τ-TILTING FINITENESS OF GROUP ALGEBRAS AND p-HYPERFOCAL SUBGROUPS

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ABSTRACT. For an algebraically closed field k of positive characteristic p and a finite group G, we give a sufficient condition for a group algebra kG to be τ-tilting finite in terms of a p-hyperfocal subgroup of G. Moreover, we show that this condition is also necessary in some cases.

1. Introduction

This report is based on [10]. Throughout this report, k denotes an algebraically closed field and algebras mean finite dimensional k-algebras. Modules are assumed to be left and finitely generated.

τ-Tilting finiteness of algebras, introduced by Demonet–Iyama–Jasso in [7], has been actively studied in recent years since it relates to many properties concerning certain finiteness: brick finiteness, functorially finiteness of all the torsion classes, completeness of g-fans, and silting-discreteness (see [7] for the first two, [5, 7] for the third, and [4] for the last). Especially in the study of symmetric algebras, τ-tilting finiteness is considered to be important because it is conjectured that over symmetric algebras τ-tilting finiteness is equivalent to tilting-discreteness, i.e., the property that any tilting complexes can be obtained from the given tilting complex by iterative irreducible mutations (see [2, 3]). Indeed, this conjecture was verified for symmetric algebras of polynomial growth [12] and Brauer graph algebras [1]. For these reasons, it is significant to consider τ-tilting finiteness of algebras, particularly symmetric algebras.

In this report, we shall consider τ-tilting finiteness of group algebras. Assume that k has a positive characteristic p and let G be a finite group. Given the classical result that the representation type of a group algebra kG is determined by a p-Sylow subgroup of G, it is natural to ask what structure of G controls τ-tilting finiteness of kG. As a positive answer to this question, we will give a sufficient condition for kG to be τ-tilting finite in terms of a so-called p-hyperfocal subgroup of G. Furthermore, we will show that this condition is also necessary in the case G is a semidirect product $P \rtimes H$ of an abelian p-group P and an abelian p'-group H, where a p'-group means a finite group of order not divisible by p.

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2. Main results

2.1. **τ-Tilting finite algebras.** First, we recall the definition of τ-tilting finite algebras. Let Λ be an algebra. We say that Λ is *τ-tilting finite* if one of the following equivalent conditions is satisfied:

- There exist only finitely many isoclasses of basic support τ-tilting modules over Λ.
- There exist only finitely many isoclasses of bricks over Λ.
- Every torsion class in the module category over Λ is functorially finite.

See [7] for more details. For simplicity, we shall only define bricks. We say that a Λ-module M is a *brick* if $\mathrm{End}_\Lambda(M)$ is isomorphic to k. It can be easily shown that every surjective algebra homomorphism reflects τ-tilting infiniteness. Hence, it is the basic method of proving τ-tilting infiniteness to construct a quotient algebra already known to be τ-tilting infinite, such as a path algebra of an extended Dynkin quiver.

2.2. **A sufficient condition for τ-tilting finiteness of kG.** In the rest of this section, we assume that k has a positive characteristic p and let G be a finite group. We shall give a sufficient condition for kG to be τ-tilting finite in terms of a p-hyperfocal subgroup of G.

Definition 1. A *p-hyperfocal subgroup* of G is the intersection of a p-Sylow subgroup of G and $O^p(G)$, where $O^p(G)$ denotes the smallest normal subgroup of G such that its quotient is a p-group.

Proposition 2 ([10, Proposition 2.15])**.** *Let R be a p-hyperfocal subgroup of G. Then kG is τ-tilting finite if one of the following holds:*

(1) *R is cyclic.*
(2) *$p = 2$ and R is isomorphic to a dihedral, semidihedral, or generalized quaternion group.*

Note that Proposition 2 is a direct consequence of [11, Theorem 3.10] and [8, Theorem 16]. We expect that the converse of Proposition 2 also holds.

2.3. **The main theorem.** In this subsection, we consider the case where G is a semidirect product $P \rtimes H$ of an abelian p-group P and an abelian p'-group H. We shall give a necessary and sufficient condition for τ-tilting finiteness of kG in terms of the (unique) p-hyperfocal subgroup R of G, which verifies that the converse of Proposition 2 holds in this case.

Theorem 3 ([10, Theorem 3.10])**.** *Let P be an abelian p-group, H be an abelian p'-group acting on P, and $G := P \rtimes H$. Denote by R the p-hyperfocal subgroup of G. Then a group algebra kG is τ-tilting finite if and only if one of the following holds:*

(1) *$p = 2$ and R is trivial or isomorphic to $C_2 \times C_2$.*
(2) *p is odd and R is cyclic.*

Remark 4. In the setting of Theorem 3, the converse of Proposition 2 holds since $C_2 \times C_2$ is exactly the dihedral group of order 4. We should also remark that the p-hyperfocal subgroup R of G cannot be nontrivial and cyclic when $p = 2$.

