EXPLICIT CONSTRUCTIONS OF SIMPLE COMODULE ALGEBRAS OVER A POINTED HOPF ALGEBRA

DAISUKE NAKAMURA

Abstract. We study right simple left comodule algebras over a Hopf algebra H, which correspond to indecomposable exact module categories over $\mathrm{Rep}(H)$. Particularly, we show that in the graded case where the degree-zero part of H is a group algebra $\Bbbk G$, any right simple left comodule algebra can be obtained as a homogeneous H-comodule subalgebra of the biproduct of the coinvariant subspace of H and a cocycle deformation of a group algebra of some subgroup of G. As an application, we construct a nontrivial example of a right simple left comodule algebra.

1. Introduction

An abelian category equipped with an action of a tensor category is called a module category, which is a categorical version of the notion of a module over a ring. In [3], P. Etingof and V. Ostrik introduced a class of module categories called exact in order to provide a framework to handle module categories over not only semisimple but also non-semisimple tensor categories, such as the representation categories of finite groups in positive characteristic. Particularly, an exact module category can be decomposed into a direct sum of indecomposable exact module categories, and the classification of indecomposable exact module categories over a fixed tensor category has been studied, but much remains unknown.

In [1], N. Andruskiewitsch and M. Mombelli showed indecomposable exact module categories over the category $\mathrm{Rep}(U)$ of finite-dimensional representations of a finite-dimensional Hopf algebra U correspond to right U-simple left U-comodule algebras. For certain Hopf algebras with tractable structures such as the small quantum group $U = u_q(\mathfrak{sl}_2)$, this result has already led to a complete classification of indecomposable exact module categories [5]. However, in the case $U = u_q(\mathfrak{sl}_3)$, it is not possible to obtain all right U-simple left U-comodule algebras in the same way. One of the reasons is that the method depends on the specificity of $u_q(\mathfrak{sl}_2)$, whose group-like elements form a cyclic group. More precisely, in the case $U = u_q(\mathfrak{sl}_2)$, any right $\mathrm{gr}(U)$-simple left $\mathrm{gr}(U)$-comodule algebra of the form $\mathrm{gr}(A)$ can be provided as a left coideal subalgebra of $\mathrm{gr}(U)$, but this is not the case.

In this paper, we suggest what should be considered instead of a left coideal subalgebra. To be more specific, we show that any right H-simple left H-comodule algebra over a graded Hopf algebra H whose degree-zero part $H(0)$ is a group algebra $\Bbbk G$ can be provided as a homogeneous H-comodule subalgebra of the biproduct $H^{\mathrm{coinv}}\#({}_{\psi}\Bbbk F)$ for some subgroup F of G and some 2-cocycle of F. As an application, we construct a nontrivial

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example of a right H-simple graded left H-comodule algebra which does not appear in the case $H = \mathrm{gr}(u_q(\mathfrak{sl}_2))$. The example is a right H-simple graded left comodule algebra L over a graded Hopf algebra H with $H(0) = \Bbbk G$, satisfying the following properties:

- L is not a left coideal subalgebra of H,
- L cannot be written as a biproduct $\mathcal{B}\#L(0)$ for any left coideal subalgebra $\mathcal{B}$ of H^{coinv} viewed as an object in the left-left Yetter-Drinfeld category over $\Bbbk G$.

2. Preliminaries

Throughout this paper, we work over a field $\Bbbk$. We denote the tensor product over $\Bbbk$ simply by $\otimes$. We assume basic familiarity with Hopf algebras, as presented in [6]. When we say "comodule", we mean left comodule for simplicity.

Definition 1. Let H be a Hopf algebra.

- An algebra is an H-comodule whose H-coaction is an algebra map called an *H-comodule algebra*.
- A nonzero H-comodule algebra L is called *right H-simple* if any right ideal of L that is an H-subcomodule of L is trivial.

Let G be a group. A map $\psi\colon G \times G \to \Bbbk^\times$ is said to be a 2-*cocycle* of G if it satisfies $\psi(a,b)\psi(ab,c) = \psi(b,c)\psi(a,bc)$ and $\psi(a,e) = 1 = \psi(e,a)$ for $a,b,c \in G$. We can define the $\Bbbk G$-comodule algebra ${}_\psi\Bbbk G$ over the group algebra $\Bbbk G$ as follows. It is the same $\Bbbk G$ as a vector space, a multiplication is given by $g * h := \psi(g,h)gh$ for $g,h \in G$ where gh denotes the product in G and a $\Bbbk G$-coaction is given by $\rho(g) = g \otimes g$ for $g \in G$.

