ON THE AUSLANDER–REITEN THEORY FOR EXTENDED HEARTS OF PROPER CONNECTIVE DG-ALGEBRAS

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Abstract. In this paper, we generalize the Auslander–Reiten theory in module categories of finite dimensional algebras to the extended hearts of proper connective DG-algebras. More precisely, we prove the existence of Auslander–Reiten sequences in the extended hearts as extriangulated categories and the first Brauer-Thrall conjecture for them.

1. Introduction

DG-algebras (differential graded algebras) are a generalization of associative algebras, and they have been studied in various areas of mathematics. Recently, the representation theory of connective DG-algebras has been actively developed from the viewpoint of silting theory [1] and derived equivalences [5].

In this note, we mainly focus on connective DG-algebras. The derived categories of proper connective DG-algebras are sometimes equivalent to the derived categories of finite dimensional algebras. However, it is difficult to find such equivalences for general proper connective DG-algebras. Thus, we need to develop tools to study some representation theoretic properties of proper connective DG-algebras without derived equivalences to finite dimensional algebras.

In the paper [6], we introduced the Auslander–Reiten theory for the d-extended module category $\mathrm{mod}^d\Lambda$ of a proper connective d-truncated DG-algebra Λ as a generalization of the classical Auslander–Reiten theory in the module category over a finite dimensional algebra. By using this theory, we can draw the Auslander–Reiten quiver of extended module categories of some proper connective DG-algebras without derived equivalences to finite dimensional algebras.

2. Main results

2.1. **Extended hearts of proper connective DG-algebras.** In this note, we fix an algebraically closed field k and a DG-algebra Λ over k.

Definition 1. A *perfect derived category* of Λ is the smallest thick subcategory of the derived category $D(\Lambda)$ of DG-Λ-modules containing Λ. We denote it by $\mathrm{per}\Lambda$. A *perfectly valued derived category* of Λ is the full subcategory of $D(\Lambda)$ consisting of DG-Λ-modules M such that $\sum_{i\in\mathbb{Z}} \dim_k H^iM < \infty$. We denote it by $\mathrm{pvd}\Lambda$.

Definition 2. Let $d \geq 1$ be an integer.

(1) Λ is said to be *proper* if $\sum_{i\in\mathbb{Z}} \dim_k H^i\Lambda < \infty$.

The detailed version of this paper has been submitted for publication elsewhere.

(2) Λ is said to be *connective* if $H^i\Lambda = 0$ for all $i > 0$.
(3) Λ is said to be *d-truncated* if $H^i\Lambda = 0$ for all $i < -d+1$.

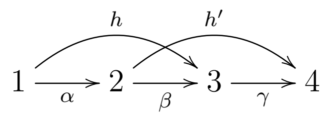
Example 3. We give some examples of proper connective d-truncated DG-algebras.

(1) Let Λ be a proper connective 1-truncated DG-algebra. Then the natural map $\Lambda \to H^0\Lambda$ is a quasi-isomorphism. Thus Λ is quasi-isomorphic to a finite dimensional algebra. Conversely, any finite dimensional algebra can be regarded as a proper connective 1-truncated DG-algebra.
(2) Let Γ be a finite dimensional algebra. If there is a silting object P in $\mathrm{per}\Gamma$, then the derived endomorphism DG-algebra $\Lambda := \mathbb{R}\mathrm{End}_\Gamma(P)$ is a proper connective DG-algebra. Furthermore, if $\mathrm{per}\Gamma(P, P[\leq -d]) = 0$, then Λ is d-truncated.

Definition 4. Let $Q = (Q_0, Q_1)$ be a graded quiver with grading $Q_1 = \coprod_{i\in\mathbb{Z}} Q_1^i$. Then we have a graded algebra kQ. We denote its grading by $kQ = \bigoplus_{i\in\mathbb{Z}}(kQ)^i$ where $(kQ)^i$ is the vector space generated by paths of total degree i. Note that $(kQ)^i$ does not correspond to $k(Q_1^i)$.