3. Sketch of the proof of Theorem 3

In this section, we keep the notation and the setting in subsection 2.3. We only need to show the “only if” part of Theorem 3 thanks to Proposition 2. In the setting of Theorem 3, the Gabriel quiver and relations of kG are completely known. We will show τ-tilting infiniteness of kG by constructing a surjective algebra homomorphism from kG to a path algebra of an extended Dynkin quiver of type $\widetilde{A}$.

3.1. **The Gabriel quiver and relations of kG.** By [9, Theorem 2.2 in Chapter 5], we can decompose P into H-invariant homocyclic summands, that is, we can assume that $P = \prod_{i\geq 1}(C_{p^i})^{t_i}$ for some $t_i \geq 0$ and $h(C_{p^i})^{t_i}h^{-1} = (C_{p^i})^{t_i}$ for all $h \in H$ and $i \geq 1$. Then let $P_i := (C_{p^i})^{t_i} \leq P$ and we regard $M_i := J_{kP_i}/J_{kP_i}^2$ as a t_i-dimensional kH-module by conjugation. We denote by $\operatorname{Irr} H$ the set of ordinary irreducible characters of H and by S_χ the simple kH-module corresponding to $\chi \in \operatorname{Irr} H$.

Proposition 5 (See [6] or [10, Proposition 3.3])**.** *Let $M_i \cong \oplus_{j=1}^{t_i} S_{\chi_{ij}}$ as a kH-module for some $\chi_{ij} \in \operatorname{Irr} H$. Then kG is isomorphic to kQ/I, where a quiver Q and an admissible ideal I are given by the following:*

- *The vertex set of Q is $\operatorname{Irr} H$.*
- *The arrow set of Q is $\{\alpha_{ij\lambda} : \lambda \to \chi_{ij} \otimes \lambda\}_{i\geq 1,\, 1\leq j\leq t_i,\, \lambda\in \operatorname{Irr} H}$.*
- *$I = \langle \alpha_{ij}\alpha_{i'j'} - \alpha_{i'j'}\alpha_{ij},\ \alpha_{ij}^{p^i} \mid i, i' \geq 1,\ 1 \leq j \leq t_i,\ 1 \leq j' \leq t_{i'}\rangle$, where the relations mean that the following equations hold for all $\lambda \in \operatorname{Irr} H$:*

$$(\lambda \xrightarrow{\alpha_{i'j'\lambda}} \chi_{i'j'} \otimes \lambda \xrightarrow{\alpha_{ij,\chi_{i'j'}\otimes\lambda}} \chi_{ij} \otimes \chi_{i'j'} \otimes \lambda) = (\lambda \xrightarrow{\alpha_{ij\lambda}} \chi_{ij} \otimes \lambda \xrightarrow{\alpha_{i'j',\chi_{ij}\otimes\lambda}} \chi_{i'j'} \otimes \chi_{ij} \otimes \lambda),$$

$$(\lambda \xrightarrow{\alpha_{ij\lambda}} \chi_{ij} \otimes \lambda \xrightarrow{\alpha_{ij,\chi_{ij}\otimes\lambda}} \chi_{ij}^{\otimes 2} \otimes \lambda \xrightarrow{\alpha_{ij,\chi_{ij}^{\otimes 2}\otimes\lambda}} \cdots \xrightarrow{\alpha_{ij,\chi_{ij}^{\otimes(p^i-1)}\otimes\lambda}} \chi_{ij}^{\otimes p^i} \otimes \lambda) = 0.$$

Example 6. Let $p = 3$, $P := (C_3)^2 \times C_9$ with generators $a, b \in (C_3)^2$, $c \in C_9$, and $H := C_4$ with a generator d. Recall that H has four irreducible characters χ_i ($i = 0, 1, 2, 3$) sending d to ζ^i, where ζ denotes a primitive fourth root of unity in k. We define $G := P \rtimes H$ via the following action of H on P:

$$dad^{-1} = ab,\ dbd^{-1} = ab^2,\ dcd^{-1} = c^8,$$

which is represented by the following matrix with respect to the generators a, b, c:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Hence the action of $d \in H$ on $M_1 \oplus M_2$ is represented by the following matrix with respect to a k-basis $\{1 - a + J_{kP}^2,\ 1 - b + J_{kP}^2,\ 1 - c + J_{kP}^2\}$:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

which is congruent over k to the matrix

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^3 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}.$$