Example 2. Let F be a subgroup of G and let ψ be a 2-cocycle of F. Then ${}_\psi\Bbbk F$ is $\Bbbk F$-comodule algebra. In addition, it is also $\Bbbk G$-comodule algebra since $\Bbbk F \subset \Bbbk G$. Moreover, ${}_\psi\Bbbk F$ is right $\Bbbk G$-simple.

Indeed, assume that $I \neq 0$ is a right ideal and a $\Bbbk G$-subcomodule of $\Bbbk F$. Since I is a $\Bbbk G$-subcomodule, we can write $I = \bigoplus_{g\in G} I(g)$ where $I(g) = \{v \in I \mid \rho(v) = g \otimes v\}$. So there exist $0 \neq v \in I$ and $g \in G$ such that $\rho(v) = g \otimes v$. We can also write $v = \sum_{f\in F} r_f f$ for some $r_f \in \Bbbk$ as $I \subset \Bbbk F$. Thus, we have $\sum_{f\in F} r_f f \otimes f = g \otimes v$ which implies that $g \in F$ and $r_g g = v$. Since I is a right ideal,

$$r_g\psi(g,g^{-1}) = r_g\psi(g,g^{-1})gg^{-1} = v * g^{-1} \in I.$$

Hence, it follows that $1 \in I$ as $r_g, \psi(g,g^{-1}) \in \Bbbk^\times$. Therefore, we have $I = \Bbbk F$.

Conversely, any right $\Bbbk G$-simple $\Bbbk G$-comodule algebra can be obtained in the form of ${}_\psi\Bbbk F$ if $\Bbbk$ is algebraically closed (see [2, Corollary 7.12.20]).

3. Adjunction for the biproduct of comodule algebras in the Yetter-Drinfeld category

For a Hopf algebra H, we denote by ${}^H\mathrm{Alg}$ the category of left H-comodule algebras, and by ${}^H_H\mathcal{YD}$ the category of left-left Yetter-Drinfeld modules over H. Let H be a Hopf algebra with comultiplication Δ_H and bijective antipode S_H and let R be a Hopf algebra with comultiplication Δ_R and counit ε_R. We shall use the Sweedler notation for comultiplications $\Delta_H(h) = h_{(1)} \otimes h_{(2)}, \Delta_R(r) = r_{(1)} \otimes r_{(2)}$. Let $\pi\colon R \to H$ and $\iota\colon H \to R$ be Hopf algebra maps satisfying $\pi \circ \iota = \mathrm{id}$. We define the coinvariant subspace

$$R^{\mathrm{coinv}} := \{r \in R \mid (\mathrm{id} \otimes \pi) \circ \Delta_R(r) = r \otimes 1\},$$

which is the image of the linear map

$$\Pi : R \longrightarrow R; \quad r \longmapsto r_{(1)}(\iota \circ S_H \circ \pi)(r_{(2)}).$$

The coinvariant subspace R^{coinv} is a left-left Yetter-Drinfeld module over H with action $h.r = \iota(h_{(1)})r\iota(S_H(h_{(2)}))$ and coaction $(\pi \otimes \mathrm{id}) \circ \Delta_R$. It also becomes a Hopf algebra in ${}^H_H\mathcal{YD}$ with multiplication inherited from R and comultiplication $(\Pi \otimes \mathrm{id}) \circ \Delta_R$. It is known that R can be decomposed as the Radford biproduct, i.e., $R \cong R^{\mathrm{coinv}} \# H$ as a Hopf algebra via $r \mapsto \Pi(r_{(1)}) \otimes \pi(r_{(2)})$ and $b\iota(h) \leftarrow\!\shortmid b \otimes h$ (see [6, Theorem 11.7.1] for example).