A *DG-quiver* is a pair (Q, d_Q) of a graded quiver (Q_0, Q_1) where $Q_1 = \coprod_{i\in\mathbb{Z}} Q_1^i$ and $d_Q^i \colon Q_1^i \to (kQ)^i$. The differential d_Q is extended to kQ by the Leibniz rule. Thus we obtain a DG-algebra structure on graded algebra kQ. We call it the *path DG-algebra* of the DG-quiver (Q, d_Q) and denote it simply by kQ.

Example 5. Consider the following DG-quiver (Q, d_Q):



where $|\alpha| = 0$, $|\beta| = 1$, $|\gamma| = 0$, $|h| = -1$ and $|h'| = -1$. The differentials are given by $d_Q(\alpha) = 0$, $d_Q(\beta) = 0$, $d_Q(\gamma) = 0$, $d_Q(h) = \beta\alpha$ and $d_Q(h') = \beta\gamma$. Then the path DG-algebra

$$kQ^{\leq -2} = 0, \quad kQ^{-1} := kh \oplus kh' \oplus k\gamma h \oplus kh'\alpha,$$
$$kQ^0 := ke_1 \oplus ke_2 \oplus ke_3 \oplus ke_4 \oplus k\alpha \oplus k\beta \oplus k\gamma \oplus k\beta\alpha \oplus k\gamma\beta \oplus k\gamma\beta\alpha$$

and the differential d_{kQ} is defined as follows:

$$d_{kQ}(\gamma h) := d_Q(\gamma)h + \gamma d_Q(h) = \gamma\beta\alpha$$
$$d_{kQ}(h'\alpha) := d_Q(h')\alpha - h' d_Q(\alpha) = \gamma\beta\alpha$$

Since $Z^{-1}kQ = k(\gamma h - h'\alpha)$ and $B^{-1}kQ = 0$, we have $H^{-1}kQ \simeq k$. Thus the path DG-algebra kQ is a proper connective 2-truncated DG-algebra which is not quasi-isomorphic to any finite dimensional algebra.

Definition 6. Let Λ be a proper connective d-truncated DG-algebra. The *d-extended module category* $\mathrm{mod}^d\Lambda$ is defined as the full subcategory of $D(\Lambda)$ consisting of DG-Λ-modules M such that $H^i(M) = 0$ for all $i \neq 0, -1, \ldots, -d+1$ and $\dim_k H^i(M) < \infty$ for all i.

Remark 7. We remark some properties of the d-extended module category $\mathrm{mod}^d\Lambda$.

(1) Since Λ is proper connective d-truncated, the DG-Λ-module Λ is in its d-extended module category $\mathrm{mod}^d\Lambda$.

(2) If $d = 1$, then the 1-extended module category $\mathrm{mod}^1\Lambda$ is equivalent to the heart of the canonical t-structure on $\mathrm{pvd}\Lambda$ which is equivalent to the category of finite dimensional $H^0\Lambda$-modules $\mathrm{mod}H^0\Lambda$.

Thanks to [7], the d-extended module category $\mathrm{mod}^d\Lambda$ has good structures for studying Auslander–Reiten theory.

Proposition 8. *Let Λ be a proper connective d-truncated DG-algebra. Then the d-extended module category $\mathrm{mod}^d\Lambda$ is a Hom-finite Krull–Schmidt extriangulated category with enough projectives and enough injectives. Moreover, the full subcategory $\mathcal{P}_\Lambda$ of projective objects corresponds to $\mathrm{add}\Lambda \subset \mathrm{mod}^d\Lambda$. Dually, the full subcategory $\mathcal{I}_\Lambda$ of injective objects corresponds to $\mathrm{add}(D\Lambda)[d-1] \subset \mathrm{mod}^d\Lambda$.*

Remark 9. We call the triangle $L \to M \to N \dashrightarrow$ in $\mathrm{pvd}\Lambda$ a *conflation* in $\mathrm{mod}^d\Lambda$ if L, M, N are objects in $\mathrm{mod}^d\Lambda$. This notion corresponds to the general notion of conflations in extriangulated categories.

