NONCOMMUTATIVE HIRZEBRUCH SURFACES

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ABSTRACT. A noncommutative $\mathbb{P}^1$-bundle over a commutative scheme defined in [16] is one of the major objects of study in noncommutative algebraic geometry. Last year in this symposium [7], we classified locally free sheaf bimodules on $\mathbb{P}^1$ of rank 2 in order to classify noncommutative Hirzebruch surfaces, which are defined to be noncommutative $\mathbb{P}^1$-bundles over $\mathbb{P}^1$. In this article, we give an idea on how to define a noncommutative $\mathbb{P}^1$-bundle over a commutative smooth projective scheme from the view point of representation theory of algebras, extending the construction of the preprojective algebra of the 2-Kronecker quiver. Then we will show that every noncommutative $\mathbb{P}^1$-bundle over a commutative smooth projective scheme has a suitable semi-orthogonal decomposition. Moreover, we will show that every noncommutative Hirzebruch surface even has a full strong exceptional sequence so that its derived category is equivalent to the derived category of a bound quiver algebra. We also compute quivers for some of such bound quiver algebras.

1. Quasi-schemes

Throughout, we fix an algebraically closed field k. By [4] and [11], every scheme X can be reconstructed from the category $\operatorname{Mod} X$ of quasi-coherent sheaves on X, so we will extend the notion of scheme as follows.

Definition 1. [10], [15] A **quasi-scheme** X is a Grothendieck category $\operatorname{Mod} X$. We say that a quasi-scheme X is noetherian if $\operatorname{Mod} X$ is locally noetherian, that is, $\operatorname{Mod} X$ has a small set of noetherian generators. In this case, we denote by $\operatorname{mod} X \subset \operatorname{Mod} X$ the full subcategory consisting of noetherian objects.

By [13], $\operatorname{Mod} X$ is a Grothendieck category for a usual scheme X, so we will view a scheme X as a quasi-scheme by $\operatorname{Mod} X$. Note that if X is a noetherian scheme, then $\operatorname{mod} X$ is the category of coherent sheaves on X.

The above notion of quasi-scheme includes noncommutative schemes. For example, the **noncommutative affine scheme** $X = \operatorname{Spec}_{nc} R$ associated to a ring R is defined to be a quasi-scheme where $\operatorname{Mod} X = \operatorname{Mod} R$ is the category of right R-modules. Our main object of study is a noncommutative projective scheme defined as follows. Let A be a graded ring. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{Tors} A \subset \operatorname{GrMod} A$ the full subcategory consisting of direct limits of right bounded modules (i.e. $M_n = 0$ for all $n \gg 0$).

Definition 2. [2] The **noncommutative projective scheme** $X = \operatorname{Proj}_{nc} A$ associated to a graded ring A is a quasi-scheme where

$$\operatorname{Mod} X = \operatorname{Tails} A := \operatorname{GrMod} A / \operatorname{Tors} A$$

The detailed version of this paper will be submitted for publication elsewhere.

is the quotient category.

The above definition can be justified by the following classical theorem.

Theorem 3. [12] *If A is a commutative graded algebra finitely generated in degree 1 over k and $X = \operatorname{Proj} A$ in the usual sense, then*