The biproduct construction for a comodule algebra object in ${}^H_H\mathcal{YD}$ and an H-comodule algebra has also been established in [4, Proposition 7.2]. We use this construction. Since R^{coinv} is a Hopf algebra in ${}^H_H\mathcal{YD}$, we can regard it as an R^{coinv}-comodule algebra in ${}^H_H\mathcal{YD}$ with the comultiplication serving as the R^{coinv}-coaction. Applying the biproduct construction for comodule algebras to R^{coinv} and an H-comodule algebra F with H-coaction ρ_F, a tensor product $R^{\mathrm{coinv}} \otimes F$ becomes a comodule algebra over the Hopf algebra $R^{\mathrm{coinv}} \# H \cong R$. We denote this by $R^{\mathrm{coinv}} \# F$ whose multiplication and R-coaction ρ are given by

$$(a \otimes \ell)(b \otimes m) = a\iota(\ell_{(-2)})b\iota(S_H(\ell_{(-1)})) \otimes \ell_{(0)}m,$$
$$\rho(a \otimes \ell) = a_{(1)}\iota(\ell_{(-1)}) \otimes a_{(2)} \otimes \ell_{(0)}$$

for $a, b \in R^{\mathrm{coinv}}$ and $\ell, m \in F$. Here, we use the Sweedler notation $\rho_F(\ell) = \ell_{(-1)} \otimes \ell_{(0)}$. Thus, we can obtain an R-comodule algebra from each H-comodule algebra. On the other hand, given an R-comodule algebra L with coaction ρ_L, L is also an H-comodule algebra with its original multiplication and H-coaction $(\pi \otimes \mathrm{id}) \circ \rho_L$. We write L^π for this H-comodule algebra. Then, we can consider following functors

$$R^{\mathrm{coinv}} \#(-)\colon {}^H\mathrm{Alg} \longrightarrow {}^R\mathrm{Alg}; \quad F \longmapsto R^{\mathrm{coinv}} \# F, \quad f\colon F \to G \longmapsto \mathrm{id} \otimes f.$$

$$(-)^\pi\colon {}^R\mathrm{Alg} \longrightarrow {}^H\mathrm{Alg}; \quad L \longmapsto L^\pi, \quad f\colon L \to M \longmapsto f.$$

It is a key observation that they are adjoint functors $(-)^\pi \dashv R^{\mathrm{coinv}} \#(-)$.

Proposition 3. *The following is an isomorphism and is natural in $L \in {}^R\mathrm{Alg}$ and $F \in {}^H\mathrm{Alg}$.*

$$\begin{array}{ccc} {}^H\mathrm{Alg}(L^\pi, F) & \cong & {}^R\mathrm{Alg}(L, R^{\mathrm{coinv}} \# F); \\ f & \mapsto & (\Pi \otimes f) \circ \rho_L, \\ \varepsilon_R \otimes g & \leftarrow\!\shortmid & g. \end{array}$$

The essential part of the proof of this proposition is well-definedness of the correspondence from left to right. So it is enough to show the following lemma.

Lemma 4. *For an R-comodule algebra L, an H-comodule algebra F and an H-comodule algebra map $f\colon L^\pi \to F$, the map $(\Pi \otimes f) \circ \rho_L\colon L \to R^{\mathrm{coinv}} \# F$ is an R-comodule algebra map.*

Proof. For simplicity, we denote the R-coaction of $R^{\mathrm{coinv}}\#F$ by ρ and $(\Pi\otimes f)\circ\rho_L$ by φ. First, we show that φ is an R-comodule map. For any $\ell\in L$,

$$\begin{aligned}\rho\circ\varphi(\ell) &= \rho(\Pi(\ell_{(-1)})\otimes f(\ell_{(0)}))\\ &= (\Pi(\ell_{(-1)}))_{(1)}\iota((f(\ell_{(0)}))_{(-1)})\otimes(\Pi(\ell_{-1}))_{(2)}\otimes(f(\ell_{(0)}))_{(0)}\\ &= \ell_{(-1)_{(1)}}\iota(S_H(\pi(\ell_{(-1)_{(3)}})))\iota((f(\ell_{(0)}))_{(-1)})\otimes\Pi(\ell_{(-1)_{(2)}})\otimes(f(\ell_{(0)}))_{(0)}\\ &= \ell_{(-1)_{(1)}}\iota(S_H(\pi(\ell_{(-1)_{(3)}})))\iota((\pi(\ell_{(0)_{(-1)}})))\otimes\Pi(\ell_{(-1)_{(2)}})\otimes f(\ell_{(0)_{(0)}})\\ &= \ell_{(-4)}\iota(S_H(\pi(\ell_{(-2)})))\iota((\pi(\ell_{(-1)})))\otimes\Pi(\ell_{(-3)})\otimes f(\ell_{(0)})\\ &= \ell_{(-3)}\varepsilon_R(\ell_{(-1)})\otimes\Pi(\ell_{(-2)})\otimes f(\ell_{(0)})\\ &= \ell_{(-2)}\otimes\Pi(\ell_{(-1)})\otimes f(\ell_{(0)})\\ &= (\mathrm{id}\otimes\varphi)\circ\rho_L(\ell).\end{aligned}$$

The third equation follows from $\Delta_R\circ\Pi(r)=r_{(1)}\iota(S_H(\pi(r_{(3)})))\otimes\Pi(r_{(2)})$ for any $r\in R$, and the fourth equation holds because $f\colon L^\pi\to F$ is an H-comodule map.

