ is of the form KQ/I, where $Q = (Q_0, Q_1)$ is a finite quiver and I is an admissible ideal of KQ. Then for each dimension vector $d = (d_i)_{i \in Q_0} \in (\mathbb{Z}_{\geq 0})^{Q_0}$, we set the *representation scheme* $\mathsf{rep}(\Lambda, d)$ by

$$\left\{ (\phi_\alpha)_{\alpha \in Q_1} \in \prod_{\alpha \in Q_1} \mathsf{Hom}_K(K^{s(\alpha)}, K^{t(\alpha)}) \mid \phi_\alpha\text{'s satisfy the relations corresponding to } I \right\},$$

where $\alpha \in Q_1$ starts at $s(\alpha)$ and ends at $t(\alpha)$. Thus $\mathsf{rep}(\Lambda, d)$ can be seen as the set of all Λ-modules whose dimension vectors are d.

By the Zariski topology, we write $\mathsf{Irr}(\Lambda, d)$ for the set of irreducible components of $\mathsf{rep}(\Lambda, d)$. We also define $\mathsf{Irr}(\Lambda)$ as the disjoint union of $\mathsf{Irr}(\Lambda, d)$'s for all $d \in (\mathbb{Z}_{\geq 0})^{Q_0}$.

Moreover there exists an action on $\mathsf{rep}(\Lambda, d)$ by the connected algebraic group $\mathsf{GL}(d) := \prod_{i \in Q_0} \mathsf{GL}(d_i)$. For each point $M \in \mathsf{rep}(\Lambda, d)$, its $\mathsf{GL}(d)$-orbit is denoted by $\mathcal{O}_M$. Then $\mathcal{O}_M$ is nothing but the subset $\{N \in \mathsf{rep}(\Lambda, d) \mid M \text{ and } N \text{ are isomorphic as } \Lambda\text{-modules}\}$. Since $\mathsf{GL}(d)$ is connected, every irreducible component $\mathcal{Z} \in \mathsf{Irr}(\Lambda, d)$ is a union of $\mathsf{GL}(d)$-orbits. In other words, if $M \in \mathcal{Z}$, then $\mathcal{O}_M$ and its closure $\overline{\mathcal{O}_M}$ are contained in $\mathcal{Z}$. Moreover $\mathcal{O}_M$ is open in $\overline{\mathcal{O}_M}$. See [4] for detail.

2. Open bricks and main results

To explain our results, we introduce a new notion on bricks.

Definition 2. Let $B \in \mathsf{brick}\,\Lambda$. Then B is called an *open brick* if its orbit closure $\overline{\mathcal{O}_B}$ is an irreducible component of $\mathsf{rep}(\Lambda, d)$; or equivalently, there exists $\mathcal{Z} \in \mathsf{Irr}(\Lambda, d)$ such that $\mathcal{O}_M$ is open dense in $\mathcal{Z}$.

Then we can state our main result. The part (2) is "an extension theorem of semibricks" in the title, and (1) follows from this.

Theorem 3. [2, Theorems 1.1, 1.2] *Let $\mathcal{S} \in \mathsf{sbrick}\,\Lambda$ be a finite semibrick.*

(1) *If $\mathcal{S}$ is maximal, then every brick $B \in \mathcal{S}$ is an open brick.*
(2) *If $B \in \mathcal{S}$ and $\mathcal{Z} \in \mathsf{Irr}(\Lambda, d)$ satisfy $\overline{\mathcal{O}_B} \subsetneq \mathcal{Z}$, then there exists $B' \in \mathcal{Z}$ such that $B' \notin \mathcal{S}$ and that $\mathcal{S} \sqcup \{B'\}$ is a semibrick.*

This result is used to show that the Jordan-Hölder property of the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ in the case Λ is hereditary [6, Theorem 3.8]. The following is the key proposition in our proof.

Proposition 4. [2, Proposition 2.5] *Let* $B \in \mathsf{brick}\,\Lambda$ *and* $\mathcal{Z} \in \mathsf{Irr}(\Lambda, d)$ *satisfy* $\overline{\mathcal{O}_B} \subsetneq \mathcal{Z}$. *Then* $\{X \in \mathcal{Z} \mid \mathsf{Hom}_\Lambda(B, X) = 0\}$ *is an open dense subset of* $\mathcal{Z}$.

Sketch of the proof. We use the same strategy as [5, Theorem 1.5, (iv)⇒(i)]. Since $\{X \in \mathcal{Z} \mid \mathsf{Hom}_\Lambda(B, X) = 0\}$ is open by upper-semicontinuity, it suffices to show that this set is nonempty.

Assume the contrary. Then the smallest value of the dimension of the K-vector space $\mathsf{Hom}_\Lambda(B, X)$ for $X \in \mathcal{Z}$ is 1, because B is a brick. Thus $\mathcal{U} := \{X \in \mathcal{Z} \mid \mathsf{dim}_K \mathsf{Hom}_\Lambda(B, X) = 1\}$ is an open dense subset of $\mathcal{Z}$. Set $d = (d_i)_{i \in Q_0}$ as the dimension vector of B. By [3, Lemma 2.1], the set

$$\mathcal{Y} := \{(X, f) \in \mathcal{U} \times \prod_{i \in Q_0} \mathsf{Hom}_K(K^{d_i}, K^{d_i}) \mid f \in \mathsf{Hom}_\Lambda(B, X)\}$$

is a one-dimensional vector bundle on $\mathcal{U}$. In particular, the natural projection $\mathcal{Y} \to \mathcal{U}$ is an open map.

The set $\mathcal{V} := \{(X, f) \in \mathcal{U} \mid f \text{ is an isomorphism}\}$ is an open subset of $\mathcal{U}$, and it is nonempty because $(B, 1_B) \in \mathcal{V}$. Thus $p(\mathcal{V})$ is an open and nonempty subset of $\mathcal{U}$. Since $\mathcal{U}$ is an open dense subset of $\mathcal{Z}$, we get that $p(\mathcal{V})$ is open dense in $\mathcal{Z}$. On the other hand, $p(\mathcal{V})$ is just the orbit $\mathcal{O}_B$ by definition. Thus, $\mathcal{O}_B$ is open dense in $\mathcal{Z}$, which contradicts the assumption $\overline{\mathcal{O}_B} \subsetneq \mathcal{Z}$. □

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