$$\operatorname{Tails} A \cong \operatorname{Mod} X.$$

2. The First Definition of a Noncommutative $\mathbb{P}^1$-bundle

For the rest of this paper, we assume that X is a smooth projective scheme over k. First, we will review one of the characterizations of a (commutative) $\mathbb{P}^1$-bundle over X. A $\mathbb{P}^1$-bundle over X can be characterized as a scheme $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} S(\mathcal{E})$ where $\mathcal{E}$ is a locally free sheaf on X of rank 2, and $S(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_X$. Note that $S(\mathcal{E}) = T(\mathcal{E})/(\mathcal{Q})$ where $T(\mathcal{E})$ is the tensor algebra of $\mathcal{E}$ over $\mathcal{O}_X$, and $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is the invertible subsheaf locally generated by the sections of the form $x \otimes y - y \otimes x$. We want to extend this characterization of a $\mathbb{P}^1$-bundle over X to a noncommutative setting. What we will replace is a locally free sheaf $\mathcal{E}$ by a locally free sheaf bimodule. If R is a commutative ring, then R-bimodules are the same as $R \otimes R$-modules. If $X = \operatorname{Spec} R$, then $\operatorname{Spec}(R \otimes R) = X \times X$, so we may think of sheaf bimodules on X as sheaves on $X \times X$.

Definition 4. [1] A **sheaf bimodule** on X is a coherent sheaf $\mathcal{M}$ on $X \times X$ such that

$$u := pr_1|_W, v := pr_2|_W : W := \operatorname{Supp} \mathcal{M} \subset X \times X \to X$$

are finite morphisms where $pr_i : X \times X \to X$ are projections. A sheaf bimodule $\mathcal{E}$ on X is called **locally free of rank** r if $pr_{i*}\mathcal{E}$ are locally free of rank r on X for $i = 1, 2$.

A sheaf bimodule $\mathcal{M}$ on X defines functors

$$- \otimes_X \mathcal{M},\ \mathcal{M} \otimes_X - : \operatorname{Mod} X \longrightarrow \operatorname{Mod} X$$

where

$$- \otimes_X \mathcal{M} := pr_{2*}(pr_1^*(-) \otimes_{X \times X} \mathcal{M})$$
$$\mathcal{M} \otimes_X - := pr_{1*}(\mathcal{M} \otimes_{X \times X} pr_2^*(-)).$$

The following is the key lemma which makes it possible to define a noncommutative $\mathbb{P}^1$-bundle.

Lemma 5. [16] *If $\mathcal{E}$ is a locally free sheaf bimodule on X of rank r, then there exist locally free sheaf bimodules $\mathcal{E}^*$ and ${}^*\mathcal{E}$ on X of rank r such that $- \otimes_X \mathcal{E}^*$ is a right adjoint to $- \otimes_X \mathcal{E}$ and $- \otimes_X {}^*\mathcal{E}$ is a left adjoint to $- \otimes_X \mathcal{E}$.*

Definition 6. [16] An invertible sheaf subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is called **non-degenerate** if the composition

$$\mathcal{E}^* \otimes_X \mathcal{Q} \to \mathcal{E}^* \otimes_X \mathcal{E} \otimes_X \mathcal{E} \to \mathcal{E}$$

is an isomorphism where the first map is induced by the inclusion $\mathcal{Q} \to \mathcal{E} \otimes_X \mathcal{E}$ and the second map is induced by the adjoint map $\mathcal{E}^* \otimes_X \mathcal{E} \to \mathcal{O}_X$.

The below is the first definition of a noncommutative $\mathbb{P}^1$-bundle over X.

Definition 7. [14], [9] A **noncommutative $\mathbb{P}^1$-bundle** over X is a quasi-scheme $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} \mathcal{A}$ where $\mathcal{E}$ is a locally free sheaf bimodule on X of rank 2, $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ is a non-degenerate invertible sheaf subbimodule, and $\mathcal{A} = T(\mathcal{E})/(\mathcal{Q})$.

Unfortunately, the above definition of a noncommutative $\mathbb{P}^1$-bundle over X has some disadvantages. For example, given a locally free sheaf bimodule $\mathcal{E}$ on X of rank 2, it is not clear if a non-degenerate invertible sheaf subbimodule $\mathcal{Q} \subset \mathcal{E} \otimes_X \mathcal{E}$ exists. Even if it exists, it is not clear if $\mathbb{P}_X(\mathcal{E})$ is independent of the choice of $\mathcal{Q}$. In order to avoid these problems, we will redefine a noncommutative $\mathbb{P}^1$-bundle in terms of a quiver, which is one of the main tools in representation theory of finite dimensional algebras.