Definition 10. Let Λ be a proper connective d-truncated DG-algebra. We define the *category of projective presentations* $\mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda)$ as the following subcategory of $\mathrm{per}\Lambda$:

$$\mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda) := \mathcal{P}_\Lambda * (\mathcal{P}_\Lambda)[1] * \cdots * (\mathcal{P}_\Lambda)[d] \subset \mathrm{per}\Lambda.$$

Dually, we define the *category of injective presentations* $\mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda)$ as the following subcategory of $\mathrm{thick}(D_d\Lambda)$:

$$\mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda) := (\mathcal{I}_\Lambda)[-d] * (\mathcal{I}_\Lambda)[-d+1] * \cdots * \mathcal{I}_\Lambda \subset \mathrm{thick}(D_d\Lambda).$$

Remark 11. Since we have a Nakayama equivalence $\nu := - \otimes^{\mathbb{L}}_\Lambda D\Lambda\colon \mathrm{per}\Lambda \to \mathrm{thick}(D\Lambda)$ and the functor $\nu \circ [-1]$ sends an object in $\mathcal{P}_\Lambda$ to an object in $\mathcal{I}_\Lambda[-d]$, the Nakayama equivalence induces an equivalence

$$\nu \circ [-1]\colon \mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda) \to \mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda).$$

To consider Auslander–Reiten theory, we need the concept of Auslander–Reiten translations. The following proposition enables us to define them.

Theorem 12. *The functor $t_{>-d}\colon \mathrm{R}^{[-d,0]}(\mathcal{P}_\Lambda) \to \mathrm{mod}^d\Lambda$ induces an equivalence*

$$\mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda)/\mathcal{P}_\Lambda[d] \xrightarrow{\sim} \mathrm{mod}^d\Lambda.$$

Dually, the functor $t_{<0}\colon \mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda) \to \mathrm{mod}^d\Lambda$ induces an equivalence

$$\mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda)/\mathcal{I}_\Lambda[-d] \xrightarrow{\sim} \mathrm{mod}^d\Lambda.$$

Here, we denote the ideal quotient associated with a subcategory by the quotient mark "/".

This theorem is already known in the case of finite dimensional algebras [2].

2.2. **Auslander–Reiten duality and existence of Auslander–Reiten sequences.**

Definition 13. We define the *stable d-extended module category* $\underline{\mathrm{mod}}^d\Lambda$ and the *costable d-extended module category* $\overline{\mathrm{mod}}^d\Lambda$ by the following ideal quotients:

$$\underline{\mathrm{mod}}^d\Lambda := \mathrm{mod}^d\Lambda/\mathcal{P}_\Lambda, \quad \overline{\mathrm{mod}}^d\Lambda := \mathrm{mod}^d\Lambda/\mathcal{I}_\Lambda.$$

By using Theorem 12, we have the following corollary.

Corollary 14. *The functor $t_{>-d}$ induces an equivalence*

$$\mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda)/(\mathcal{P}_\Lambda[d]\oplus\mathcal{P}_\Lambda)\xrightarrow{\sim}\underline{\mathrm{mod}}^d\Lambda.$$

where $\mathcal{P}_\Lambda[d]\oplus\mathcal{P}_\Lambda$ is the additive subcategory generated by objects in $\mathcal{P}_\Lambda[d]$ and $\mathcal{P}_\Lambda$.

Dually, the functor $t_{<0}$ induces an equivalence

$$\mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda)/(\mathcal{I}_\Lambda[-d]\oplus\mathcal{I}_\Lambda)\xrightarrow{\sim}\overline{\mathrm{mod}}^d\Lambda.$$

where $\mathcal{I}_\Lambda[-d]\oplus\mathcal{I}_\Lambda$ is the additive subcategory generated by objects in $\mathcal{I}_\Lambda[-d]$ and $\mathcal{I}_\Lambda$.

Definition 15. We define the *Auslander–Reiten translation* $\tau\colon\underline{\mathrm{mod}}^d\Lambda\to\overline{\mathrm{mod}}^d\Lambda$ as the composition of the following equivalences:

$$\underline{\mathrm{mod}}^d\Lambda\xrightarrow{\sim}\mathrm{K}^{[-d,0]}(\mathcal{P}_\Lambda)/(\mathcal{P}_\Lambda[d]\oplus\mathcal{P}_\Lambda)\xrightarrow{\nu\circ[-1]}\mathrm{K}^{[0,d]}(\mathcal{I}_\Lambda)/(\mathcal{I}_\Lambda[-d]\oplus\mathcal{I}_\Lambda)\xrightarrow{\sim}\overline{\mathrm{mod}}^d\Lambda.$$

Remark 16. We note that there exists explicit calulations of the Auslander–Reiten translation τ. Details are given in [6, Section 4].

