ON THE UBIQUITY OF DOMINANT LOCAL RINGS

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Abstract. Let R be a d-dimensional Cohen–Macaulay complete local ring with infinite residue field k. In this article, we provide upper bounds for the dominant index of R in each of the following cases:

- R has codimension at most two and is not a complete intersection,
- R has multiplicity at most five and is not Gorenstein,
- R is Burch.

This recovers and generalizes a result on hypersurfaces due to Ballard, Favero and Katzarkov, and refines a result on Burch rings due to Takahashi.

We begin with recalling the notation for generation in a triangulated category introduced by Bondal and Van den Bergh [5].

Definition 1. Let $\mathcal{T}$ be a triangulated category. In what follows, all subcategories are assumed to be strictly full.

(1) For a subcategory $\mathcal{X}$ of $\mathcal{T}$, we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums, direct summands and shifts. When $\mathcal{X}$ is given by a single object X, we simply write $\langle X \rangle$.

(2) For two subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$, we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects $E \in \mathcal{T}$ which fits into an exact triangle $X \to E \to Y \to \Sigma X$ in $\mathcal{T}$ such that $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

(3) For an object $X \in \mathcal{T}$ and a nonnegative integer n, we set

$$\langle X \rangle_n = \begin{cases} 0 & (n = 0), \\ \langle X \rangle & (n = 1), \\ \langle \langle X \rangle_{n-1} * \langle X \rangle \rangle & (n \geq 2). \end{cases}$$

Next we recall the definition of a singularity category.

Definition 2. Let R be a noetherian ring. Let $\mathrm{D^{sg}}(R)$ stand for the *singularity category* of R in the sense of Buchweitz [7] and Orlov [16], which is defined to be the Verdier quotient

$$\mathrm{D^{sg}}(R) = \frac{\mathrm{D^b}(\operatorname{mod} R)}{\mathrm{K^b}(\operatorname{proj} R)},$$

where $\mathrm{D^b}(\operatorname{mod} R)$ denotes the bounded derived category of finitely generated R-modules, and $\mathrm{K^b}(\operatorname{proj} R)$ the bounded homotopy category of finitely generated projective R-modules. Note by definition that $\mathrm{D^{sg}}(R)$ is a triangulated category.

The detailed version [11] of this article will be submitted for publication elsewhere.

Now we can state the definitions of a dominant index and a uniformly dominant local ring, which are introduced by Takahashi [23].

Definition 3. Let R be a commutative noetherian local ring with residue field k. We define the *dominant index* $\mathrm{dx}(R)$ of R by

$$\mathrm{dx}(R) = \inf\{n \in \mathbb{Z}_{\geq -1} \mid k \in \langle X \rangle_{n+1} \text{ for every } 0 \neq X \in \mathrm{D^{sg}}(R)\} \in \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$

We say that R is *uniformly dominant* if $\mathrm{dx}(R) < \infty$.

Remark 4.

(1) By definition, a commutative noetherian local ring R is regular if and only if the equality $\mathrm{dx}(R) = -1$ holds.
(2) A uniformly dominant local ring is *dominant* in the sense of Takahashi [22]. A dominant local ring is *Tor/Ext-friendly* in the sense of Avramov, Iyengar, Nasseh and Sather-Wagstaff [2], so in particular, the Auslander–Reiten conjecture holds for such a ring.

The purpose of this article is to consider the following three questions.

Question 5. Let R be a commutative noetherian local ring.

(1) Under what conditions, is R uniformly dominant?
(2) If R is uniformly dominant, then how big is its dominant index $\mathrm{dx}(R)$?
(3) If R is Golod, is then R uniformly dominant?

To state our result, we need to recall several definitions from commutative algebra.

Definition 6. Let R be a commutative noetherian local ring with residue field k. For a finitely generated R-module M, we denote by $\nu(M)$ and $\mathrm{P}_M(t)$ the minimal number of generators and Poincaré series of M, respectively, that is:

$$\nu(M) = \dim_k(M \otimes_R k) = \dim_k(\mathrm{Hom}_R(M,k)) \in \mathbb{N},$$
$$\mathrm{P}_M(t) = \sum_{i\in\mathbb{Z}} \dim_k(\mathrm{Tor}_i^R(M,k))t^i = \sum_{i\in\mathbb{Z}} \dim_k(\mathrm{Ext}_R^i(M,k))t^i \in \mathbb{Z}[\![t]\!].$$

For a commutative noetherian local ring, we denote by $\dim R$ the Krull dimension of R. Also, $\ell(-)$ stands for the length of a composition series.