3. Noncommutative $\mathbb{P}^1$-bundles Defined by Quivers

A quiver $Q = (Q_0, Q_1, h, t)$ consists of a set of vertices Q_0, a set of arrows Q_1, and two maps $h, t : Q_1 \to Q_0$. A path of length n is a sequence of arrows $x_1 x_2 \cdots x_n$ where $h(x_i) = t(x_{i+1})$ for all $i = 1, \ldots, n-1$. Each vertex i can be regarded as a path e_i of length 0. The **path algebra** kQ of a quiver Q is a vector space spanned by all paths with the multiplication defined by the concatenation of paths. For a quiver Q, we define the double of Q by

$$\begin{aligned} \overline{Q} &:= (Q_0, \{x, x^* \mid x \in Q_1\}, \overline{h}, \overline{t}) \\ \overline{h}(x) &:= h(x) =: \overline{t}(x^*) \\ \overline{t}(x) &:= t(x) =: \overline{h}(x^*). \end{aligned}$$

The **preprojective algebra** $\Pi_k Q$ of a quiver Q is the path algebra of the quiver $\overline{Q}$ modulo the ideal generated by

$$\sum_{x \in Q_1} (xx^* - x^*x).$$

Fortunately, in this article, we need to understand only one example below.

Example 8. If

$$Q = \bullet \; \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \end{array} \; \bullet$$

is a quiver, called the 2-Kronecker quiver, then the path algebra is $R = kQ \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$. Moreover,

$$\overline{Q} = \bullet \; \begin{array}{c} \xrightarrow{x} \\ \xrightarrow{y} \\ \xleftarrow{x^*} \\ \xleftarrow{y^*} \end{array} \; \bullet$$

is the double of Q, and the preprojective algebra is

$$A = \Pi_k Q = k\overline{Q}/(xx^* + yy^*, x^*x + y^*y).$$

In this case, it is classical that $\operatorname{Tails} A \cong \operatorname{Mod} \mathbb{P}^1$ that is, $\operatorname{Proj}_{nc} A \cong \mathbb{P}^1$, and $\mathcal{D}^b(\operatorname{mod} R) \cong \mathcal{D}^b(\operatorname{mod} \mathbb{P}^1)$.

By the above example, we want to define a noncommutative $\mathbb{P}^1$-bundle over X to be the noncommutative projective scheme associated to the preprojective algebra of the quiver $Q := \mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X$. In order to do it, we need to extend the notion of quiver Q so that Q_0 is a set of k-linear categories, Q_1 is a set of k-linear functors, etc. We will explain it by an example.

Let V be a finite dimensional vector space over k. (In the above example, $V = kx+ky$.) For a quiver

$$Q = k \xrightarrow{V} k := \operatorname{Mod} k \xrightarrow{-\otimes_k V} \operatorname{Mod} k,$$

the path algebra of Q can be defined so that $kQ \cong \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$, and the preprojective algebra of Q can be defined by $\Pi_k Q := k\overline{Q}/(\operatorname{Im}\varphi, \operatorname{Im}\psi)$ where $\overline{Q} = k \underset{V^*}{\overset{V}{\rightleftarrows}} k$ is the double of Q, and $\varphi : k \to V \otimes_k V^*, \psi : k \to V^* \otimes_k V^{**} \cong V^* \otimes_k V$ are the adjoint maps. If we want to extend the above construction to the quiver

$$Q = \mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X := \operatorname{Mod} X \xrightarrow{-\otimes_X \mathcal{E}} \operatorname{Mod} X$$

