DEFINING RELATIONS OF 3-DIMENSIONAL CUBIC AS-REGULAR ALGEBRAS WHOSE POINT SCHEMES ARE REDUCIBLE

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Abstract. In noncommutative algebraic geometry, classification of Artin-Schelter regular algebras is one of the most important projects. Recently, the first and third author extend the notion of geometric algebra for cubic algebras and classify 3-dimensional cubic Artin-Schelter regular algebras whose point schemes are $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or a union of two irreducible conics. In this report, we study the four types when the point scheme E is either (i) quadrangle, (ii) a conic and two lines in a triangle, (iii) a conic and two lines intersecting in one point, or (iv) a double conic. For this four types, we give a complete list of defining relations of 3-dimensional cubic Artin-Schelter regular algebras and classify them up to isomorphism and graded Morita equivalence in terms of their defining relations. By the results of [8] and our main theorems, the classification of 3-dimensional cubic Artin-Schelter regular algebras whose point schemes are reducible will be completed.

1. Preliminaries

1.1. **Artin-Schelter regular algebras.** Throughout this report, let k be an algebraically closed field of characteristic 0. We assume that A is a connected graded algebra finitely generated in degree 1 over k, that is, $A = k\langle x_1, \dots, x_n\rangle/I$ where $\deg x_i = 1\,(i = 1, \dots, n)$ and I is a homogeneous two-sided ideal of $k\langle x_1, \dots, x_n\rangle$. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules. Morphisms in $\operatorname{GrMod} A$ are right A-module homomorphisms preserving degrees. We say that two graded algebras A and B are *graded Morita equivalent* if the categories $\operatorname{GrMod} A$ and $\operatorname{GrMod} B$ are equivalent. We denote by $\mathbb{P}^n_k$ the projective space of dimension n over k. We recall that

$$\operatorname{GKdim} A := \inf \{\alpha \in \mathbb{R} \mid \dim_k (\textstyle\sum_{i=0}^n A_i) \le n^\alpha \text{ for all } n \gg 0\}$$

is called the *Gelfand-Kirillov dimension* of A. In noncommutative algebraic geometry, Artin-Schelter regular algebras are main objects to study.

Definition 1 ([1])**.** A graded algebra A is called a *d-dimensional Artin-Schelter regular* (*simply AS-regular*) *algebra* if A satisfies the following conditions:

(1) $\operatorname{gldim} A = d < \infty$,
(2) $\operatorname{GKdim} A < \infty$,
(3) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \ne d. \end{cases}$

Note that AS-regular algebras are noncommutative analogues of commutative polynomial algebras. In fact, A is a d-dimensional commutative AS-regular algebra if and only if $A \cong k[x_1, \dots, x_d]$.

The detailed version of this paper will be submitted for publication elsewhere.

Example 2. (1) A is a 1-dimensional AS-regular algebra if and only if $A \cong k[x]$ as graded algebras.

(2) A is a 2-dimensional AS-regular algebra if and only if A is isomorphic to one of the following forms as graded algebras:

$$k\langle x,y\rangle/(\lambda xy - yx) \text{ or } k\langle x,y\rangle/(-x^2 + xy - yx)$$

where $\lambda \in k\backslash\{0\}$. Moreover, $k\langle x,y\rangle/(\lambda' xy - yx)$ is isomorphic to $k\langle x,y\rangle/(\lambda xy - yx)$ as graded algebras if and only if $\lambda' = \lambda^{\pm 1}$.

Classification of 3-dimensional AS-regular algebras is one of the most important projects in noncommutative algebraic geometry. Any 3-dimensional AS-regular algebra is isomorphic to a graded algebra of the form

$$k\langle x,y,z\rangle/(f_1,f_2,f_3) \text{ (quadratic case) or } k\langle x,y\rangle/(g_1,g_2) \text{ (cubic case)},$$

where f_i are homogeneous elements of degree 2 and g_j are homogeneous elements of degree 3 (see [1, Theorem 1.5 (i)]). In [5], [6] and [7], we gave a complete list of defining relations of 3-dimensional quadratic AS-regular algebras and classified them up to isomorphism and graded Morita equivalence. The next project is to determine the defining relations of 3-dimensional cubic AS-regular algebras. For the rest paper, we focus on the cubic case.

1.2. **twisted superpotentials and derivation-quotient algebras.** In this subsection, we recall the definitions of twisted superpotentials and derivation-quotient algebras. They play an important role to study AS-regular algebras.