Definition 7. Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $\mathrm{edim}\, R$, $\mathrm{codim}\, R$, $\mathrm{e}(R)$ and $\mathrm{r}(R)$ stand for the embedding dimension, (embedding) codimension, (Hilbert–Samuel) multiplicity and type of R, respectively, that is:

$$\mathrm{edim}\, R = \nu(\mathfrak{m}) \in \mathbb{Z}_{\geq 0},$$
$$\mathrm{codim}\, R = \mathrm{edim}\, R - \dim R \in \mathbb{Z}_{\geq 0},$$
$$\mathrm{e}(R) = \lim_{n\to\infty} \frac{d!}{n^d}\ell(R/\mathfrak{m}^{n+1}) \in \mathbb{Z}_{>0},$$
$$\mathrm{r}(R) = \dim_k(\mathrm{Ext}_R^u(k,R)) \in \mathbb{Z}_{>0},$$

where $d = \dim R$ and $u = \mathrm{depth}\, R := \inf\{i \in \mathbb{Z}_{\geq 0} \mid \mathrm{Ext}_R^i(k,R) \neq 0\}$.

Definition 8. Let R be a commutative noetherian local ring with residue field k.

(1) We say that R is a *quasi-fiber product* over k if there exist a regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ on R and commutative noetherian local rings A, B whose residue fields are isomorphic to k such that the factor ring $R/(\boldsymbol{x})$ is isomorphic to the fiber product

$$A \times_k B := \{(a, b) \in A \times B \mid f(a) = g(b)\},$$

where $f : A \to k$ and $g : B \to k$ are the canonical surjections.

(2) We say that R is a *Burch* ring if there exist a regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ on the completion $\widehat{R}$ of R, a regular local ring S with maximal ideal $\mathfrak{n}$ and an ideal I of S with $\mathfrak{n}(I : \mathfrak{n}) \neq \mathfrak{n}I$ such that the factor ring $\widehat{R}/(\boldsymbol{x})$ is isomorphic to the factor ring S/I, where $(I : \mathfrak{n}) = \{a \in S \mid \mathfrak{n}a \subseteq I\}$.

(3) We say that R is *G-regular* if every totally reflexive R-module is free, or equivalently, if for every finitely generated R-module, its G-dimension is equal to its projective dimension.

(4) A Cohen–Macaulay local ring R is said to have *finite representation type* if there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay R-modules.

Remark 9.

(1) The notion of a quasi-fiber product is introduced by Freitas, Jorge Pérez, R. Wiegand and S. Wiegand [9]. A commutative noetherian local ring R with maximal ideal $\mathfrak{m}$ and residue field k is a quasi-fiber product over k if and only if $\mathfrak{m}$ is *quasi-decomposable* in the sense of Nasseh and Takahashi [14].

(2) The notion of a Burch ring is introduced by Dao, Kobayashi and Takahashi [8]. This notion is a common generalization of those of a local hypersurface and a Cohen–Macaulay local ring with minimal multiplicity.

(3) The notion of a G-regular local ring is introduced by Takahashi [19]. A G-regular local ring is Gorenstein if and only if it is regular. Every Golod local ring is G-regular. For details, we refer the reader to [19].

Now we can state our result, which gives some partial answers to Question 5.

Theorem 10. *Let R be a Cohen–Macaulay complete local ring with infinite residue field k. Put $d = \dim R$, $c = \operatorname{codim} R$ and $r = \mathrm{r}(R)$. Then the following statements hold true.*

(1) *R is uniformly dominant with $\mathrm{dx}(R) \leq d$ in each of the following cases:*
- *R is a quasi-fiber product over k (e.g., R has minimal multiplicity, i.e., $\mathrm{e}(R) = c + 1$).*
- *R is a non-Gorenstein ring with $\mathrm{e}(R) \leq 5$.*
- *R is such that $\mathrm{e}(R) \leq c + 2$ and $\mathrm{P}_k(t) \neq \frac{(1+t)^d}{1-ct+rt^2}$.*

(2) *R is uniformly dominant with $\mathrm{dx}(R) \leq d + 1$ in each of the following cases:*
- *R is a Burch ring (e.g., R is a hypersurface).*
- *R is a non-Gorenstein G-regular ring with $\mathrm{e}(R) \leq 6$.*
- *R is a non-Gorenstein ring with $c = 2$ and $\mathrm{e}(R) \leq 11$.*

(3) *Assume that the dimension d is at most 2 and the residue field k is algebraically closed. If the Cohen–Macaulay local ring R has finite representation type, R is uniformly dominant with* $\mathrm{dx}(R) \le \max\{1, d\}$.