where X is a smooth projective scheme and $\mathcal{E}$ is a locally free sheaf bimodule on X, then there is no problem to define the path algebra of Q so that $\mathcal{O}_X Q \cong \begin{pmatrix} \mathcal{O}_X & \mathcal{E} \\ 0 & \mathcal{O}_X \end{pmatrix}$, however, there is a problem to define the preprojective algebra of Q by $\Pi_{\mathcal{O}_X} Q := \mathcal{O}_X\overline{Q}/(\operatorname{Im}\varphi, \operatorname{Im}\psi)$ where $\overline{Q} := \mathcal{O}_X \underset{\mathcal{E}^*}{\overset{\mathcal{E}}{\rightleftarrows}} \mathcal{O}_X$ is the double of Q, and $\varphi : \mathcal{O}_X \to \mathcal{E} \otimes_X \mathcal{E}^*, \psi : \mathcal{O}_X \to \mathcal{E}^* \otimes_X \mathcal{E}$ because there is no canonical map $\psi : \mathcal{O}_X \to \mathcal{E}^* \otimes_X \mathcal{E}$ unless $\mathcal{E} \cong \mathcal{E}^{**}$, so we will modify the definition. The definition below was essentially given by Van den Bergh without using quivers.

Definition 9. [16] Let $Q = \mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X$ be a quiver where X is a smooth projective scheme and $\mathcal{E}$ is a locally free sheaf bimodule on X of rank 2. The preprojective algebra $\mathcal{A} = \Pi_{\mathcal{O}_X} Q$ of a quiver Q is the path algebra of the quiver

$$\overline{Q} := \cdots \xrightarrow{{}^{**}\mathcal{E}} \mathcal{O}_X \xrightarrow{{}^{*}\mathcal{E}} \mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X \xrightarrow{\mathcal{E}^*} \mathcal{O}_X \xrightarrow{\mathcal{E}^{**}} \mathcal{O}_X \xrightarrow{\mathcal{E}^{***}} \cdots$$

modulo the ideal generated by $\mathcal{Q}_i := \operatorname{Im}\varphi_i$ where $\varphi_i : \mathcal{O}_X \to \mathcal{E}^{i*} \otimes \mathcal{E}^{(i+1)*}$ are the adjoint maps. In this setting, $\mathbb{P}_X(\mathcal{E}) := \operatorname{Proj}_{nc} \mathcal{A}$ is called a **noncommutative $\mathbb{P}^1$-bundle** over X. If X is a curve, then $\mathbb{P}_X(\mathcal{E})$ is called a **noncommutative ruled surface**. If $X = \mathbb{P}^1$, then $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ is called a **noncommutative Hirzebruch surface**.

Note that $\mathbb{P}_X(\mathcal{E})$ is well-defined for every $\mathcal{E}$, independent of the choice of $\mathcal{Q}$, and agrees with the old definition when Q exsits [16].

4. Semi-orthogonal Decomposition

In this section, we show that every noncommutative $\mathbb{P}^1$-bundle has a semi-orthogonal decomposition defined below.

Definition 10. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{N}_1, \mathcal{N}_2$ strictly full triangulated subcategories. We say that $\mathcal{T} = \langle \mathcal{N}_1, \mathcal{N}_2 \rangle$ is a **semi-orthogonal decomposition** if

(1) the inclusion functors $\mathcal{N}_1, \mathcal{N}_2 \to \mathcal{T}$ have both left and right adjoints,
(2) $\mathrm{Hom}_{\mathcal{T}}(N_2, N_1) = 0$ for every $N_1 \in \mathcal{N}_1, N_2 \in \mathcal{N}_2$, and
(3) the smallest strictly full triangulated subcategory containing $\mathcal{N}_1, \mathcal{N}_2$ is $\mathcal{T}$.

Below, we denote by $\pi : \mathrm{GrMod}\, \mathcal{A} \to \mathrm{Tails}\, \mathcal{A}$ the quotient functor, which is an exact functor, and by $\omega : \mathrm{Tails}\, \mathcal{A} \to \mathrm{GrMod}\, \mathcal{A}$ the section functor, which is the right adjoint to π.

