THICK REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. We give the classification of thick representations and dense representations of the symmetric group S_n over $\mathbb{C}$.

Key Words: Thick representation, Dense representation, Symmetric group.

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1. Introduction

Let V be a finite-dimensional vector space over a field k. For a group representation $\rho : G \to \mathrm{GL}(V)$, we say that ρ is *thick* if, for any subspaces V_1, V_2 of V with $\dim_k V_1 + \dim_k V_2 = \dim_k V$, there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$. We also say that ρ is *dense* if the exterior representation $\Lambda^m \rho : G \to \mathrm{GL}(\Lambda^m V)$ is irreducible for any $0 < m < \dim_k V$ (see Definition 3). If ρ is dense, then it is thick; and if it is thick, then it is irreducible (Proposition 5). In this paper, we discuss the classification of thick representations and dense representations of the symmetric group over $\mathbb{C}$.

Let S_n be the symmetric group of degree n. For a partition λ of n, we denote by V_λ the irreducible representation of S_n over $\mathbb{C}$ corresponding to λ. The main theorems of this paper are the following:

Theorem 1 ([7])**.** *For a finite-dimensional irreducible representation ρ of S_n over $\mathbb{C}$, ρ is thick if and only if it is dense.*

Theorem 2 ([7])**.** *The thick (or dense) representations of the symmetric group S_n over $\mathbb{C}$ are those on the following list:*

(1) *the trivial representation $V_{(n)}$ of S_n for $n \geq 1$,*
(2) *the sign representation $V_{(1^n)}$ of S_n for $n \geq 2$,*
(3) *the standard representation $V_{(n-1,1)}$ of S_n for $n \geq 3$,*
(4) *the product of the standard and sign representation $V_{(2,1^{n-2})}$ of S_n for $n \geq 4$,*
(5) *the 2-dimensional irreducible representation $V_{(2^2)}$ of S_4,*
(6) *the 5-dimensional irreducible representations $V_{(2^3)}$ and $V_{(3^2)}$ of S_6.*

Regarding the classification problem of thick representations, in the case of finite-dimensional irreducible representations over $\mathbb{C}$ of connected semi-simple Lie groups over $\mathbb{C}$, the notion of thickness fits well within the framework of weight theory and the classification has been successfully achieved (for details, see Theorems 9, 10, 11, and [4, Theorem 3.12]). In contrast, for the symmetric group, no progress had been made so far.

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However, once we realized that the irreducible representations of the symmetric group can be treated concretely through Specht modules, we were able to complete the classification of its thick representations.

The organization of this paper is as follows: in Section 2, we review thick representations and dense representations for arbitrary groups. In Section 3, we review the known classification of thick representations and dense representations. In Section 4, we give a brief outline of the proofs of Theorems 1 and 2.

2. Preliminaries

In this section, we give a review of thick representations and dense representations for arbitrary groups. For details, see [3] and [4]. As stated in the Introduction of [3], we believe that as the image $\rho(G)$ of a representation $\rho : G \to \mathrm{GL}(V)$ becomes larger, ρ tends to become thick or dense. This is the reason for the names "thick" and "dense".

Definition 3 ([3, Definitions 2.1 and 2.3] and [4, Definitions 2.1 and 2.2])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. We say that ρ is *m-thick* if, for any subspaces V_1, V_2 of V with $\dim_k V_1 = m$ and $\dim_k V_2 = \dim_k V - m$, there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$. If ρ is m-thick for any $0 < m < \dim_k V$, then we say that ρ is *thick*. We also say that ρ is *m-dense* if $\Lambda^m \rho : G \to \mathrm{GL}(\Lambda^m V)$ is irreducible. If ρ is m-dense for any $0 < m < \dim_k V$, then we say that ρ is *dense*.

Proposition 4 ([3, Proposition 2.6] and [4, Proposition 2.7])**.** *For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,*

$$m\text{-thick} \Longleftrightarrow (n-m)\text{-thick},$$
$$m\text{-dense} \Longleftrightarrow (n-m)\text{-dense}.$$

Proposition 5 ([3, Proposition 2.7 and Corollary 2.8] and [4, Proposition 2.8 and Corollary 2.9])**.** *For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$ and $0 < m < n$,*

$$\begin{array}{ccccc} m\text{-dense} & \Longrightarrow & m\text{-thick} & & \\ & & \Downarrow & & \\ 1\text{-dense} & \Longleftrightarrow & 1\text{-thick} & \Longleftrightarrow & \text{irreducible.} \end{array}$$

In particular,

$$\text{dense} \Rightarrow \text{thick} \Rightarrow \text{irreducible.}$$

Corollary 6 ([3, Corollary 2.9] and [4, Corollary 2.10])**.** *When $n \le 3$,*

$$\text{dense} \Leftrightarrow \text{thick} \Leftrightarrow \text{irreducible.}$$

The following is a more refined definition of m-thickness.