The following theorem is a generalization of the Auslander–Reiten duality in module categories of finite dimensional algebras. This recovers the classical Auslander–Reiten duality when $d=1$.

Theorem 17. *The Auslander–Reiten translation τ induces the following functorial isomorphism for all $M,N\in\mathrm{mod}^d\Lambda$:*

$$\underline{\mathrm{mod}}^d\Lambda(M,N)\simeq D(\mathbb{E}(N,\tau M))$$

To state the existence of Auslander–Reiten sequences, we recall the definition of Auslander–Reiten sequences in extriangulated categories introduced in [3].

Definition 18. Let $f\colon M\to N$ be a morphism in $\mathrm{mod}^d\Lambda$.

(1) f is called a *left almost split morphism* if f is not a split monomorphism and for any morphism $g\colon M\to N'$ which is not a split monomorphism, there exists a morphism $h\colon N\to N'$ such that $g=h\circ f$.
(2) f is called a *left minimal morphism* if for any endomorphism $h\colon N\to N$, $f=f\circ h$ implies that h is an isomorphism. We call f a *source morphism* if f is left almost split and left minimal.
(3) f is called a *right almost split morphism* if f is not a split epimorphism and for any morphism $g\colon M'\to N$ which is not a split epimorphism, there exists a morphism $h\colon M'\to M$ such that $g=f\circ h$.
(4) f is called a *right minimal morphism* if for any endomorphism $h\colon M\to M$, $f=h\circ f$ implies that h is an isomorphism. We call f a *sink morphism* if f is right almost split and right minimal.

Definition 19 ([3, Theorem 2.9.])**.** A conflation $M\xrightarrow{f}N\xrightarrow{g}L\overset{\delta}{\dashrightarrow}$ in $\mathrm{mod}^d\Lambda$ is called an *Auslander–Reiten sequence* if the following equivalent conditions are satisfied:

(1) f is a left almost split morphism and L is indecomposable.
(2) g is a right almost split morphism and M is indecomposable.
(3) f is a source morphism.
(4) g is a sink morphism.

Since the existence of Auslander–Reiten sequences is equivalent to the Auslander–Reiten duality [3], we can conclude the following corollary:

Corollary 20. *For any indecomposable non-projective object M in* $\mathrm{mod}^d\Lambda$*, there exists an Auslander–Reiten sequence:*

$$\tau M \to E \to M \dashrightarrow .$$

Dually, for any indecomposable non-injective object N in $\mathrm{mod}^d\Lambda$*, there exists an Auslander–Reiten sequence:*

$$N \to E' \to \tau^{-1}N \dashrightarrow .$$

This corollary is already known in the following cases:

(1) For finite dimensional algebras [8]
(2) For d-self-injective DG-algebras [4]

As in the module categories of finite dimensional algebras, we have a sink morphism ending at indecomposable projective objects and a source morphism starting from indecomposable injective objects.

Proposition 21. *Let P be an indecomposable projective object in* $\mathrm{mod}^d\Lambda$*. We define* $\mathrm{rad}P$ *as the cocone of* $P \to \mathrm{top}H^0P$ *in* $\mathrm{pvd}\Lambda$*. Then the natural morphism* $\mathrm{rad}P \to P$ *is a sink morphism. Dually, let I be an indecomposable injective object in* $\mathrm{mod}^d\Lambda$*. We define* $\mathrm{corad}I$ *as the cone of* $\mathrm{soc}H^{-d+1}I \to I$ *in* $\mathrm{pvd}\Lambda$*. Then the natural morphism* $I \to \mathrm{corad}I$ *is a source morphism.*

If there is an indecomposable projective-injective object P in $\mathrm{mod}^d\Lambda$, we have a special Auslander–Reiten sequence associated with P.