Definition 11. [6] Let $\mathbb{P}_X(\mathcal{E})$ be a noncommutative $\mathbb{P}^1$-bundle over a smooth projective scheme X. The **structure maps** $f_i : \mathbb{P}_X(\mathcal{E}) \to X$ are the adjoint pairs of functors

$$f_i^* : \mathrm{Mod}\, X \xrightarrow{-\otimes e_i \mathcal{A}} \mathrm{GrMod}\, \mathcal{A} \xrightarrow{\pi} \mathrm{Tails}\, \mathcal{A}$$

$$f_{i*} : \mathrm{Tails}\, \mathcal{A} \xrightarrow{\omega} \mathrm{GrMod}\, \mathcal{A} \xrightarrow{(-)_i} \mathrm{Mod}\, X.$$

Theorem 12. [3] *(Serre Duality) If $\mathbb{P}_X(\mathcal{E})$ is a noncommutative $\mathbb{P}^1$-bundle over a smooth projective scheme X of dimension d, then there exists an equivalence functor*

$$- \otimes_{\mathbb{P}_X(\mathcal{E})} \omega_{\mathbb{P}_X(\mathcal{E})} : \mathcal{D}^b(\mathrm{mod}\, \mathbb{P}_X(\mathcal{E})) \to \mathcal{D}^b(\mathrm{mod}\, \mathbb{P}_X(\mathcal{E}))$$

such that

$$\mathrm{Ext}^i_{\mathbb{P}_X(\mathcal{E})}(\mathcal{M}, \mathcal{N} \otimes_{\mathbb{P}_X(\mathcal{E})} \omega_{\mathbb{P}_X(\mathcal{E})}) \cong \mathrm{Ext}^{d+1-i}_{\mathbb{P}_X(\mathcal{E})}(\mathcal{N}, \mathcal{M})^*$$

for all $\mathcal{M}, \mathcal{N} \in \mathrm{mod}\, \mathbb{P}_X(\mathcal{E})$ and $i = 0, \ldots, d+1$ where $(-)^*$ is the functor taking the k-vector space dual.

Theorem 13. [8] *If $\mathbb{P}_X(\mathcal{E})$ is a noncommutative $\mathbb{P}^1$-bundle over a smooth projective scheme X, then*

$$\mathcal{D}^b(\mathrm{mod}\, \mathbb{P}_X(\mathcal{E})) = \langle f_{i+1}^* \mathcal{D}^b(\mathrm{mod}\, X), f_i^* \mathcal{D}^b(\mathrm{mod}\, X) \rangle$$

is a semi-orthogonal decomposition for every $i \in \mathbb{Z}$.

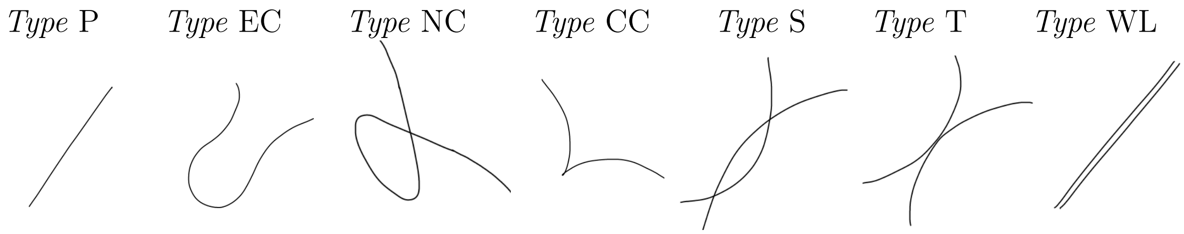
5. Noncommutative Hirzebruch Surfaces

We now focus on the case $X = \mathbb{P}^1$. It is known that if $\mathcal{E}$ is a locally free sheaf on $\mathbb{P}^1$ of rank 2, then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some unique $a \le b \in \mathbb{Z}$, and $\mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a+m) \oplus \mathcal{O}_{\mathbb{P}^1}(b+m)$ for every $m \in \mathbb{Z}$. We say that $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ is $\mathbb{F}_d$ if $b - a = d \in \mathbb{N}$. We define a noncommutative analogue of $\mathbb{F}_d$ below.