Let V be a 2-dimensional vector space with a basis $\{x_1, x_2\}$. For $w \in V^{\otimes 4}$, there exist unique $w_i \in V^{\otimes 3}$ such that $w = x_1 \otimes w_1 + x_2 \otimes w_2$. Then the *partial derivative* of w with respect to x_i is $\partial_{x_i}(w) := w_i$ where $i = 1, 2$, and the *derivation-quotient algebra* of w is

$$\mathcal{D}(w) := k\langle x,y\rangle/(\partial_{x_1}(w), \partial_{x_2}(w)).$$

We define the k-linear map $\varphi : V^{\otimes 4} \to V^{\otimes 4}$ by $\varphi(v_1 \otimes v_2 \otimes v_3 \otimes v_4) := v_4 \otimes v_1 \otimes v_2 \otimes v_3$.

Definition 3 ([3], [10])**.** Let $w \in V^{\otimes 4}$.

(1) If $\varphi(w) = w$, then w is called a *superpotential*

(2) If there exists $\theta \in \mathrm{GL}_2(k)$ such that $(\theta \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})\varphi(w) = w$, then w is called a *twisted superpotential*.

Remark 4. (1) By [4, Theorem 5] and [3, Theorem 6.8], every 3-dimensional cubic AS-regular algebra A is isomorphic to a derivation-quotient algebra $\mathcal{D}(w)$ of a twisted superpotential w, and by [10, Proposition 2.12], such w is unique up to nonzero scalar multiples.

(2) Mori and Ueyama [11] classified superpotentials w such that $\mathcal{D}(w)$ are 3-dimensional cubic Calabi-Yau AS-regular algebras.

Let V be a 2-dimensional vector space over k with a basis $\{x_1, x_2\}$ and $w \in V^{\otimes 4}$ a twisted superpotential. We write $\mathbf{x} := (x_1, x_2)^t$ and $\mathbf{f} := (\partial_{x_1}(w), \partial_{x_2}(w))^t$ where, for a matrix N, we denote by N^t the transpose of N. There is a unique 2×2 matrix M with entries in $V^{\otimes 2}$ such that $\mathbf{f} = M\mathbf{x}$.

Proposition 5 (cf. [2],[11])**.** *Let V be an two-dimensional vector space with a basis $\{x_1, x_2\}$ and $w \in V^{\otimes 4}$ a twisted superpotential. Then $\mathcal{D}(w)$ is AS-regular if and only if $\partial_{x_1}(w), \partial_{x_2}(w)$*

are linearly independent and the common zero locus in $\mathbb{P}^1_k \times \mathbb{P}^1_k$ of entries of the matrix M in the above is empty.

1.3. **geometric algebras.** In this subsection, we recall the definition of geometric algebra and useful results to classify geometric algebras up to isomorphism and graded Morita equivalent. Let $E \subset \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k$ be a projective variety and $\pi_i : \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k \to \mathbb{P}^{n-1}_k$ be the i-th projection $(i = 1, 2)$. We set the following notation:

$$\mathrm{Aut}^G_k E := \{\sigma \in \mathrm{Aut}_k E \mid (\pi_1 \circ \sigma)(p_1, p_2) = \pi_2(p_1, p_2), \quad \forall (p_1, p_2) \in E\}.$$

We say that a pair (E, σ) is a *geometric pair* if $\sigma \in \mathrm{Aut}^G_k E$. Let $A = k\langle x_1, \dots, x_n\rangle/(R)$ be a cubic algebra where $R \subset k\langle x_1, \dots, x_n\rangle_3$ is a subspace. We define Γ_A as follows:

$$\Gamma_A := \{(p_1, p_2, p_3) \in \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k \times \mathbb{P}^{n-1}_k \mid f(p_1, p_2, p_3) = 0, \quad \forall f \in R\}.$$

Definition 6 ([8, Definition 3.3], cf. [9])**.** Let $A = k\langle x_1, \dots, x_n\rangle/(R)$ be a cubic algebra where $R \subset k\langle x_1, \dots, x_n\rangle_3$ is a subspace. We say that A is a *geometric algebra* if there exists geometric pair (E, σ) such that

(**G1**) $\Gamma_A = \{(p, q, \pi_2\sigma(p, q)) \mid (p, q) \in E\}$,

(**G2**) $R = \{f \in k\langle x_1, \cdots, x_n\rangle_3 \mid f(p, q, \pi_2\sigma(p, q)) = 0, \ \forall (p, q) \in E\}$.

