MINIMAL PROJECTIVE RESOLUTION AND MAGNITUDE HOMOLOGY OF GEODETIC METRIC SPACES

SHUN WAKATSUKI, YASUHIKO ASAO, AND AARON CHAN

ABSTRACT. Asao-Ivanov showed that the magnitude homology of a finite metric space X is isomorphic to the value of the derived functor $\mathrm{Tor}^{\sigma X}$ over a ring σX. In this article, we give an application of the theory of minimal projective resolution to this derived functor. Especially in the case of a geodetic graph, freeness and a criterion for diagonality of the magnitude homology (i.e., Koszulity of σX) are established. Moreover, we give computational examples including cyclic graphs. This is a joint work with Yasuhiko Asao and Aaron Chan.

1. DEFINITION OF MAGNITUDE HOMOLOGY

Magnitude homology $\mathsf{MH}^{\ell}_{*}(X)$ is an invariant of a metric space $X = (X, d)$, introduced by Hepworth–Willerton [4] and Leinster–Shulman [5] as a categorification of the magnitude, introduced by Leinster in 2000's. It reflects geometric information such as a certain kind of convexity in the metric space and has been studied as an interesting invariant. It can be described as a Tor over a graded ring σX, which is obtained as a quotient of a certain path algebra. In this article, we show some results on $\mathsf{MH}^{\ell}_{*}(X)$ obtained by explicitly constructing a minimal projective resolution over σX. Koszulness of σX is also discussed.

We begin by giving the definition of the magnitude homology of a metric space (X, d). Let $\mathbb{K}$ be a commutative ring. We denote the free module spanned by the Cartesian product of X (as a set) by $\mathsf{MC}_n(X) = \mathbb{K}X^{n+1}$. For $0 \le i \le n$, define the map $\partial_{n,i}\colon \mathsf{MC}_n(X) \to \mathsf{MC}_{n-1}(X)$ by

$$\partial_{n,i}(x_0, \dots, x_n) = \begin{cases} (x_0, \dots, \hat{x}_i, \dots, x_n) & \text{if } x_{i-1} \le x_i \le x_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

where $x_{-1} = x_1$ and $x_{n+1} = x_{n-1}$. Here the condition $x_{i-1} \le x_i \le x_{i+1}$ is defined by $d(x_{i-1}, x_i) + d(x_i, x_{i+1}) = d(x_{i-1}, x_{i+1})$, which means that the triangle inequality holds with equality. Roughly speaking, this corresponds to the condition that the points x_{i-1}, x_i, x_{i+1} lie on a straight line in X. Using this, define $\partial_n = \sum_{0 \le i \le n} (-1)^i \partial_{n,i}$, which yields a chain complex $(\mathsf{MC}_*(X), \partial_*)$. Its homology $\mathsf{MH}_n(X) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}$ is called the *magnitude homology*. Additionally, by considering the sum $\ell = \sum_i d(x_i, x_{i+1})$ of distances, we obtain a secondary grading $\mathsf{MH}_*(X) = \bigoplus_{\ell} \mathsf{MH}^{\ell}_{*}(X)$.

When a (connected, undirected, simple) graph G is given, its vertex set inherits a metric space structure by assigning length 1 to each edge and measuring shortest-path distances.

The detailed version of this paper has been submitted for publication elsewhere.

The first instance of magnitude homology appeared in this setting in [4], and it has been actively studied as an important case.

Example 1. Let C_N be the cyclic graph of N (≥ 3) vertices and label the vertices as in Figure 1. As mentioned above, the vertex set of C_N has the structure of a metric space.

- In C_5, we have $v_1 \leq v_2 \leq v_3$ since $d(v_1, v_2) + d(v_2, v_3) = 1 + 1 = 2 = d(v_1, v_3)$.
- In C_5, the condition $v_1 \leq v_2 \leq v_4$ does not hold since $d(v_1, v_2) + d(v_2, v_4) = 1 + 2 > 2 = d(v_1, v_4)$. In other words, v_2 lies on no shortest path from v_1 to v_4.
- In C_6, we have $w_1 \leq w_2 \leq w_4$ since $d(w_1, w_2) + d(w_2, w_4) = 1 + 2 = 3 = d(w_1, w_4)$. Note that there are two shortest paths (clockwise and counterclockwise) from w_1 to w_4.

Hepworth-Willerton [4] computed the first several terms of the magnitude homology $\mathsf{MH}_*(C_N)$ of the cyclic graph by using computer, and formulated a conjectural recursive formula. This conjecture was solved by Gu [3], but this illustrates that the magnitude homology is difficult to compute even for cyclic graphs.

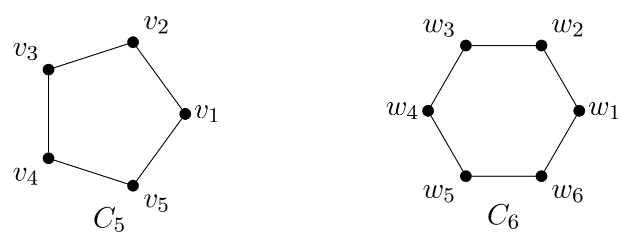

FIGURE 1. Cyclic graphs C_5 and C_6

2. Magnitude homology as Tor

As in the previous section, magnitude homology is defined as the homology of the complex $\mathsf{MC}_*(X)$, which is defined by an explicit formula. Asao–Ivanov [1] described $\mathsf{MH}_*(X)$ in terms of Tor (assuming X is finite):

$$\mathsf{MH}_*(X) \cong \mathrm{Tor}_*^{\sigma X}(\mathbb{K}^{\# X}, \mathbb{K}^{\# X}).$$

