EMBEDDINGS INTO MODULES OF FINITE PROJECTIVE DIMENSIONS AND THE n-TORSIONFREENESS OF SYZYGIES

YUYA OTAKE

Abstract. Let R be a commutative noetherian ring. In this article, we find out close relationships between the module M being embedded in a module of projective dimension at most n and the $(n+1)$-torsionfreeness of the nth syzygy of M. As an application, we consider the n-torsionfreeness of syzygies of the residue field k over a local ring R.

Key Words: n-torsionfree module, n-syzygy module, projective dimension, Gorenstein ring.

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1. Introduction

Throughout this article, let R be a commutative noetherian ring. We assume that all modules are finitely generated ones. It is a natural and classical question to ask when a given R-module can be embedded in an R-module of finite projective dimension. Auslander and Buchweitz [2] proved that over a Gorenstein local ring any module admits a *finite projective hull*, which is a dual notion of a *Cohen–Macaulay approximation*.

Theorem 1 (Auslander–Buchweitz). *Let R be a Gorenstein local ring and M an R-module. Then there exists an exact sequence $0 \to M \to Y^M \to X^M \to 0$ of R-modules such that Y^M has finite projective dimension and X^M is maximal Cohen-Macaulay.*

In particular, every module over a Gorenstein local ring can be embedded in a module of finite projective dimension. Conversely, Foxby [5] proved that if R is a Cohen–Macaulay local ring and every R-module can be embedded in an R-module of finite projective dimension, then R is Gorenstein. Takahashi, Yassemi and Yoshino [13] succeeded in removing from Foxby's theorem the assumption of Cohen–Macaulayness of the ring R.

Theorem 2 (Foxby, Takahashi–Yassemi–Yoshino). *Let R be a local ring of depth t. Let k be the residue field of R. Then the following are equivalent.*

(1) *The ring R is Gorenstein.*
(2) *Any R-module can be embedded in an R-module of finite projective dimension.*
(3) *The module $\operatorname{Tr}\Omega^t k$ can be embedded in an R-module of finite projective dimension.*

Here, we denote by $\operatorname{Tr}(-)$ and $\Omega^n(-)$ the (Auslander–Bridger) transpose and n-th syzygy, respectively. In the present article, for a fixed integer n, we consider embedding a given module in a module of projective dimension at most n. Our answer to this question is Theorem 3, which says that the question is closely related to the $(n+1)$-torsionfreeness of nth syzygies. The notion of n-torsionfree modules was introduced by Auslander and

The detailed version [11] of this article has been submitted for publication elsewhere.

Bridger [1] as a generalization of the notion of torsionfree modules over integral domains: An R-module M is called *n-torsionfree* if $\mathrm{Ext}_R^i(\mathrm{Tr}\,M, R) = 0$ for all $1 \le i \le n$. Various studies on the n-torsionfreeness have been done so far; see [1, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13]. As an application of Theorem 3, we can recover Theorems 1 and 2.

Next, let us consider the case where R is local with residue field k, and has depth t. Recently, Dey and Takahashi [3] studied the torsionfreeness of syzygies of k. They especially proved in [3, Theorems 4.1(2) and 4.5(1)] that $\Omega^t k$ is $(t+1)$-torsionfree, and it is a $(t+2)$nd syzygy if and only if the local ring R has type one. Motivated by their results, as another application of our main theorem, we consider the n-torsionfreeness of syzygies of the residue field k.

2. Modules embedded in modules of finite projective dimension

The following theorem is the first main result of this article. The following theorem gives an answer to the question of when a given R-module can be embedded in an R-module of projective dimension at most n, under the assumption that the given module is locally of finite Gorenstein dimension. Let M be an R-module. We denote by $\mathrm{Gdim}_R\,M$ the *Gorenstein dimension* of M; see [1] for details.

Theorem 3. *Let M be an R-module and n a nonnegative integer. Consider the following conditions.*

(1) *The module $\Omega^n M$ is $(n+1)$-torsionfree.*
(2) *There exists an exact sequence $0 \to M \to Y \to X \to 0$ of R-modules such that Y has projective dimension at most n and $\mathrm{Ext}_R^i(X, R) = 0$ for all $1 \le i \le n+1$.*
(3) *The module M can be embedded in an R-module of projective dimension at most n.*

Then the implications (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3) hold. If $\mathrm{Gdim}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for all prime ideals $\mathfrak{p}$ of R with $\mathrm{depth}\, R_\mathfrak{p} < n$, then all the three conditions are equivalent.

Let us consider an application of the above theorem. We can deduce Theorem 2 due to Foxby [5] and Takahashi, Yassemi and Yoshino [13] directly from Theorem 3.

