POINT MODULES OF A FAMILY CONSISTING OF THE ENVELOPING ALGEBRAS OF COLOR LIE ALGEBRAS

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ABSTRACT. A point scheme is an important object in Noncommutative algebraic geometry. There exists a one-to-one correspondence between the points of the underlying topological space of the point scheme of an algebra and the isomorphism classes of point modules of one. Then the underlying topological space of a point scheme can be determined by the set of all point modules. In this report, we determine the set of all point modules of a family of the enveloping algebras of color Lie algebras.

1. INTRODUCTION

Throughout this report, we assume that k is an algebraically closed field with characteristic zero. Artin–Schelter [1] introduced an important object called an AS-regular algebra in Noncommutative algebraic geometry.

Definition 1 ([1])**.** Let $A = \bigoplus_{i \geq 0} A_i$ be a connected $\mathbb{N}$-graded algebra over k. An algebra A is called an *AS-regular algebra of dimension d and of Gorenstein parameter of ℓ* if A satisfies following conditions:

(i) A has a finite global dimension; gl.dim $A = d < \infty$,
(ii) A satisfies Gorenstein condition;

$$\underline{\mathrm{Ext}}_A^n(k, A) \cong \begin{cases} k(\ell) & \text{if } n = d, \\ 0 & \text{if } n \neq d; \end{cases}$$

(iii) A has a finite Gelfand–Kirillov dimension;

$$\mathrm{GKdim} A := \inf \left\{ \alpha \in \mathbb{R} \mid \dim_k \left(\sum_{i=0}^n A_i \right) \leq n^\alpha \text{ for all } n \gg 0 \right\} < \infty.$$

Note that a commutative AS-regular algebra is a polynomial ring. Because of this fact, AS-regular algebras are often considered as "noncommutative polynomial rings". Levasseur [5] proved that a connected $\mathbb{N}$-graded Auslander-regular algebra with finite Gelfand–Kirillov dimension is an AS-regular algebra.

One of the ways of studying an AS-regular algebra is to determine the point scheme of it. This way can relate an AS-regular algebra to a projective scheme. Artin–Tate–Van den Bergh [2] classified a 3-dimensional AS-regular algebra using the point schemes of it. Also, Belmans–De Laet–La Bruyn [3] studied the point varieties of skew polynomial algebras S on n variables for $n \geq 3$. In [3], they determined the point variety structures of

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S by using the variables of S as an explicit normal element to classify the point modules of S. This way inspires us to study an algebra having some explicit normal elements.

A color Lie (super) algebra was introduced by Ree [8] as a generalized Lie algebra. The universal enveloping algebra of a color Lie (super) algebra satisfies some homological properties like those of (super) Lie algebras (see Price [7]). In particular, the universal enveloping algebra of a color Lie algebra can be regarded as an AS-regular algebra under some conditions. In this report, we will show the sets of all point modules and the point variety structures of a family consisting of the enveloping algebras of color Lie algebras (Corollary 12).

2. Universal enveloping algebras of color Lie algebras

Throughout this report, for a graded vector space $V = \bigoplus_{i \in I} V_i$ with indexed by a nonempty set, we denote the set of homogeneous elements of V as $h(V) = \bigcup_{i \in I} V_i$, and $|\ | : h(V) \setminus \{0\} \to k^\times; x \mapsto i$ such that $x \in V_i$. In this section, we recall the definition and some properties of the universal enveloping algebra of a color Lie algebra.

Definition 2 ([8]). Let G be an abelian group and $\varepsilon : G \times G \to k^\times$ a map. The map ε is called a *skew symmetric bicharacter* if, for any $\alpha, \beta, \gamma \in G$, the following equations hold:

(i) $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$,

(ii) $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$,

(iii) $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$.

Definition 3 ([8]). Let G be an abelian group and $\varepsilon : G \times G \to k^\times$ be a skew symmetric bicharacter. We denote a G-graded k-vector space as $L = \bigoplus_{\gamma \in G} L_\gamma$ and G-graded bilinear form as a $\langle\ ,\ \rangle : L \times L \to L$. The tuple $L = (L, \langle\ ,\ \rangle)$ is called the (G, ε)-*color Lie algebra* if, for any $a, b, c \in h(G)$, the following equations hold:

(i) $\langle a, b \rangle = -\varepsilon(|a|, |b|)\langle b, a \rangle$,

(ii) $\varepsilon(|c|, |a|)\langle a, \langle b, c \rangle\rangle + \varepsilon(|a|, |b|)\langle b, \langle c, a \rangle\rangle + \varepsilon(|b|, |c|)\langle c, \langle a, b \rangle\rangle = 0$.