Proposition 22. *Let P be an indecomposable projective-injective object in* $\mathrm{mod}^d\Lambda$*. Then there exists an Auslander–Reiten sequence:*

$$\mathrm{rad}P \to P \oplus Q \to \mathrm{corad}P \dashrightarrow$$

where Q is defined as the cone of the morphism $\mathrm{soc}H^{-d+1}P \to \mathrm{rad}P$*.*

Definition 23. The *Auslander–Reiten quiver* of $\mathrm{mod}^d\Lambda$ is defined as the quiver whose vertices are isomorphism classes of indecomposable objects in $\mathrm{mod}^d\Lambda$ and for any two indecomposable objects M, N in $\mathrm{mod}^d\Lambda$, the number of arrows from vertex M to vertex N is given by the dimension of the vector space of $\mathrm{Irr}(M, N)$ (see [3, Corollary 3.3.] or [6, Corollary 3.3.]).

The following theorem is a first Brauer-Thrall conjecture type theorem for extended hearts of proper connective DG-algebras.

Theorem 24. *Let Λ be a proper connective d-truncated DG-algebra. Suppose there exist connected components $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ of the Auslander–Reiten quiver of* $\mathrm{mod}^d\Lambda$ *as a quiver (not considering the Auslander–Reiten translations). If each $\mathcal{C}_i$ contains finitely many isomorphism classes of indecomposable objects in* $\mathrm{mod}^d\Lambda$ *and $\bigcup_i \mathcal{C}_i$ contains the following set:*

$$\{t_{>-d}(P[i]) \mid P \text{ is an indecomposable projective module}\},$$

then $\bigcup_i \mathcal{C}_i$ corresponds to the entire Auslander–Reiten quiver of $\mathrm{mod}^d\Lambda$*.*

Example 25. We consider the graded quiver $Q := [\ 1 \xrightarrow{\alpha} 2\ ]$ where $|\alpha| = -1$ and we regard the graded path algebra kQ as a DG-algebra with trivial differential. Then kQ is a proper connective 2-truncated DG-algebra.

We can see that the indecomposable projective DG-kQ-modules are given by $P_1 := e_1 kQ$ and $P_2 := e_2 kQ$. Dually, the indecomposable injective DG-kQ-modules are given by $I_1 := D(kQe_1)[1]$ and $I_2 := D(kQe_2)[1]$. Then we get the following isomorphisms in $\mathrm{mod}^2 kQ$:

$$P_1 \cong I_2, \quad P_2 \cong S_2, \quad I_1 \cong S_1[1]$$

where S_1 and S_2 are simple $H^0(kQ)$-modules corresponding to vertices 1 and 2 respectively. Thus by Proposition 22, we have an Auslander–Reiten sequence:

$$S_2[1] \to P_1 \to S_1 \dashrightarrow .$$

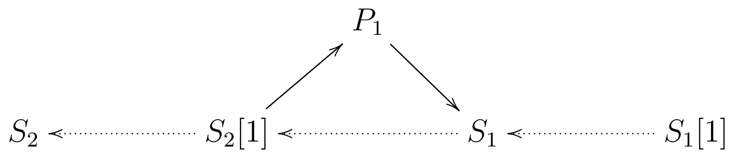
Since $S_2[1]$ is not projective and there are no irreducible morphisms to non-projective indecomposable objects from the above Auslander–Reiten sequence, we conclude that the morphism $0 \to S_2[1]$ is a sink morphism. Therefore, we have the following Auslander–Reiten quiver of $\mathrm{mod}^2 kQ$:

$$S_2 \to 0 \to S_2[1] \dashrightarrow .$$

Dually, we also have the following Auslander–Reiten quiver of $\mathrm{mod}^2 kQ$:

$$S_1 \to 0 \to S_1[1] \dashrightarrow .$$

Since $S_2 \cong P_2$ and $S_1[1] \cong I_1$, we have a connected component of the Auslander–Reiten quiver of $\mathrm{mod}^2 kQ$ as follows:



and this connected component satisfies the condition in Theorem 24. Thus this is the entire Auslander–Reiten quiver of $\mathrm{mod}^2 kQ$.

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