Next, we show that φ is an algebra map. It is obvious that φ preserves the identity element since the maps $\Delta_R, S_H, \rho_L, \pi$ and ι preserve it. For any $k,\ell\in L$,

$$\begin{aligned}\varphi(k)\varphi(\ell) &= (\Pi(k_{(-1)})\otimes f(k_{(0)}))(\Pi(\ell_{(-1)})\otimes f(\ell_{(0)}))\\ &= \Pi(k_{(-1)})(\iota((f(k_{(0)}))_{(-2)})\Pi(\ell_{(-1)})\iota(S_H((f(k_{(0)}))_{(-1)}))\otimes(f(k_{(0)}))_{(0)}f(\ell_{(0)})\\ &= \Pi(k_{(-1)})(\iota(\pi(k_{(0)_{(-2)}})))\Pi(\ell_{(-1)})\iota(S_H(\pi(k_{(0)_{(-1)}})))\otimes f(k_{(0)_{(0)}})f(\ell_{(0)})\\ &= \Pi(k_{(-3)})(\iota(\pi(k_{(-2)})))\Pi(\ell_{(-1)})\iota(S_H(\pi(k_{(-1)})))\otimes f(k_{(0)})f(\ell_{(0)})\\ &= k_{(-2)}\Pi(\ell_{(-1)})\iota(S_H(\pi(k_{(-1)})))\otimes f(k_{(0)})f(\ell_{(0)})\\ &= k_{(-2)}\ell_{(-2)}\iota(S_H(\pi(\ell_{(-1)})))\iota(S_H(\pi(k_{(-1)})))\otimes f(k_{(0)})f(\ell_{(0)})\\ &= k_{(-2)}\ell_{(-2)}\iota(S_H(\pi(k_{(-1)}\ell_{(-1)})))\otimes f(k_{(0)}\ell_{(0)})\\ &= (k\ell)_{(-2)}\iota(S_H(\pi((k\ell)_{(-1)})))\otimes f((k\ell)_{(0)})\\ &= \Pi((k\ell)_{(-1)})\otimes f((k\ell)_{(0)})\\ &= \varphi(k\ell).\end{aligned}$$

The second equation follows from the definition of the multiplication of $R^{\mathrm{coinv}}\#F$, the third equation holds because $f\colon L^\pi\to F$ is an H-comodule map, the fifth equation follows from $\Pi(r_{(1)})\iota(\pi(r_{(2)}))=r$ for $r\in R$, the sixth equation follows from the definition of Π, and the seventh equation holds because S_H is an anti-algebra map and f is an algebra map. The proof is done. □

As an example, let us consider in the case $H=\Bbbk$ of Proposition 3.

Example 5. Let R be a Hopf algebra with unit $\mu\colon\Bbbk\to R$ and counit $\varepsilon\colon R\to\Bbbk$. Then $\varepsilon\circ\mu=\mathrm{id}$. So we can apply Proposition 3 with $\pi:=\varepsilon$ and $\iota:=\mu$. We have

$$\mathrm{Alg}(L,F)={}^{\Bbbk}\mathrm{Alg}(L^\pi,F)\cong{}^{R}\mathrm{Alg}(L,R^{\mathrm{coinv}}\#F)={}^{R}\mathrm{Alg}(L,R\otimes F)$$

for an R-comodule algebra L and an algebra F. In the case of $F=\Bbbk$, it means that giving an *augmentation* of L, which is an algebra map $L\to\Bbbk$, is the same as giving an R-comodule algebra map $L\to R$. If L is also right R-simple then $L\to R$ is injective, which implies that L is a left coideal subalgebra of R. Conversely, by [5, Lemma 3.13], every left coideal subalgebra is right simple. Therefore, we have

$$\{\text{right } R\text{-simple augmented comodule algebras}\} = \{\text{left coideal subalgebras of } R\},$$

where an *augmented* comodule algebra refers to a comodule algebra with an augmentation.