Additionally, the subspaces of L are defined as $L_- := \bigoplus_{\gamma \in G_-} L_\gamma$ (resp. $L_+ := \bigoplus_{\gamma \in G_+} L_\gamma$) with $G_+ := \{\gamma \in G \mid \varepsilon(\gamma, \gamma) = -1\}$ (resp. $G_- := \{\gamma \in G \mid \varepsilon(\gamma, \gamma) = -1\}$).

Definition 4 ([8]). For a color Lie algebra L, let $T(L)$ denote the tensor algebra of L and $J(L)$ the two-sided ideal of $T(L)$ generated by $a \otimes b - \varepsilon(|a|, |b|)b \otimes a - \langle a, b \rangle$ for any $a, b \in h(L)$. The (universal) enveloping algebra of L is defined by $U(L) := T(L)/J(L)$.

The universal enveloping algebra of a color Lie algebra has some homological properties like a Lie (super) algebra. We recall an important proposition that the universal enveloping algebra of a color Lie algebra has conditions of an AS-regular algebra as follows:

Theorem 5 (see [7]). *Let L be a finite dimensional color Lie algebra with $L_- = 0$ and $U(L)$ be the universal enveloping algebra of L. Then $U(L)$ is Auslander-regular algebra.*

Also, for a finite dimensional color Lie algebra L with $L_- = 0$ and the universal enveloping algebra $U(L)$ of L, $\mathrm{GKdim}U(L) = \dim_k L$. By [5], the universal enveloping algebra $U(L)$ of a color Lie algebra L is an AS-regular algebra, if $\dim_k L < \infty$, $L_- = 0$, and $U(L)$ can be regarded as an $\mathbb{N}$-graded connected algebra.

Example 6. (1) We set $G := \mathbb{Z}$, $\varepsilon(1,1) := -1$, $L := \langle x \rangle_k$, and $|x| := -1$. Then the universal enveloping algebra of the (G, ε)-color Lie algebra L is isomorphic to $k[x]/(x^2)$.

(2) Let S be a skew polynomial algebra on n-variables, that is,

$$S := k\langle x_1, x_2, \cdots, x_n \rangle / \langle x_i x_j - q_{i,j} x_j x_i \mid 1 \leq i, j \leq n \rangle$$

where $q_{i,j} \in k^\times$ with $q_{i,i} = 1$ and $q_{i,j} q_{j,i} = 1$ for $1 \leq i, j \leq n$. Then there exists the (G, ε)-color Lie algebra L where $G = \mathbb{Z}^n$, $L = \langle y_1, y_2, \cdots, y_n \rangle_k$, and $|y_i| = (\delta_{i,1}, \delta_{i,2}, \cdots, \delta_{i,n})$ for $1 \leq i \leq n$ where $\delta_{a,b}$ is the Kronecker delta. We have $S \cong U(L)$.

3. Main results

In this section, we show our main results in this report. First, we recall the definitions of a point module and a point variety. Throughout this section, assume that A is an $\mathbb{N}$-graded connected finitely presented algebra generated in degree 1. That is, there exist positive integers s, t, generators $x_1, x_2, \cdots, x_s$ with $\deg x_i = 1$ ($1 \leq i \leq s$), and homogeneous relations $f_1, \cdots, f_t$ with $\deg f_j \geq 2$ ($1 \leq j \leq t$), presented as $A \cong k\langle x_1, \cdots, x_s \rangle / \langle f_1, f_2, \cdots, f_t \rangle$.

Definition 7 ([2, 6]). Suppose that A is the $\mathbb{N}$-graded connected finitely presented algebra generated in degree 1 as above. For a (left) graded module $P = \bigoplus_{i \in \mathbb{Z}} P_i$, P is called a *point module* if the following conditions:

(i) $\dim_k P_i = \begin{cases} 1 & \text{for } i \geq 0, \\ 0 & \text{for } i < 0; \end{cases}$

(ii) $P = AP_0$.

Definition 8 ([2]). Suppose that A be the $\mathbb{N}$-graded connected finitely presented algebra generated in degree 1 as above. We set the k-vector space $V := \langle x_1, \cdots, x_s \rangle_k$ generated by the generators of A and two-sided ideal $I = \bigoplus_{i \geq 0} I_i$ of $k\langle x_1, \cdots, x_s \rangle$ generated by the relations $f_1, \cdots, f_t$ of A. We fix an integer $i \geq 2$ and regard the elements f of I_i as multilinear function from $(V^*)^{\times i}$ to k. The zero scheme of I_i is denoted as $(\Gamma_A)_i \subset (\mathbb{P}^s)^{\times i}$, and the restriction of the projection $\pi_{1i} : (\mathbb{P}^s)^{\times i+1} \to (\mathbb{P}^s)^{\times i}$ of the first i elements as $\mathrm{Pr}_{1i}^{i+1} : (\Gamma_A)_{i+1} \to (\Gamma_A)_i$. The *point scheme* Γ_A of A is defined the inverse limit of $(\Gamma_A)_i$:

$$\Gamma_A := \varprojlim (\Gamma_A)_i.$$

When we consider the reduced structure of Γ_A, it is called the *point variety*.