Definition 7 ([6])**.** For a group representation $\rho : G \to \mathrm{GL}(V)$, we say that ρ is *(i,j)-thick* if, for any subspaces V_1, V_2 of V with $\dim_k V_1 = i$ and $\dim_k V_2 = j$, there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$.

The following proposition will be used in Section 4.

Proposition 8 ([6])**.** *If $i + j \le \dim V$ and $\rho : G \to \mathrm{GL}(V)$ is not (i,j)-thick, then ρ is not thick.*

3. Known classification of thick representations

In this section, we review the known classification of thick representations and dense representations. (For the background of thick representations, see [5, Section 2].)

The classification of thick representations was first successfully completed in the case of connected semi-simple Lie groups over $\mathbb{C}$. Let G be a connected semi-simple Lie group over $\mathbb{C}$. Let $\mathfrak{g}, \mathfrak{h}, \Delta^+ (\subset \mathfrak{h}^*)$ be the Lie algebra of G, a Cartan subalgebra, the set of positive roots, respectively. For a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$, denote by $W(V)$ the set of weights of V. We can regard $W(V)$ as a partially ordered set by using Δ^+. We say that $\rho : G \to \mathrm{GL}(V)$ is *weight multiplicity-free* if the dimension of the φ-eigenspace is 1 for any $\varphi \in W(V)$.

Theorem 9 ([4, Theorem 3.5]). *For a connected semi-simple Lie group G over $\mathbb{C}$, a finite-dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$ is thick if and only if ρ is weight multiplicity-free and the weight poset $W(V)$ is a totally ordered set.*

Theorem 10 ([4, Theorem 3.6]). *The thick representations of connected simple Lie groups are those on the following list:*

(1) *the trivial* 1*-dimensional representation for any groups,*
(2) A_n $(n \geq 1)$
  - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1,*
  - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n,*
  - *the symmetric tensor $S^m(V)$ $(m \geq 2)$ of V for A_1 with highest weight $m\omega_1$,*

(3) B_n $(n \geq 2)$
  - *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1,*
  - *the spin representation for B_2 with highest weight ω_2,*

(4) C_n $(n \geq 3)$
  - *the standard representation V for C_n $(n \geq 3)$ with highest weight ω_1,*

(5) G_2
  - *the* 7*-dimensional representation V for G_2 with highest weight ω_1.*

Theorem 11 ([4, Theorem 3.7]). *The dense representations of connected simple Lie groups are those on the following list:*

(1) *the trivial* 1*-dimensional representation for any groups,*
(2) A_n $(n \geq 1)$
  - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1,*
  - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n,*
  - *the symmetric tensor $S^2(V)$ of V for A_1 with highest weight $2\omega_1$,*

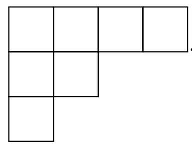
(3) B_n $(n \geq 2)$
  - *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1.*

Remark 12. Through the theorems above, we can see that irreducible representations, thick representations, and dense representations do not necessarily coincide. However, for the symmetric group, thick representations and dense representations coincide by Theorem 1.

4. Main Theorem

In this section, we give a brief outline of the proofs of Theorems 1 and 2.

Definition 13 (*cf.* [2] and [1])**.** Let n be a positive integer. To each partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ of n with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_s$, we can associate a Young diagram in the following way. For example, for $\lambda = (4, 2, 1)$, the Young diagram is



Definition 14 (*cf.* [2] and [1])**.** For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$ of n, a *Young tableau* is a tableau in which the entries are the numbers from 1 to n, each occurring once. A *standard Young tableau* is a Young tableau which satisfies

(1) strictly increasing across each row,
(2) strictly increasing down each column.

For example, for the partition $\lambda = (4, 2, 1)$ of $7 = 4 + 2 + 1$, let us consider the following Young tableaux

(a)

3	1	6	2
7	5		
4			

, (b)

1	2	5	6
3	4		
7			

.

(a) is a Young tableau, but not standard. (b) is a standard Young tableau.