Therefore, by Proposition 5, kG is isomorphic to kQ/I, where

$$Q := \begin{tikzcd}[row sep=huge, column sep=huge] \chi_0 \arrow[r, shift left, "\alpha_{30}"] \arrow[d, shift left, "\alpha_{10}"] \arrow[dr, shift left, "\alpha_{22}" near end] & \chi_3 \arrow[l, shift left, "\alpha_{13}"] \arrow[d, shift left, "\alpha_{33}"] \arrow[dl, shift left, "\alpha_{21}" near end] \\ \chi_1 \arrow[u, shift left, "\alpha_{31}"] \arrow[r, shift left, "\alpha_{11}"] \arrow[ur, shift left, "\alpha_{23}" near end] & \chi_2 \arrow[l, shift left, "\alpha_{32}"] \arrow[u, shift left, "\alpha_{12}"] \arrow[ul, shift left, "\alpha_{20}" near end] \end{tikzcd},$$

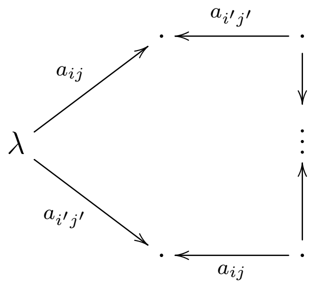
$$I := \langle \alpha_{j,i+l}\alpha_{il} - \alpha_{i,j+l}\alpha_{jl},\ \alpha_{1l}^3,\ \alpha_{3l}^3,\ \alpha_{2l}^9 \mid i,j,l \in \mathbb{Z}/4\mathbb{Z} \rangle.$$

3.2. **Sketch of the proof.** Let R be the p-hyperfocal subgroup of G. Then the crucial fact is that as far as we consider τ-tilting finiteness of kG, we can assume $P = R$ (see [10, Corollary 3.6 and Lemma 3.9]). Moreover, the Gabriel quiver of kG has no loops in the case $P = R$ (see [10, Lemma 3.7]).

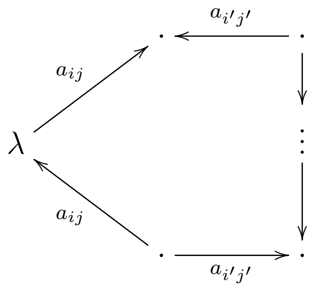
If p is odd and $P = R$ has rank ≥ 2, then we can take distinct pairs $(i,j) \neq (i',j')$ $(i, i' \geq 1,\ 1 \leq j \leq t_i,\ 1 \leq j' \leq t_{i'})$ and consider the following sequence of arrows in the Gabriel quiver Q of kG starting from any $\lambda \in \operatorname{Irr} H$:

$$\lambda \xrightarrow{\alpha_{ij}} \chi_{ij} \otimes \lambda \xleftarrow{\alpha_{i'j'}} \chi_{i'j'}^* \otimes \chi_{ij} \otimes \lambda \xrightarrow{\alpha_{ij}} \cdots,$$

where χ^* means the dual character of $\chi \in \operatorname{Irr} H$. Since Q has no loops, we obtain a *zigzag cycle* γ of length ≥ 2 with distinct vertices as shown in the following figure.



zigzag cycle of even length



zigzag cycle of odd length

Then by annihilating all vertices and arrows outside γ, we can obtain from kG a quotient path algebra of an extended Dynkin quiver of type $\widetilde{A}$ since a_{ij}^2 does not vanish after taking the quotient (see [10, Proposition 2.18] for more details).

Example 7. Let $p = 5$, $P := C_5 \times C_5$ with generators a, b, and $H := C_3$ with a generator c. We define $G := P \rtimes H$ via the following action of H on P:

$$cac^{-1} = b,\ cbc^{-1} = a^{-1}b^{-1}.$$

Then by Proposition 5, kG is isomorphic to kQ/I, where

$$Q := \begin{tikzcd} & \chi_0 \arrow[dl, shift left, "\alpha"] \arrow[dr, shift left, "\beta"] & \\ \chi_1 \arrow[ur, shift left, "\beta"] \arrow[rr, shift left, "\alpha"] & & \chi_2 \arrow[ul, shift left, "\alpha"] \arrow[ll, shift left, "\beta"] \end{tikzcd},$$

$$I := \langle \alpha\beta - \beta\alpha,\, \alpha^5,\, \beta^5 \rangle.$$

We can obtain a quotient path algebra of the following extended Dynkin quiver of type $\widetilde{A_2}$:

$$\begin{tikzcd} & \chi_0 \arrow[dl, "\alpha"'] & \\ \chi_1 & & \chi_2 \arrow[ul, "\alpha"'] \arrow[ll, "\beta"] \end{tikzcd}.$$

If $p = 2$, then the above argument does not work because a_{ij}^2 can be zero. In the proof of [10, Theorem 3.10], we construct a zigzag cycle of even length by considering the action of the Frobenius map on the vertex set of the Gabriel quiver. We omit the detail since it is just a case-by-case proof.

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