Definition 14. For a locally free sheaf bimodule $\mathcal{E}$ on $\mathbb{P}^1$ of rank 2 and $m \in \mathbb{Z}$, we define $a_m \le b_m \in \mathbb{Z}$ by $\mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_m) \oplus \mathcal{O}_{\mathbb{P}^1}(b_m)$. We say that $\mathcal{E}$ **commutes with shifts** if $\mathcal{O}_{\mathbb{P}^1}(m) \otimes_{\mathbb{P}^1} \mathcal{E} \cong (\mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{P}^1} \mathcal{E})(m)$ (i.e. $a_m = a_0 + m, b_m = b_0 + m$) for every $m \in \mathbb{Z}$. We say that $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ is a **noncommutative** $\mathbb{F}_d$ if $\mathcal{E}$ commutes with shifts and $b_0 - a_0 = d \in \mathbb{N}$ (i.e. $a_m = m, b_m = d + m$ for every $m \in \mathbb{Z}$).

Remark 15. A first example of $\mathcal{E}$ which does not commute with shifts was given in [5].

Theorem 16. [9], [8] *For every locally free sheaf bimodule* $\mathcal{E}$ *on* $\mathbb{P}^1$ *of rank 2,* $W = \operatorname{Supp}\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$ *is a Cartier divisor of bidegree* $(1,1)$ *or* $(2,2)$*. In fact, it is one of the following types:*



If W is of Type P, then $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ is isomorphic to a commutative Hirzebruch surface, so we will ignore Type P for the rest of this article. If $\mathcal{E}$ is a sheaf bimodule on $\mathbb{P}^1$, and $\iota : W := \operatorname{Supp}\mathcal{E} \to \mathbb{P}^1 \times \mathbb{P}^1$ is the embedding, then there is a unique $\mathcal{U} \in \operatorname{mod} W$ such that $\iota_*\mathcal{U} \cong \mathcal{E}$. Using the explicit classification of $\mathcal{U}$ for each type in [8], [7], all possible $\{(a_m, b_m)\}_{m\in\mathbb{Z}}$ are computed in terms of $\mathcal{U}$.

Theorem 17. *Let* $\mathcal{E}$ *be a locally free sheaf bimodule on* $\mathbb{P}^1$ *of rank 2.*

(Case 1) If W *is irreducible (Type EC, NC, CC, WL), and* $\mathcal{U} \in \operatorname{Pic} W$*, then*

$$
\begin{aligned}
(a_m, b_m) &= \left(\frac{\deg\mathcal{U}}{2} + m - 1, \frac{\deg\mathcal{U}}{2} + m - 1\right),\ b_m - a_m = 0\\
&\quad \textit{if}\ \mathcal{U} \not\simeq u^*\mathcal{O}_{\mathbb{P}^1}(-m) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}\left(\frac{\deg\mathcal{U}}{2} + m\right),\ \deg\mathcal{U} \equiv 0 \mod 2,\\
(a_m, b_m) &= \left(\frac{\deg\mathcal{U}}{2} + m - 2, \frac{\deg\mathcal{U}}{2} + m\right),\ b_m - a_m = 2\\
&\quad \textit{if}\ \mathcal{U} \simeq u^*\mathcal{O}_{\mathbb{P}^1}(-m) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}\left(\frac{\deg\mathcal{U}}{2} + m\right),\ \deg\mathcal{U} \equiv 0 \mod 2,\\
(a_m, b_m) &= \left(\frac{\deg\mathcal{U} - 1}{2} + m - 1, \frac{\deg\mathcal{U} - 1}{2} + m\right),\ b_m - a_m = 1\\
&\quad \textit{if}\ \deg\mathcal{U} \equiv 1 \mod 2.
\end{aligned}
$$