Definition 15 ([8, pages 88–89])**.** Let us consider variables $x_1, x_2, \dots, x_n$. For a partition λ of n, we define a *λ-tableau* and a *standard λ-tableau* by changing $\{1, 2, \dots, n\}$ into $\{x_1, x_2, \dots, x_n\}$. For example,

(a)

x_3	x_1	x_6	x_2
x_7	x_5		
x_4			

, (b)

x_1	x_2	x_5	x_6
x_3	x_4		
x_7			

.

(a) is a λ-tableau, but not standard. (b) is a standard λ-tableau.

Definition 16 ([8, pages 88–89])**.** Suppose that the variables $a_1, \dots, a_t$ occur in j-th column of the λ-tableau y, with a_i in the i-th row. We form the difference product

$$\Delta(a_1, \dots, a_t) = \prod_{i<j} (a_i - a_j)$$

if $t > 1$, and $\Delta(a_1) = 1$ if $t = 1$. Multiplying these difference products for all the columns of y, we obtain a polynomial $f(y)$. For example, if y is

(a)

x_3	x_1	x_6	x_2
x_7	x_5		
x_4			

or (b)

x_1	x_2	x_5	x_6
x_3	x_4		
x_7			

,

then

$$(a)\ f(y) = (x_3 - x_7)(x_3 - x_4)(x_7 - x_4)(x_1 - x_5) \quad \text{and}$$
$$(b)\ f(y) = (x_1 - x_3)(x_1 - x_7)(x_3 - x_7)(x_2 - x_4).$$

Definition 17 ([8, page 89])**.** For a partition λ of n, let

$$S^\lambda = \mathbb{Z}\{f(y) \mid y \text{ is a } \lambda\text{-tableau } \} \subset \mathbb{Z}[x_1, \ldots, x_n].$$

Then S_n acts on S^λ. We call S^λ the Specht module corresponding to λ.

Theorem 18 ([8, Theorem 1.1])**.** *For a partition λ of n, S^λ has a $\mathbb{Z}$-basis*

$$B^\lambda = \{f(y) \mid y \text{ is a standard } \lambda\text{-tableau } \}.$$

Definition 19. For a partition λ of n, we denote by V_λ the Specht module $S^\lambda \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem 20 ([8, page 90 and Section 4] and [1, Section 7.2, Proposition 1])**.** *For each partition λ of n, V_λ is an irreducible representation of S_n over $\mathbb{C}$. Every irreducible representation of S_n over $\mathbb{C}$ is isomorphic to exactly one V_λ.*

In the sequel, let us prove Theorems 1 and 2. First, we deal with dense representations of the symmetric group. We prove the following theorem.

Theorem 21 ([7])**.** *The dense representations of the symmetric group S_n over $\mathbb{C}$ are those on the following list:*

(1) *the trivial representation $V_{(n)}$ of S_n for $n \geq 1$,*
(2) *the sign representation $V_{(1^n)}$ of S_n for $n \geq 2$,*
(3) *the standard representation $V_{(n-1,1)}$ of S_n for $n \geq 3$,*
(4) *the product of the standard and sign representation $V_{(2,1^{n-2})}$ of S_n for $n \geq 4$,*
(5) *the 2-dimensional irreducible representation $V_{(2^2)}$ of S_4,*
(6) *the 5-dimensional irreducible representations $V_{(2^3)}$ and $V_{(3^2)}$ of S_6.*

Lemma 22 ([7])**.** *Let V_λ be an irreducible representation of S_n over $\mathbb{C}$. If there exists $2 \leq r \leq \dim_{\mathbb{C}} V_\lambda - 1$ such that $\dim_{\mathbb{C}} \Lambda^r V_\lambda > \sqrt{n!}$, then V_λ is not dense.*

proof. Note that $\dim_{\mathbb{C}} V_\mu \leq \sqrt{n!}$ for any irreducible representation V_μ of S_n. Indeed,

$$(\dim_{\mathbb{C}} V_\mu)^2 \leq \sum_\rho (\dim_{\mathbb{C}} V_\rho)^2 = \sharp S_n = n!$$

by the Wedderburn-Artin Theorem. Hence, $\dim_{\mathbb{C}} V_\mu \leq \sqrt{n!}$. If $\dim_{\mathbb{C}} \Lambda^r V_\lambda > \sqrt{n!}$, then $\Lambda^r V_\lambda$ is not irreducible. Therefore, V_λ is not dense. □

For the proof of Theorem 21, we divide the proof into several steps:
(Step 1) All irreducible representations in the list of Theorem 21 are dense.
(Step 2) Discuss the case S_n $(n \geq 9)$.
(Step 3) Discuss the case S_n $(n \leq 8)$.

Here, we explain Step 2. We omit Step 1 and Step 3. For details, see [7].