Proof of Theorem 2. Assume that R is Gorenstein. Then for any R-module M the tth syzygy $\Omega^t M$ is maximal Cohen–Macaulay, in particular, $(t+1)$-torsionfree. The implication (1) $\Rightarrow$ (2) follows from Theorem 3. The implication (2) $\Rightarrow$ (3) is clear. Suppose that $\mathrm{Tr}\,\Omega^t k$ is a submodule of an R-module of finite projective dimension. It follows from Theorem 3 that $\Omega^t\,\mathrm{Tr}\,\Omega^t k$ is $(t+1)$-torsionfree. In particular, $\mathrm{Ext}^1(\Omega^t\,\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R) = \mathrm{Ext}^{t+1}(\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R) = 0$. Since $\mathrm{Ext}^1(\Omega^t k, R)$ is a direct summand of $\mathrm{Ext}^1(\Omega^t\,\mathrm{Tr}\,\Omega^t\,\mathrm{Tr}\,\Omega^t k, R)$, we have $\mathrm{Ext}^{t+1}(k, R) = \mathrm{Ext}^1(\Omega^t k, R) = 0$ and the implication (3) $\Rightarrow$ (1) holds. □

Grades of Ext modules are one of the main subjects of the theory of Auslander and Bridger; see [1, Chapters 2 and 4]. Recall that the *grade* of an R-module M is defined to be the infimum of integers i such that $\mathrm{Ext}_R^i(M, R) \ne 0$, and denoted by $\mathrm{grade}_R\,M$. We state the relationship between Theorem 3 and the grade condition given by Auslander and Bridger.

Corollary 4. *Let $n \geq 0$ be an integer and M an R-module. If $\Omega^n M$ is $(n+1)$-torsionfree, then* $\operatorname{grade}_R \operatorname{Ext}_R^i(M,R) \geq i$ *for all integers $1 \leq i \leq n$.*

3. The n-torsionfreeness of syzygies of the residue field of local rings

Let M and N be R-modules. By $M \approx N$ we mean that there are projective modules P and Q such that $M \oplus P \cong N \oplus Q$.

The following corollary is necessary to prove Theorem 7, which is one of the main theorems in this article. For a local ring $(R, \mathfrak{m}, k)$ we denote by $\mathrm{r}(R)$ the *type* of R, that is, $\mathrm{r}(R)$ is the dimension of the vector space $\operatorname{Ext}_R^{\operatorname{depth} R}(k,R)$ over the residue field k of R.

Corollary 5. *Suppose that R is local and with depth t. Let k be the residue field of R. Then the following hold.*

1. [3, Theorem 4.1(2)] *The module $\Omega^t k$ is $(t+1)$-torsionfree.*
2. *There exists an exact sequence $0 \to k \to Y^k \to X^k \to 0$ such that Y^k has projective dimension t and $X^k \approx \operatorname{Tr}\Omega^{t+1}\operatorname{Tr}\Omega^t k$. Moreover, if $t > 0$, then $Y^k \approx \operatorname{Tr}\Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$.*

Proof. We note that the residue field k can be embedded in a module of finite projective dimension. Hence, by Theorem 3, the module $\Omega^t k$ is $(t+1)$-torsionfree, and there exists an exact sequence $0 \to k \to Y^k \to X^k \to 0$ such that Y^k has projective dimension at most t and $X^k \approx \operatorname{Tr}\Omega^{t+1}\operatorname{Tr}\Omega^t k$. We assume that t is positive. Then since $\operatorname{Ext}^i(k,R) = 0 = \operatorname{Ext}^i(X^k,R)$ for all $1 \leq i \leq t-1$, so does Y^k. Also, we have $\operatorname{Ext}^t(Y^k,R) \cong \operatorname{Ext}^t(k,R) \cong k^{\oplus \mathrm{r}(R)}$. By the following lemma, we obtain that $Y^k \approx \operatorname{Tr}\Omega^{t-1}\operatorname{Ext}^t(Y^k,R) \cong \operatorname{Tr}\Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$. ☐

Lemma 6. [9, Theorem 2.7] *Let Y be an R-module and $s > 0$ an integer. If $\operatorname{Ext}_R^i(Y,R) = 0$ for all $1 \leq i < s$ and Y has projective dimension at most s, then $Y \approx \operatorname{Tr}\Omega^{s-1}\operatorname{Ext}_R^s(Y,R)$.*

Theorem 7. *Let $(R, \mathfrak{m}, k)$ be local and with depth t. The following hold.*

1. *The local ring R has type one if and only if the module $\Omega^t k$ is $(t+2)$-torsionfree.*
2. *The local ring R is Gorenstein if and only if the module $\Omega^t k$ is $(t+3)$-torsionfree, if and only if one has $\operatorname{Ext}_R^i(\operatorname{Tr}\Omega^t k, R) = 0$ for some integer $i \geq t+3$*

Proof. We only need to prove the case where $t > 0$. In this case, by Corollary 5, there exists an exact sequence $0 \to \operatorname{Tr} X^k \to \operatorname{Tr} Y^k \to \operatorname{Tr} k \to 0$, and we have $\operatorname{Tr} X^k \approx \Omega^{t+1}\operatorname{Tr}\Omega^t k$ and $\operatorname{Tr} Y^k \approx \Omega^{t-1}(k^{\oplus \mathrm{r}(R)})$. So we obtain the long exact sequence

$$0 \to \operatorname{Ext}^1(\operatorname{Tr} k, R) \to \operatorname{Ext}^1(\operatorname{Tr} Y^k, R) \to \operatorname{Ext}^1(\operatorname{Tr} X^k, R) \to \operatorname{Ext}^2(\operatorname{Tr} k, R) \to \cdots.$$

Since the module $\operatorname{Tr} k$ has projective dimension one, the assertions follow. ☐

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Graduate School of Mathematics, Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, Japan
Email address: `m21012v@math.nagoya-u.ac.jp`