4. Right simple graded comodule algebra

First, we recall the definition of graded objects. We call a vector space V a *graded vector space* if it can be written as a direct sum $V = \bigoplus_{n\in\mathbb{N}_0} V(n)$ of subspaces $(V(n))_{n\in\mathbb{N}_0}$ indexed by the set of non-negative integers $\mathbb{N}_0$. We say that a linear map f from a graded vector space $V = \bigoplus_{n\in\mathbb{N}_0} V(n)$ to a graded vector space $W = \bigoplus_{n\in\mathbb{N}_0} W(n)$ *preserves the grading* if $f(V(n)) \subset W(n)$ for $n \in \mathbb{N}_0$. A *graded Hopf algebra* is a graded vector space that is a Hopf algebra whose structure maps, which are multiplication, unit, comultiplication, counit, preserve the grading. Note the grading on the base field is taken in the trivial way $\Bbbk(0) = \Bbbk, \Bbbk(n) = 0$ for $n \neq 0$ and the grading on the tensor product of $V = \bigoplus_{n\in\mathbb{N}_0} V(n)$ and $W = \bigoplus_{n\in\mathbb{N}_0} W(n)$ is defined by $(V \otimes W)(n) = \bigoplus_{i=0}^{n} V(n-i) \otimes W(i)$. Similarly, we can define a *graded comodule algebra* as a comodule algebra with a grading whose multiplication, unit and coaction preserve the grading. We say a subspace W of graded vector space $V = \bigoplus_{n\in\mathbb{N}_0} V(n)$ is *homogeneous* if $W = \sum_{n\in\mathbb{N}_0}(V(n) \cap W)$. This condition holds if and only if W is graded by $W(n) = V(n) \cap W$ for $n \in \mathbb{N}_0$.

Let $H = \bigoplus_{n\in\mathbb{N}_0} H(n)$ be a graded Hopf algebra. We denote by $\pi\colon H \to H(0)$ the canonical projection, and by $\iota\colon H(0) \to H$ the canonical injection. It is clear that $\pi \circ \iota = \mathrm{id}$, so H can be decomposed as $H \cong H^{\mathrm{coinv}} \# H(0)$. Let $L = \bigoplus_{n\in\mathbb{N}_0} L(n)$ be a graded H-comodule algebra with coaction ρ_L. We denote by $p\colon L \to L(0)$ the canonical projection. We can regard $H(0)$ and $L(0)$ as graded with trivial grading given by

$$(H(0))(n) := \begin{cases} H(0) & (n=0), \\ 0 & (otherwise), \end{cases} \qquad (L(0))(n) := \begin{cases} L(0) & (n=0), \\ 0 & (otherwise). \end{cases}$$

So $\pi\colon H \to H(0), \iota\colon H(0) \to H$ and $p\colon L \to L(0)$ preserve the grading with respect to this grading. Thus, $\Pi\colon H \to H$ preserves the grading, as it is a composition of maps that preserve the grading. Since $H^{\mathrm{coinv}} = \mathrm{Im}(\Pi)$ is a homogeneous subspace of H, $H^{\mathrm{coinv}} \otimes L(0)$ also a graded vector space with grading given

$$(H^{\mathrm{coinv}} \otimes L(0))(n) = \sum_{i=0}^{n} (H^{\mathrm{coinv}}(n-i)) \otimes (L(0)(i)) = (H(n) \cap H^{\mathrm{coinv}}) \otimes L(0).$$

Furthermore, since the multiplication and the H-coaction of $H^{\mathrm{coinv}} \# L(0)$ can be written as compositions of maps that preserve the grading, the following holds.

Proposition 6. *The H-comodule algebra $H^{\mathrm{coinv}} \# L(0)$ is a graded H-comodule algebra with grading given by $(H^{\mathrm{coinv}} \# L(0))(n) := (H(n) \cap H^{\mathrm{coinv}}) \# L(0)$.*

It is easily verified by calculation that $\rho_L \circ p = (\pi \otimes p) \circ \rho_L$. So we have the following proposition.