(Case 2) If $W = W_1 \cup W_2$ *is reducible (Type S, T), and* $\mathcal{U} \in \operatorname{Pic} W$ *with* $\deg(\mathcal{U}|_{W_1}) \leq \deg(\mathcal{U}|_{W_2})$*, then*

$$
\begin{aligned}
&\{a_m, b_m\} = \{\deg(\mathcal{U}|_{W_1}) + m - 1, \deg(\mathcal{U}|_{W_2}) + m - 1\},\ b_m - a_m = 0\\
&\qquad \textit{if}\ \mathcal{U} \not\simeq u^*\mathcal{O}_{\mathbb{P}^1}(-m) \otimes_W v^*\mathcal{O}_{\mathbb{P}^1}(\deg(\mathcal{U}|_{W_1}) + m),\ \deg(\mathcal{U}|_{W_1}) = \deg(\mathcal{U}|_{W_2}),\\
&\{a_m, b_m\} = \{\deg(\mathcal{U}|_{W_1}) + m, \deg(\mathcal{U}|_{W_2}) + m - 2\},\ b_m - a_m = |\deg(\mathcal{U}|_{W_2}) - \deg(\mathcal{U}|_{W_1}) - 2|\\
&\qquad \textit{otherwise.}
\end{aligned}
$$

(Case 3) If W *is reduced (Type NC, CC, S, T) and* $\mathcal{U} \notin \operatorname{Pic} W$*, then*

Type	$\widetilde{W}$	$\widetilde{\mathcal{U}}$ $(i \leq j)$	$\{a_m, b_m\}$	$b_m - a_m$
NC, CC	$\mathbb{P}^1$	$\mathcal{O}(2i)$	$\{i+m-1, i+m\}$	1
NC, CC	$\mathbb{P}^1$	$\mathcal{O}(2i+1)$	$\{i+m, i+m\}$	0
S, T	$\mathbb{P}^1 \cup \mathbb{P}^1$	$\mathcal{O}(i,j)$	$\{i+m, j+m-1\}$	$\lvert j-i-1\rvert$
S, T	$\mathbb{P}^1 \sqcup \mathbb{P}^1$	$\mathcal{O}(i) \sqcup \mathcal{O}(j)$	$\{i+m, j+m\}$	$j-i$

where $\nu : \widetilde{W} := \mathcal{S}pec\, \mathcal{E}nd_{\mathcal{O}_W}(\mathcal{U}) \to W$*,* $\widetilde{\mathcal{U}} \in \operatorname{Pic} \widetilde{W}$ *such that* $\nu_* \widetilde{\mathcal{U}} = \mathcal{U}$*,* $\mathbb{P}^1 \cup \mathbb{P}^1$ *denotes a pair of crossing lines, and* $\mathbb{P}^1 \sqcup \mathbb{P}^1$ *denotes a pair of disjoint lines.*

(Case 4) If W *is non-reduced (Type WL) and* $\mathcal{U} \notin \operatorname{Pic} W$*, then*

$$\{a_m, b_m\} = \left\{ \frac{\chi(\mathcal{U}) - \#(\mathrm{Ninv}(\mathcal{U}))}{2} + m, \frac{\chi(\mathcal{U}) + \#(\mathrm{Ninv}(\mathcal{U}))}{2} + m - 2 \right\},$$
$$b_m - a_m = |\#(\mathrm{Ninv}(\mathcal{U})) - 2|$$

where $\mathrm{Ninv}(\mathcal{U}) := \{p \in W \mid \mathcal{U}_p \not\cong \mathcal{O}_{W,p}\} \subset \operatorname{Sing} W$*.*