Artin–Tate–Van den Bergh [2] proved that there exists the nice relationship between the point scheme Γ_A and point modules of A.

Theorem 9 ([2]). *Suppose that A be the $\mathbb{N}$-graded connected finitely presented algebra generated in degree 1 as above. Let Γ_A be the point scheme of A. Then there exist a one-to-one correspondence between the points of underlying topological space of Γ_A and the isomorphism classes of point modules of A.*

Belmans–De Laet–Le Bruyn [3] determined the point modules and point variety structures of skew polynomial algebras. So, we have a one question; what is a point module (or a point variety) of the enveloping algebra of a color Lie algebra, which is not a skew polynomial algebra?

The result of Belmans–De Laet–Le Bruyn [3] inspires us to study algebras which have explicit normal elements. We consider a color Lie algebra defined by some properties "far from" skew polynomial algebra. For studying the point modules, we consider the following algebras $\mathcal{E}_u^{(n)}$.

Let u be in $k^\times$. For $G = \mathbb{Z} \times \mathbb{Z}$, the map ε is defined by $\varepsilon : G \times G \to k^\times$, $((\alpha, \beta), (\alpha', \beta')) \mapsto u^{(\alpha\beta' - \alpha'\beta)}$. For a positive integer n, the (G, ε)-color Lie algebra $\mathcal{L}_n$ is defined as

$$\mathcal{L}_n = \langle x, e_1, e_2, \ldots e_n \rangle_k \text{ with } x \in (\mathcal{L}_n)_{(1,0)}, e_i \in (\mathcal{L}_n)_{(i-1,1)} \text{ for } 1 \leq i, j \leq n.$$

For any $1 \leq i, j \leq n$, $\langle x, e_i \rangle = \begin{cases} e_{i+1} & i < n, \\ 0 & i = n, \end{cases}$ $\langle e_i, e_j \rangle = 0.$ The algebra $\mathcal{E}_u^{(n)}$ is defined as

the enveloping algebra of $\mathcal{L}_n$, that is, $\mathcal{E}_u^{(n)} := U(\mathcal{L}_n)$. When $u = 1$, L_n is called a *simplest filiform Lie algebra* (see [4]). By the definition, $\mathcal{E}_u^{(n)}$ holds some following properties.

Proposition 10. *For $n \geq 2$ and $u \in k^\times$, we denote the above color Lie algebra as $\mathcal{E}_u^{(n)}$. The following statements hold:*

(1) $\mathcal{E}_u^{(n)}$ *is an $(n + 1)$-dimensional AS-regular algebra generated in degree 1.*
(2) e_n *is a normal element of $\mathcal{E}_u^{(n)}$, that is, $e_n \mathcal{E}_u^{(n)} = \mathcal{E}_u^{(n)} e_n$.*
(3) *There is a graded algebra isomorphism $\mathcal{E}_u^{(n)}/\langle e_n \rangle \cong \mathcal{E}_u^{(n-1)}$.*
(4) *Hilbert series of $\mathcal{E}_u^{(n)}$ forms $\dfrac{1}{(1-t)} \displaystyle\prod_{i=1}^{n} \dfrac{1}{(1-t^i)}$.*

Remark that, for an $\mathbb{N}$-graded connected finitely presented algebra generated in degree 1 and a normal element g in A, the point module P of A is g-torsionfree or $gP = 0$. On the other hand, by Proposition 10, there exists an explicit normal element e_n of $\mathcal{E}_u^{(n)}$ for $n \geq 2$. Then a point module P of $\mathcal{E}_u^{(n)}$ satisfies the condition whether e_n-torsionfree or $e_n P = 0$.

Theorem 11. *For each $n \geq 2$, $\mathcal{E}_u^{(n)}$ has no e_n-torsionfree point module.*

By Theorem 11, any point module P of $\mathcal{E}_u^{(n)}$ satisfies $e_n P = 0$ for $n \geq 2$. Hence, we have the following corollary by inductively.

Corollary 12. *Let u be in $k^\times$. For each $n \geq 2$, there exists the one-to-one correspondence between the isomorphism classes of point modules of $\mathcal{E}_u^{(n)}$ and those of $\mathcal{E}_u^{(1)}$:*

$$\{P \mid P \text{ is a point module of } \mathcal{E}_u^{(n)}\}/\cong \xleftrightarrow{1:1} \{P \mid P \text{ is a point module of } \mathcal{E}_u^{(1)}\}/\cong.$$

So, the point variety structure of $\mathcal{E}_u^{(n)}$ is same as that of $\mathcal{E}_u^{(1)} \cong k\langle x, y \rangle/\langle xy - uyx \rangle$.

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