In this case, we write $A = \mathcal{A}(E, \sigma)$ and E is called the *point scheme* of A.

The following theorem tells us that the classification of geometric algebras reduces to the classification of geometric pairs.

Theorem 7 ([8, Theorem 3.5, 3.6])**.** *Let $A = \mathcal{A}(E, \sigma)$ and $A' = \mathcal{A}(E', \sigma')$ be geometric algebras.*

(1) *$A \cong A'$ as graded algebras if and only if there exists an automorphism τ of $\mathbb{P}^{n-1}_k$ such that $(\tau \times \tau)(E) = E'$ and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau\times\tau} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow[\tau\times\tau]{} & E' \end{array}$$

commutes.

(2) $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$ *if and only if there exists a sequence $\{\tau_i\}_{i\in\mathbb{Z}}$ of automorphisms of $\mathbb{P}^{n-1}_k$ such that $(\tau_i \times \tau_{i+1})(E) = E'$ and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\tau_i\times\tau_{i+1}} & E' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ E & \xrightarrow[\tau_{i+1}\times\tau_{i+2}]{} & E' \end{array}$$

commutes for all $i \in \mathbb{Z}$.

2. Main results

In [2], Artin, Tate and Van den Bergh found a nice geometric characterization of 3-dimensional cubic AS-regular algebras.

Theorem 8 ([2]). *Every 3-dimensional cubic AS-regular algebra A is a geometric algebra $A = \mathcal{A}(E, \sigma)$. Moreover, E is $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or a curve of bidegree $(2, 2)$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$.*

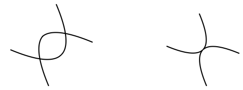
In this report,

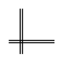
(I) we give a complete list of defining relations of 3-dimensional cubic AS-regular algebras by using Theorem 8 and (G2) condition;

(II) we classify them up to isomorphism in terms of their defining relations by using Theorem 7 (1);

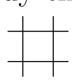
(III) we classify them up to graded Morita equivalence in terms of their defining relations by using Theorem 7 (2).

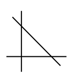
Remark 9. (1) In [8, Theorem 4.9, 4.10], for two cases when $E = \mathbb{P}^1_k \times \mathbb{P}^1_k$ or a union of two irreducible curves of bidegree $(1, 1)$ in $\mathbb{P}^1_k \times \mathbb{P}^1_k$, the classification was completed.

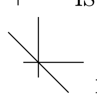


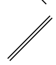
(2) In [2], for the case when $E =$  is two double lines, the classification was completed.

In this report, we study the following four cases:

(1) Type FL: $E =$  is a quadrangle.

(2) Type S′: $E =$  is a conic and two lines in a triangle.

(3) Type T′: $E =$  is a conic and two lines intersecting in one point.

(4) Type WL: $E =$  is a double conic.

The following theorem lists all possible defining relations of algebras in each type up to isomorphism.

Theorem 10. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*

(I) *the defining relations of A, and*

(II) *the conditions to be isomorphic in terms of their defining relations.*

Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X \neq Y$ or $i \neq j$, then Type X_i algebra is not isomorphic to any Type Y_j algebra.

Type	(I) defining relations ($\alpha, \beta \in k$)	(II) condition to be isomorphic
FL_1	$\begin{cases} xy^2 + \alpha y^2 x, \\ x^2 y - \alpha y x^2 \end{cases}$ $(\alpha \neq 0)$	$\alpha' = \alpha, -\alpha^{-1}$
FL_2	$\begin{cases} -\alpha x^3 + yxy, \\ \beta xyx - y^3 \end{cases}$ $(\alpha \neq \beta, \alpha\beta \neq 0)$	$(\alpha', \beta') = (\alpha, \beta)$ *in* $\mathbb{P}^1_k$
S'	$\begin{cases} xy^2 - y^2 x, \\ x^2 y + yx^2 - 2y^3 \end{cases}$	———

T'_1	$\begin{cases} xy^2 - y^2x, \\ x^2y - yx^2 + yxy - xy^2 \end{cases}$	————
T'_2	$\begin{cases} xy^2 - y^2x + 2y^3, \\ x^2y - yx^2 - \alpha xy^2 + \alpha yxy \\ +2y^2x - (\alpha+2)y^3 \qquad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha$
WL_1	$\begin{cases} \alpha^2 xy^2 + y^2x - 2\alpha yxy, \\ yx^2 + \alpha^2 x^2y - 2\alpha xyx \qquad (\alpha \neq 0) \end{cases}$	$\alpha' = \alpha^{\pm 1}$
WL_2	$\begin{cases} xy^2 + y^2x - 2yxy, \\ 4xy^2 + 2y^3 + yx^2 + x^2y \\ \qquad -4yxy - 2xyx \end{cases}$	————

The following theorem lists all possible defining relations of algebras in each type up to graded Morita equivalence.