(4) *Assume that the codimension c is at most 2. Then R is either a local complete intersection with $c = 2$, or a uniformly dominant local ring such that* $\mathrm{dx}(R) \le 6d + 5$.

Here are several remarks on the above theorem.

Remark 11.

(1) The upper bound for $\mathrm{dx}(R)$ given in the above theorem when R is a Burch ring refines the one given by Takahashi [23] and recovers the one given by Ballard, Favero and Katzarkov [3] in the case of a hypersurface.

(2) A classical result due to Scheja [18] shows that every Cohen–Macaulay local ring of codimension at most two is either a complete intersection or a Golod ring. The final assertion of the above theorem thus supports Question 5(3) in the affirmative, which is also shown in [23] under the additional assumption that the defining ideal is a monomial ideal. Furthermore, this theorem tells us that uniformly dominant local rings are ubiquitous.

(3) For more details of the above theorem, we refer the reader to [11].

References

[1] L. L. Avramov, Infinite free resolutions, *Six lectures on commutative algebra*, 1–118, Mod. Birkhäuser Class., *Birkhäuser Verlag, Basel*, 2010.

[2] L. L. Avramov; S. B. Iyengar; S. Nasseh; K. Sather-Wagstaff, Persistence of homology over commutative noetherian rings, *J. Algebra* **610** (2022), 463–490.

[3] M. Ballard; D. Favero; L. Katzarkov, Orlov spectra: bounds and gaps, *Invent. Math.* **189** (2012), no. 2, 359–430.

[4] A. Bahlekeh; S. Salarian; R. Takahashi; Z. Toosi, Spanier–Whitehead categories of resolving subcategories and comparison with singularity categories, *Algebr. Represent. Theory* **25** (2022), no. 3, 595–613.

[5] A. Bondal; M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), no. 1, 1–36, 258.

[6] W. Bruns; J. Herzog, Cohen–Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics **39**, *Cambridge University Press, Cambridge*, 1998.

[7] R.-O. Buchweitz, Maximal Cohen–Macaulay modules and Tate cohomology, With appendices and an introduction by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz, Math. Surveys Monogr. **262**, *American Mathematical Society, Providence, RI*, 2021.

[8] H. Dao; T. Kobayashi; R. Takahashi, Burch ideals and Burch rings, *Algebra Number Theory* **14** (2020), no. 8, 2121–2150.

[9] T. H. Freitas; V. H. Jorge Pérez; R. Wiegand; S. Wiegand, Auslander–Reiten and Huneke–Wiegand conjectures over quasi-fiber product rings, `arXiv:2205.01031`.

[10] C. Huneke; I. Swanson, Integral closure of ideals, rings, and modules, London Math. Soc. Lecture Note Ser. **336**, *Cambridge University Press, Cambridge*, 2006.

[11] T. Kobayashi; R. Takahashi, On the ubiquity of uniformly dominant local rings, in preparation.

[12] G. J. Leuschke; R. Wiegand, Cohen–Macaulay representations, Math. Surveys Monogr. **181**, *American Mathematical Society, Providence, RI*, 2012.

[13] H. Matsumura, Commutative ring theory, Cambridge Stud. Adv. Math. **8**, *Cambridge University Press, Cambridge*, 1989.

[14] S. Nasseh; R. Takahashi, Local rings with quasi-decomposable maximal ideal, *Math. Proc. Cambridge Philos. Soc.* **168** (2020), no. 2, 305–322.
[15] A. Neeman, Triangulated categories, Annals of Mathematics Studies **148**, *Princeton University Press, Princeton, NJ*, 2001.
[16] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, *Proc. Steklov Inst. Math.* **246** (2004), no. 3, 227–248.
[17] B. Poonen, Isomorphism types of commutative algebras of finite rank over an algebraically closed field, *Computational arithmetic geometry*, 111–120, Contemp. Math. **463**, *American Mathematical Society, Providence, RI*, 2008.
[18] G. Scheja, Über die Bettizahlen lokaler Ringe, *Math. Ann.* **155** (1964), 155–172.
[19] R. Takahashi, On G-regular local rings, *Comm. Algebra* **36** (2008), no. 12, 4472–4491.
[20] R. Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.
[21] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.
[22] R. Takahashi, Dominant local rings and subcategory classification, *Int. Math. Res. Not. IMRN* (2023), no. 9, 7259–7318.
[23] R. Takahashi, Uniformly dominant local rings and Orlov spectra of singularity categories, Preprint (2024), `arXiv:2412.00669`.
[24] Y. Yoshino, Cohen–Macaulay modules over Cohen–Macaulay rings, London Math. Soc. Lecture Note Ser. **146**, *Cambridge University Press, Cambridge*, 1990.

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