Proof of Step 2.

By [9, Result 2], if $n \geq 9$, then the first four minimal degrees of S_n are

$$\text{(A) } 1, \quad \text{(B) } n-1, \quad \text{(C) } \frac{1}{2}n(n-3), \quad \text{(D) } \frac{1}{2}(n-1)(n-2).$$

The case (A) corresponds to (1) and (2), and the case (B) corresponds to (3) and (4). By Step 1, the representations listed in (1)–(4) are dense. We claim that any d-dimensional irreducible representation V of S_n ($n \geq 9$) is not dense if $d \geq \frac{1}{2}n(n-3)$. We prove this claim separately for the cases d even and d odd, using Lemma 23.

Lemma 23. *We have* $\binom{2m}{m} \geq 2^m$ *for* $m \geq 1$ *and* $\binom{2m+1}{m} \geq 2^{m+1}$ *for* $m \geq 2$.

Proof. For the latter inequality, let us consider how to choose m elements from the set $X = \{a_1, b_1, a_2, b_2, \ldots, a_m, b_m, c\}$. Since there are m-element subsets $\{*_i \mid 1 \leq i \leq m\}$ ($* = a$ or b) and $X_j = \{c\} \cup \{*_i \mid 1 \leq i \leq m, i \neq j\}$ ($1 \leq j \leq m$, $* = a$ or b) of X, we obtain

$$\binom{2m+1}{m} \geq 2^m + m \cdot 2^{m-1} \geq 2^{m+1}$$

for $m \geq 2$. We can prove the former inequality in the same way. $\square$

When $d = 2m$ for $m \in \mathbb{N}$,

$$\dim_{\mathbb{C}} \Lambda^m V = \binom{2m}{m} \geq 2^m \geq \sqrt{2^{\frac{1}{2}n(n-3)}}.$$

It can be shown that $2^{\frac{1}{2}(n-3)} > n$ for $n \geq 10$ by induction. If $n \geq 10$, then

$$\sqrt{2^{\frac{1}{2}n(n-3)}} > \sqrt{n^n} > \sqrt{n!}.$$

If $n = 9$, then we can verify

$$\sqrt{2^{\frac{1}{2}n(n-3)}} > \sqrt{n!}.$$

by direct calculation. Hence, for $n \geq 9$,

$$\dim_{\mathbb{C}} \Lambda^m V > \sqrt{n!},$$

which implies that V is not dense by Lemma 22.

When $d = 2m+1$ for $m \in \mathbb{N}$,

$$\dim_{\mathbb{C}} \Lambda^m V = \binom{2m+1}{m} \geq 2^{m+1} \geq \sqrt{2^{\frac{1}{2}(n-1)(n-2)}},$$

since $m+1 = \frac{d+1}{2} \geq (\frac{n(n-3)}{2}+1)/2 = \frac{(n-1)(n-2)}{4}$ and $m = \frac{d-1}{2} \geq \frac{27-1}{2} = 13$. It can be shown that $2^{\frac{1}{2}(n-2)} > n$ for $n \geq 9$ by induction. Hence, for $n \geq 9$,

$$\dim_{\mathbb{C}} \Lambda^m V \geq \sqrt{2^{\frac{1}{2}(n-1)(n-2)}} > \sqrt{n^{n-1}} > \sqrt{n!},$$

which implies that V is not dense by Lemma 22. $\square$

Theorems 1 and 2 follow from Theorem 21 and the following theorem.

Theorem 24 ([7])**.** *If V_λ is not contained in the list of Theorem 21, then V_λ is not thick.*

For proving Theorem 24, we only need to show that V_λ is not (i,j)-thick with $i + j \leq \dim V_\lambda$ if V_λ is not contained in the list of Theorem 21 by Proposition 8.

Proposition 25 ([7])**.** *If V_λ is not contained in the list of Theorem 21, then there exist subspaces W_1 and W_2 of V_λ such that*

(1) $\sigma(W_1) \cap W_2 \neq 0$ *for any* $\sigma \in S_n$,
(2) $\dim W_1 + \dim W_2 \leq \dim V_\lambda$.

Suppose that V_λ is not contained in the list of Theorem 21 for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ of n. Then $s \geq 2$. To prove Proposition 25, we divide the argument into three cases: (Case 1) $s = 2$, (Case 2) $s = 3$, and (Case 3) $s \geq 4$.

Here, we prove Case 3. We omit Case 1 and Case 2. For details, see [7].