Corollary 18. *Let* $\mathcal{E}$ *be a locally free sheaf bimodule on* $\mathbb{P}^1$ *of rank 2,* $\iota : W := \operatorname{Supp} \mathcal{E} \to \mathbb{P}^1 \times \mathbb{P}^1$ *the embedding, and* $\mathcal{U} \in \operatorname{mod} W$ *such that* $\iota_* \mathcal{U} \cong \mathcal{E}$*.*

(1) *If* W *is integral (Type EC, NC, CC), then* $b_m - a_m \in \{0, 1, 2\}$ *for every* $m \in \mathbb{Z}$*.*
(2) *If* $\mathcal{E}$ *does not commute with shifts, then* $\mathcal{U} \cong u^* \mathcal{O}_{\mathbb{P}^1}(i) \otimes_W v^* \mathcal{O}_{\mathbb{P}^1}(j)$ *for some* $i, j \in \mathbb{Z}$*.*

6. Quivers

In this last section, we show that every noncommutative Hirzebruch surface has a full strong exceptional sequence, so that it is derived equivalent to a bound quiver algebra kQ/I. We also compute the quiver Q for a noncommutative $\mathbb{F}_d$ for $d = 0, 1, 2$.

Definition 19. Let $\mathcal{T}$ be a triangulated category. A sequence $\{E_1, \dots, E_\ell\}$ of objects in $\mathcal{T}$ is a **full strong exceptional sequence** if

(1) $\operatorname{End}_{\mathcal{T}}(E_i) \cong k$ for every $1 \leq i \leq \ell$,
(2) $\operatorname{Hom}_{\mathcal{T}}(E_i, E_j[q]) = 0$ for every $q \neq 0$ and $1 \leq i, j \leq \ell$,
(3) $\operatorname{Hom}_{\mathcal{T}}(E_i, E_j[q]) = 0$ for every q and $1 \leq j < i \leq \ell$, and
(4) the smallest strictly full triangulated subcategory containing $E_1, \dots, E_\ell$ is $\mathcal{T}$.

Recall that $a_{-1} \leq b_{-1} \in \mathbb{Z}$ are defined by $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_{-1}) \oplus \mathcal{O}_{\mathbb{P}^1}(b_{-1})$.

Theorem 20. [8] *Let* $\mathcal{E}$ *be a locally free sheaf bimodule on* $\mathbb{P}^1$ *of rank 2. If we write* $\mathcal{O}(i,j) := f_{-i}^*(\mathcal{O}_{\mathbb{P}^1}(j)) \in \operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ *for* $i, j \in \mathbb{Z}$*, then*

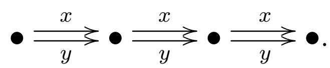
$$\mathcal{O}(-1, -j-1), \mathcal{O}(-1, -j), \mathcal{O}(0, -1), \mathcal{O}(0, 0)$$

is a full strong exceptional sequence for $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}))$ *if and only if* $j \geq -a_{-1} - 1$*. In this case,* $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) \cong \mathcal{D}^b(\operatorname{mod} R)$ *where*

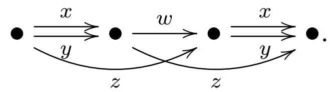
$$R = \operatorname{End}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(\mathcal{O}(-1, -j-1) \oplus \mathcal{O}(-1, -j) \oplus \mathcal{O}(0, -1) \oplus \mathcal{O}(0, 0)).$$

Theorem 21. [8] *Let* $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ *be a noncommutative* $\mathbb{F}_d$*. For* $d = 0, 1, 2$*, there is a bound quiver algebra* $R = kQ/I$ *such that* $\mathcal{D}^b(\operatorname{mod} \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) \cong \mathcal{D}^b(\operatorname{mod} kQ/I)$ *where*

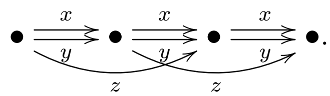
(1) $d = 0$:



(2) $d = 1$:



(3) $d = 2$:



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