Here σX is an associative graded ring defined by

- $\sigma X = \mathbb{K}X^2 = \mathbb{K}\{e_{xy} \mid x, y \in X\}$ as an abelian group with $\deg(e_{xy}) = d(x, y)$
- $e_{xy} \cdot e_{zw} = \begin{cases} e_{xw} & \text{if } y = z \text{ and } x \leq y \leq w \\ 0 & \text{otherwise} \end{cases}$
- $\sum_{x \in X} e_x$ is the unit, where $e_x = e_{xx}$

and $\mathbb{K}^{\# X}$ is a σX-module via the ring homomorphism $\varepsilon \colon \sigma X \twoheadrightarrow \bigoplus_{x \in X} \mathbb{K}e_x \cong \mathbb{K}^{\# X}$. Note that, if X is a metric space arising from a graph G, then σX is isomorphic to the path algebra of a quiver for G, modulo certain relations.

This description enables the use of homological algebra techniques in the study of magnitude homology. In particular, we can determine the magnitude homology of geodetic metric spaces. Here a metric space (X, d) is *geodetic* if any two points of X are connected by a *unique* shortest path. For example, the cyclic graph C_N is geodetic if and only if N is odd.

Theorem 2 (Asao–Wakatsuki [2]). *Let (X, d) be a finite geodetic metric space. Then there exists a minimal projective resolution of $\mathbb{K}^{\# X}$ over σX together with an explicit construction.*

Here $\mathbb{K}$ can be any commutative ring. Hence the existence of the minimal projective resolution does not follow from the well-known theorem for modules over Artinian rings (e.g., a finite dimensional algebra over a field). Since the above theorem gives an *explicit* construction of the resolution, we can describe the homology explicitly:

Theorem 3 (Asao–Wakatsuki [2]). *Let (X, d) be a geodetic metric space. Then $\mathsf{MH}^l_n(X)$ is a free module $\mathbb{K}\Theta^l_n$ spanned by the cycles $\Theta^l_n = \{(x_0, \dots, x_n) \in X^{n+1} \mid (1)\cdots(4)\} \subset X^{n+1}$ defined by the following conditions:*

(1) $\sum_i d(x_i, x_{i+1}) = l$
(2) $1 \le \forall i \le n-1$, *not* $x_{i-1} \le x_i \le x_{i+1}$
(3) $x_0 \le a \le x_1 \Rightarrow a = x_0$ *or* x_1
(4) $1 \le \forall i \le n-1$, $((x_i \le a \le x_{i+1}$ *and* $a \ne x_{i+1}) \implies x_{i-1} \le x_i \le a)$

Here $x \le y \le z$ denotes the condition $d(x, y) + d(y, z) = d(x, z)$ for $x, y, z \in X$.

3. Koszulity of σX and diagonality of $\mathsf{MH}_*(X)$

Consider the case where the metric space (X, d) is derived from a graph G (or the distance function d takes values in integers). We say that (X, d) is *diagonal* if it satisfies $\mathsf{MH}^l_n(X) = 0$ whenever $n \ne l$. This class has been investigated as an important one in the study of magnitude homology. For example, the above definition is given in [4] together with basic examples containing complete graphs, trees and joins. Note that $\mathrm{Tor}^{\sigma X}_n(\mathbb{K}^{\# X}, \mathbb{K}^{\# X})$ is a graded module since σX is a graded ring, and this grading corresponds to ℓ in $\mathsf{MH}^\ell_*(X)$. Hence (when $\mathbb{K}$ is a field) the diagonality of (X, d) is equivalent to the Koszulness of the algebra σX.

As an application of Theorem 3, we have the following theorem.

Theorem 4 (Asao–Wakatsuki [2]). *Let (X, d) be a finite geodetic metric space. Then the following are equivalent:*

- *(X, d) is diagonal (i.e., σX is Koszul).*
- *(X, d) is 2-diagonal (i.e., $\ell \ne 2$ implies $\mathsf{MH}^\ell_2(X) = 0$).*
- *There is no 4-cut in (X, d).*

Here $(x, y, z, w) \in X^4$ is a 4-cut if we have $x \le y \le z$, $y \le z \le w$, $y \ne z$ and not $x \le y \le w$.

Since the absence of 4-cuts expresses a geometric property of the metric space, this result provides a connection between algebraic and geometric properties.

References

[1] Y. Asao and S. O. Ivanov, *Magnitude homology is a derived functor*, arXiv:2402.14466.
[2] Y. Asao and S. Wakatsuki, *Minimal projective resolution and magnitude homology of geodetic metric spaces*, to appear in Homology Homotopy Appl., arXiv:2408.12147.
[3] Y. Gu, *Graph magnitude homology via algebraic morse theory*, arXiv:1809.07240.

[4] R. Hepworth and S. Willerton, *Categorifying the magnitude of a graph*, Homology Homotopy Appl. **19(2)** (2017), 31–60.

[5] Tom Leinster and Michael Shulman, *Magnitude homology of enriched categories and metric spaces.* Algebr. Geom. Topol. **21(5)** (2021), 2175–2221.

Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan
Email address: shun.wakatsuki@math.nagoya-u.ac.jp

Faculty of Applied Mathematics
Fukuoka University
Nanakuma, Jonan-ku, Fukuoka, Fukuoka 814-0180, Japan
Email address: asao@fukuoka-u.ac.jp

Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya, Aichi 464-8602, Japan
Email address: aaron.kychan@gmail.com