Proposition 7. *The projection p is an $H(0)$-comodule algebra map from L^{π} to $L(0)$.*

By Lemma 4, we obtain an H-comodule algebra map $(\Pi \otimes p) \circ \rho_L \colon L \to H^{\mathrm{coinv}} \# L(0)$. Moreover, this map preserves the grading since it is a composition of maps that preserve the grading. If L is right H-simple then this map is injective, which implies the following lemma.

Lemma 8. *If L is right H-simple then L is a homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# L(0)$.*

Conversely, we examine the condition that a homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# L(0)$ becomes right H-simple. The following is established in [5, Theorem 4.4].

Proposition 9. *Assume that H and L are finite-dimensional. Then, L is H-simple if and only if $L(0)$ is $H(0)$-simple.*

The following is the desired result.

Lemma 10. *If H and L are finite-dimensional then the following are equivalent.*

(i) *$L(0)$ is right $H(0)$-simple.*
(ii) *Any homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# L(0)$ is right H-simple.*
(iii) *$H^{\mathrm{coinv}} \# L(0)$ is right H-simple.*

Proof. We first show that (i) implies (ii). Consider a homogeneous H-comodule subalgebra A of $H^{\mathrm{coinv}} \# L(0)$. Then, we have

$$A(0) = (H^{\mathrm{coinv}} \# L(0))(0) \cap A = (H^{\mathrm{coinv}}(0) \# L(0)(0)) \cap A = (\Bbbk \# L(0)) \cap A = L(0) \cap A.$$

The third equation follows from $H^{\mathrm{coinv}}(0) = H^{\mathrm{coinv}} \cap H(0) = \Bbbk$ because $H^{\mathrm{coinv}} \cap H(0) \ni h = \pi(h) = \varepsilon(h_{(1)})\pi(h_{(2)}) = \varepsilon(h)1$. Since $L(0)$ is right $H(0)$-simple, $A(0) \subset L(0)$ is right $H(0)$-simple. Therefore, we have A is right H-simple by Proposition 9. It is obvious that (ii) implies (iii). That (iii) implies (i) follows since $L(0) = (H^{\mathrm{coinv}} \# L(0))(0)$ is right $H(0)$-simple. □

If $H(0) = \Bbbk G$ then we obtain a useful consequence. Note that the most typical case satisfying the condition is for the grading $H = \mathrm{gr}(U)$ induced by the coradical filtration of a pointed Hopf algebra U.

Theorem 11. *Suppose the base field $\Bbbk$ is algebraically closed, and assume H and L are finite-dimensional. If $H(0)$ is a group algebra $\Bbbk G$ of some group G, then the following are equivalent.*

(i) *L is right H-simple.*
(ii) *L is a homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# ({}_\psi \Bbbk F)$ for some subgroup F of G and some 2-cocycle ψ of F.*

Proof. Assume that L is right H-simple. By Lemma 8, L is a homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# L(0)$. Moreover, as $L(0)$ is right $\Bbbk G (= H(0))$-simple, $L(0)$ can be written as ${}_\psi \Bbbk F$ for some subgroup F of G and some 2-cocycle ψ of F (see Example 2). [1]

Conversely, if L is a homogeneous H-comodule subalgebra of $H^{\mathrm{coinv}} \# ({}_\psi \Bbbk F)$ then $L(0) = (H^{\mathrm{coinv}} \# ({}_\psi \Bbbk F))(0) \cap L = {}_\psi \Bbbk F \cap L$. In addition, ${}_\psi \Bbbk F$ is right $\Bbbk G$-simple (see Example 2). So $L(0) \subset {}_\psi \Bbbk F$ is right $H(0)$-simple. Proposition 9 yields that L is right H-simple. □

[1] The assumption that the base field is algebraically closed is used only at this point.

As a result, for a finite-dimensional graded Hopf algebra H with $H(0)=\Bbbk G$, assuming that the base field $\Bbbk$ is algebraically closed,

$$\{\text{finite-dimensional right } H\text{-simple graded } H\text{-comodule algebras}\}$$
$$= \left\{ \begin{array}{l} \text{homogeneous } H\text{-comodule subalgebras of } H^{\mathrm{coinv}}\#({}_{\psi}\Bbbk F), \\ \text{for some subgroup } F \text{ of } G \text{ and some 2-cocycle } \psi \text{ of } F \end{array} \right\}.$$