Lemma 26 ([7])**.** *Let ${}^t\lambda$ be the partition of n whose Young diagram coincides with the transposed diagram of the Young diagram corresponding to λ. Then $V_{{}^t\lambda}$ is thick if and only if so is V_λ.*

To prove Case 3, we may assume that $s \geq 4$ and $\lambda_1 \geq 4$, provided that Case 1 and Case 2 hold.

Definition 27 ([7])**.** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ with $s \geq 4$ and $\lambda_1 \geq 4$. Set

$$W_1 = \left\langle f(y) \;\middle|\; y = \begin{array}{|c|c|c|}\hline x_1 & * & \cdots \\ \hline x_2 & \cdots & \\ \cline{1-2} \vdots & \cdots & \\ \cline{1-2} \end{array} \text{ is a standard } \lambda\text{-tableau} \right\rangle$$

and

$$W_1' = \left\langle f(y) \;\middle|\; y = \begin{array}{|c|c|c|}\hline x_1 & x_2 & \cdots \\ \hline * & \cdots & \\ \cline{1-2} \vdots & \cdots & \\ \cline{1-2} \end{array} \text{ is a standard } \lambda\text{-tableau} \right\rangle.$$

Note that $V_\lambda = W_1 \oplus W_1'$ and that $\dim V_\lambda = \dim W_1 + \dim W_1'$.

Proof of Case 3.

If $\dim W_1' \leq \dim W_1$, then change V_λ into $V_{{}^t\lambda}$. For proving Case 3, we may assume that $\dim W_1 \leq \dim W_1'$ by Lemma 26. Note that $\dim W_1$ equals to the number of standard skew tableaux of shape $\lambda/(1,1)$ and that the number of standard skew tableaux is invariant under taking the transpose of the diagram. Then we have $2\dim W_1 \leq \dim V_\lambda$.

By the Garnir's relations, we see that

$$W_1 = \left\langle f(y) \;\middle|\; y = \begin{array}{|c|c|c|}\hline x_1 & * & \cdots \\ \hline x_2 & \cdots & \\ \cline{1-2} \vdots & \cdots & \\ \cline{1-2} \end{array} \text{ is a (not necessarily standard) } \lambda\text{-tableau} \right\rangle.$$

In particular, for any $3 \leq i \leq j \leq n$, there exists a λ-tableau y whose first column contains $\{x_1, x_2, x_i, x_j\}$ such that $f(y) \in W_1$. For $\sigma \in S_n$,

$$\sigma(W_1) = \left\langle f(y) \;\middle|\; y = \begin{array}{|c|c|c|}\hline x_{\sigma(1)} & * & \cdots \\ \hline x_{\sigma(2)} & \cdots & \\ \cline{1-2} \vdots & \cdots & \\ \cline{1-2} \end{array} \text{ is a (not nec. standard) } \lambda\text{-tableau} \right\rangle.$$

We easily verify that there exists a λ-tableau y whose first column contains $\{x_{\sigma(1)}, x_{\sigma(2)}\}$ such that $f(y) \in W_1$. Hence, $\sigma(W_1) \cap W_1 \neq 0$.

Setting $W_2 = W_1$, we see that $\sigma(W_1) \cap W_2 \neq 0$ and $\dim W_1 + \dim W_2 \leq \dim V_\lambda$, which implies Case 3. □

References

[1] W. Fulton, Young tableaux, London Math. Soc. Stud. Texts, 35, Cambridge University Press, Cambridge, 1997.

[2] N. Iwahori, Taishō-gun to ippan senkei-gun no hyōgenron, (Japanese) [Representations of symmetric groups and general linear groups], Iwanami Shoten, Tokyo, 1982.

[3] K. Nakamoto and Y. Omoda, *Thick representations and dense representations I*, Kodai Math. J. **42** (2019), 274–307.

[4] ———, *The classification of thick representations of simple Lie groups*, Kodai Math. J. **45** (2022), 259–269.

[5] ———, *Characterization of 4-dimensional non-thick irreducible representations*, Proceedings of the 54th Symposium on Ring Theory and Representation Theory, 76–83, Symposium on Ring Theory and Representation Theory Organizing Committee, Saitama, 2023

[6] ———, *Thick representations and dense representations II*, in preparation.

[7] K. Nakamoto, S. Okuyama, and Y. Omoda, *The classification of thick representations of the symmetric group*, in preparation.

[8] M. H. Peel, *Specht modules and symmetric groups*, J. Algebra **36** (1975), no. 1, 88–97.

[9] R. Rasala, *On the minimal degrees of characters of S_n*, J. Algebra **45** (1977), no. 1, 132–181.

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