# EXAMPLES OF TILTING-DISCRETE SYMMETRIC ALGEBRAS

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ABSTRACT. We give a counter example of the folklore conjecture stating any $\tau$-tilting finite symmetric algebra is tilting-discrete.

## INTRODUCTION

For a finite dimensional algebra $A$ over an algebraically closed field, the following implications follow by definition:

$$\text{representation-finite} \Longrightarrow \tau\text{-tilting finite} \Longleftarrow \text{silting-discrete.}$$

Evidently, the converses do not necessarily hold in general; see Example 5. Now, let us assume that $A$ is symmetric; then, all silting complexes are tilting [5]. We know from [3] that every representation-finite symmetric algebra is silting-discrete. Moreover, it was shown that Brauer graph algebras whose Brauer graphs have precisely one cycle of odd length and none of even length are representation-infinite and silting-discrete [2]. However, no example of silting-indiscrete $\tau$-tilting finite symmetric algebras was known; so, the $\tau$-tilting finitenss of symmetric algebras was expected to be a derived invariant.

In this note, we give an example of $\tau$-tilting finite symmetric algebras which are not silting-disctete; that is, one constructs two derived equivalent symmetric algebras of which one is $\tau$-tilting finite and the other is not.

## 1. RESULTS

The following is useful to grope for our algebras; see [4] for a proof.

**Proposition 1.** *The following conditions are equivalent:*

1. *Any $\tau$-tilting finite symmetric algebra is silting-discrete;*
2. *For every algebra $A$, the trivial extension $T(A)$ of $A$ is $\tau$-tilting finite if and only if it is silting-discrete.*

Therefore, we forcus on trivial extension algebras, and the advantage is the following: if $T(A)$ is $\tau$-tilting finite, then

1. $A$ is $\tau$-tilting finite [7];
2. Its Cartan matrix is positive definite [8].

Note that the converse does not necessarily hold (Example 5).

Now, we find out our algebras.

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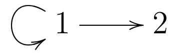
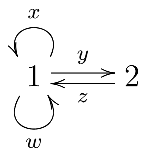
The detailed version of this paper will be submitted for publication elsewhere.

**Theorem 2.** *Let $A$ be the radical-square-zero algebra presented by the quiver:*



*Then its trivial extension $T(A)$ is $\tau$-tilting finite and silting-indiscrete.*

*Proof.* Observe that $T(A)$ is given by the following quiver with certain relations; we omit them, but give the Loewy structure.




$$, \qquad T(A) = \begin{smallmatrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \\ 2 \end{smallmatrix}$$

Moreover, we easily see that $T(A)$ is $\tau$-tilting finite (six 2-term tilting complexes).

Let $X$ be the APR tilting $A$-module with respect to the vertex 2: put $B := \mathrm{End}_A(X)$. Then, we obtain that $B$ is a cyclic Nakayama algebra and its trivial extension $T(B)$ has a multiple arrow in the Gabriel quiver; hence, it is not $\tau$-tilting finite. Since $T(A)$ and $T(B)$ are derived equivalent, it turns out that $T(A)$ is not silting-discrete. □

*Remark* 3. (1) The algebra $A$ as in the theorem is not silting-discrete, either.
(2) By the theorem, Proposition 1 fails.

Thus, we ask when a $\tau$-tilting finite symmetric algebra is silting-discrete, and a big problem is to classify such algebras. Here is a partial answer.

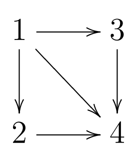
**Theorem 4.** *Let $A$ be either of the following:*

(1) *piecewise hereditary algebras;*
(2) *gentle algebras;*
(3) *cluster-tilted algebras.*

*Then the trivial extension of $A$ is silting-discrete if and only if it is $\tau$-tilting finite.*

We close this note by putting an example of (nonsymmetric) $\tau$-tilting finite algebras which are neither representation-finite nor silting-discrete.

**Example 5.** Let $A$ be the radical-square-zero algebra presented by the quiver:




Then, we observe the following:

(1) $A$ is representation-infinite [6];
(2) It is $\tau$-tilting finite [1];
(3) It is not silting-discrete [4];
(4) It has a positive definite Cartan matrix;
(5) Its trivial extension is not $\tau$-tilting finite.

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