5. A construction of simple graded comodule algebra

In this section, the base field is the field of complex numbers $\mathbb{C}$. We define a Hopf algebra $\mathcal{H}$ as follows: As an algebra, it is generated by g, h, x and y subject to the relations

$$g^2=1=h^4,\ x^2=0=y^2,\ gh=hg,\ xy=yx,\ gy=yg,\ hx=xh,\ gx=-xg,\ hy=\mathrm{i}yh$$

where i denotes the imaginary unit. A comultiplication Δ, a counit ε and an antipode are given by

$$\Delta(g)=g\otimes g,\quad \Delta(h)=h\otimes h,\quad \Delta(x)=x\otimes 1+g\otimes x,\ \Delta(y)=y\otimes 1+h^2\otimes y,$$
$$\varepsilon(g)=1=\varepsilon(h),\quad \varepsilon(x)=0=\varepsilon(y),$$
$$S(g)=g^{-1},\quad S(h)=h^{-1},\quad S(x)=-g^{-1}x,\quad S(y)=-h^{-2}y.$$

This is a graded Hopf algebra whose n-th grading $\mathcal{H}(n)$ is given by the linear span of the elements of the form $g^ih^jx^ky^l$ for $i,j,k,l\in\mathbb{N}_0, 0\le k+l\le n$. Particularly, $\mathcal{H}(0)=\mathbb{C}\langle g,h\rangle\cong\mathbb{C}(\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z})$. Let $G:=\langle g,h\rangle$. A basis of $\mathcal{H}$ is given by

$$g^ih^jx^ky^l \quad \text{for} \quad 0\le i,k,l\le 1,\ 0\le j\le 3$$

and the dimension of $\mathcal{H}$ is 32. By a direct computation using the basis, we have $\mathcal{H}^{\mathrm{coinv}}=\langle x,y\rangle$. Let $F:=\langle g,h^2\rangle\cong\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and let $\psi\colon F\times F\to\mathbb{C}^\times$ be a 2-cocycle of F defined by $\psi(g^ih^{2j},g^kh^{2l}):=(-1)^{il}$ for $i,j,k,l\in\mathbb{Z}$. We have ${}_{\psi}\mathbb{C}F\cong\langle g,h'\mid g^2=1=h'^2, gh'=-h'g\rangle$ as an algebra. By a straightforward computation, we have an isomorphism

$$\mathcal{H}^{\mathrm{coinv}}\#({}_{\psi}\mathbb{C}F)\cong\left\langle g,h',x,y\ \middle|\ \begin{array}{llll} g^2=1=h'^2, & x^2=0=y^2, & xy=yx, & \\ gy=yg, & gx=-xg, & h'y=-yh', & gh'=-h'g \end{array}\right\rangle=:\mathcal{K},$$

given by $1\otimes g\mapsto g,\ 1\otimes h^2\mapsto h',\ x\otimes 1\mapsto x,\ y\otimes 1\mapsto y$. The $\mathcal{H}$-coaction of $\mathcal{K}$ is interpreted as $\rho(g)=g\otimes g,\ \rho(h')=h^2\otimes h',\ \rho(x)=x\otimes 1+g\otimes x,\ \rho(y)=y\otimes 1+h^2\otimes y$. By Theorem 11, any homogeneous left $\mathcal{H}$-comodule subalgebra of $\mathcal{K}$ is right $\mathcal{H}$-simple.

A straightforward way to obtain homogeneous left comodule algebras is to consider the algebra generated by homogeneous elements and check whether it is closed under the coaction. We construct a right $\mathcal{H}$-simple comodule algebra using this way. We consider a subalgebra $\mathcal{L}:=\langle gh'x+y,g,h'\rangle$ of $\mathcal{K}$ and this is homogeneous. One can verify that

$$\rho(gh'x+y)=gh^2x\otimes gh'+y\otimes 1+h^2\otimes(gh'x+y)\in\mathcal{H}\otimes\mathcal{L}.$$

So $\mathcal{L}$ is closed under the $\mathcal{H}$-coaction. Hence, $\mathcal{L}$ is a right $\mathcal{H}$-simple comodule algebra. Moreover, $\mathcal{L}$ satisfies the following.