Theorem 11. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional cubic AS-regular algebra. For each type the following table describes*
(I) *the defining relations of A, and*
(III) *the conditions to be graded Morita equivalent in terms of their defining relations. Moreover, every algebra listed in the following table is AS-regular. In the following table, if $X \neq Y$, then Type X algebra is not graded Morita equivalent to any Type Y algebra.*

Type	(I) defining relations ($\alpha, \beta \in k$)	(III) condition to be graded Morita equivalent
FL	$\begin{cases} -\alpha x^3 + yxy, \\ \beta xyx - y^3 \end{cases} \quad (\alpha \neq \beta, \alpha\beta \neq 0)$	$(\alpha', \beta') = (\alpha, \beta), (\beta, \alpha)$ *in* $\mathbb{P}^1$
S'	$\begin{cases} xy^2 - y^2x, \\ x^2y + yx^2 - 2y^3 \end{cases}$	————
T'	$\begin{cases} xy^2 - y^2x, \\ x^2y - yx^2 + yxy - xy^2 \end{cases}$	————
WL	$\begin{cases} xy^2 + y^2x - 2yxy \\ yx^2 + x^2y - 2xyx \end{cases}$	————

Remark 12. (1) Theorem 10 and Theorem 11 are proved by the following five steps:

Step 1: Find all automorphisms $\sigma \in \mathrm{Aut}_k^G E$ for each E.

Step 2: Find the defining relations of $\mathcal{A}(E, \sigma)$ for each $\sigma \in \mathrm{Aut}_k^G E$ by using (G2) condition in Definition 6.

Step 3: Find a twisted superpotential w such that $\mathcal{D}(w) = \mathcal{A}(E, \sigma)$ and check its AS-regularity by using Proposition 5.

Step 4: Classify them up to isomorphism of graded algebras in terms of their defining relations by using Theorem 7 (1).

Step 5: Classify them up to graded Morita equivalence in terms of their defining relations by using Theorem 7 (2).

(2) By [8, Theorem 4.9, 4.10], Theorem 10 and Theorem 11, we give a complete list of defining relations of 3-dimensional cubic AS-regular algebras whose point schemes are reducible.

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. Math., **66** (1987), 171-216.

[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift, vol. **1**, (1990), 33-85.

[3] R. Bocklandt, T. Schedler and M. Wemyss, *Superpotentials and higher order derivations*, J. Pure Appl. Algebra **214** (2010), 1501–1522.

[4] M. Dubois-Violette, *Multilinear forms and graded algebras*, J. Algebra **317** (2007), 198–225.

[5] A. Itaba and M. Matsuno, *Defining relations of* 3*-dimensional quadratic AS-regular algebras*, Math. J. Okayama Univ. **63** (2021), 61–86.

[6] A. Itaba and M. Matsuno, *AS-regularity of geometric algebras of plane cubic curves*, J. Aust. Math. Soc. **112**, no. 2 (2022), 193–217.

[7] M. Matsuno, *A complete classification of* 3*-dimensional quadratic AS-regular algebras of Type EC*, Canad. Math. Bull. **64** (1) (2021), 123–141.

[8] M. Matsuno and Y. Saito, *Defining relations of* 3*-dimensional cubic AS-regular algebras of Type P, S and T*, submitted, (arXiv:2309.11112).

[9] I. Mori, *Noncommutative projective schemes and point schemes*, Argebras, Rings and Their Representations, World Sci., Hackensack, N.J., (2006), 215-239.

[10] I. Mori and S. P. Smith, *m-Koszul Artin-Schelter regular algebras*, J. Algebra **446** (2016), 373–399.

[11] I. Mori and K. Ueyama, The classification of 3-dimensional noetherian cubic Calabi-Yau algebras, J. Pure Appl. Algebra **223** (2019), no. 5, 1946–1965.

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