Proposition 12. *The following hold.*

- *$\mathcal{L}$ is not a left coideal subalgebra of $\mathcal{H}$,*
- *$\mathcal{B}\#\mathcal{L}(0)\to\mathcal{L}; b\otimes a\mapsto ba$ cannot be an isomorphism for any left coideal subalgebra $\mathcal{B}$ of $\mathcal{H}^{\mathrm{coinv}}$ viewed as an object in the Yetter-Drinfeld category over $\mathbb{C}G$.*

The first property holds as follows: Assume $\mathcal{L}$ is a left coideal subalgebra. Then there is an algebra map $\alpha\colon \mathcal{L} \to \mathbb{C}$ by Example 5. By the relations of $\mathcal{K}$, $\alpha(g)\alpha(h') = -\alpha(g)\alpha(h')$ and $\alpha(g), \alpha(h') \neq 0$ which is a contradiction.

The second property holds as follows: Recall that the comultiplication of $\mathcal{H}^{\mathrm{coinv}}$, which is a Hopf algebra in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ is given by $(\Pi \otimes \mathrm{id}) \circ \Delta$. So a left coideal subalgebra I of $\mathcal{H}^{\mathrm{coinv}}$ in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ means that I is a subalgebra of $\mathcal{H}^{\mathrm{coinv}}$ which satisfies $(\Pi \otimes \mathrm{id}) \circ \Delta(I) \subset \mathcal{H}^{\mathrm{coinv}} \otimes I$. Thus, we can consider $\leftharpoonup\colon I \times (\mathcal{H}^{\mathrm{coinv}})^* \longrightarrow I;\ a \leftharpoonup f := f(\Pi(a_{(1)}) \otimes a_{(2)})$, using the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$, where $(\mathcal{H}^{\mathrm{coinv}})^*$ is the dual space of $\mathcal{H}^{\mathrm{coinv}}$. Now, $\mathcal{H}^{\mathrm{coinv}} = \langle x, y \rangle = \mathbb{C} \oplus \mathbb{C}x \oplus \mathbb{C}y \oplus \mathbb{C}xy$. We denote the dual basis of the basis $1, x, y, xy$ by $1^*, x^*, y^*, (xy)^*$. By a direct computation, we see that $I \leftharpoonup x^* \subset \mathbb{C} \oplus \mathbb{C}y$. Furthermore, we have:

- If $I \leftharpoonup x^* = 0$ then $I \subset \langle y \rangle$.
- If $I \leftharpoonup x^* = \mathbb{C}$ then $I = \langle x + ty \rangle$ for some $t \in \mathbb{C}$.
- If $I \leftharpoonup x^* = \mathbb{C} \oplus \mathbb{C}y$ then $I = \langle x, y \rangle$.

Hence, all left coideal subalgebras of $\mathcal{H}^{\mathrm{coinv}}$ in ${}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ are given by $\mathbb{C},\ \langle x+ty \rangle,\ \langle y \rangle,\ \langle x, y \rangle$ for $t \in \mathbb{C}$. Assume $\mathcal{B}\#\mathcal{L}(0) \to \mathcal{L}; b \otimes a \mapsto ba$ is an isomorphism. Then we have $\mathcal{B} \cdot (\mathcal{L}(0)) = \mathcal{L}$ but this is not satisfied for any of $\mathcal{B} = \mathbb{C},\ \langle x + ty \rangle,\ \langle y \rangle,\ \langle x, y \rangle$. This is because $\mathcal{L}$ does not contain elements of the form $x + ty$ or y since $\mathcal{L} = \mathbb{C}(gh'x + y) \oplus \mathbb{C}(h'x + gy) \oplus \mathbb{C}(x - gh'y) \oplus \mathbb{C}(gx - h'y) \oplus \mathbb{C}\langle g, h' \rangle$.

Remark 13. One can present the algebra $\mathcal{L}$ in terms of generators and relations. Since $g(gh'x + y) = (gh'x + y)g,\ h'(gh'x + y) = -(gh'x + y)h',\ (gh'x + y)^2 = 0$, by letting $w := gh'x + y$,

$$\mathcal{L} = \left\langle g, h', w \,\middle|\, \begin{array}{l} g^2 = 1 = h'^2, \qquad gh' = -h'g \\ w^2 = 0, \quad gw = wg, \quad h'w = -wh' \end{array} \right\rangle$$

whose $\mathcal{H}$-coaction is given by

$$\rho(g) = g \otimes g, \quad \rho(h') = h^2 \otimes h', \quad \rho(w) = gh^2x \otimes gh' + y \otimes 1 + h^2 \otimes w.$$

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Graduate School of science and Engineering
Okayama University of science
Ridaicho, kitaku 700-0005, Japan
Email address: r25ndk3od@ous.jp