THE MODULI OF 4-DIMENSIONAL SUBALGEBRAS OF THE FULL MATRIX RING OF DEGREE 3

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Abstract. We describe the moduli $\mathrm{Mold}_{3,4}$ of 4-dimensional subalgebras of the full matrix ring of degree 3. We show that $\mathrm{Mold}_{3,4}$ has three irreducible components, whose relative dimensions over $\mathbb{Z}$ are 5, 2, 2, respectively.

Key Words: moduli of subalgebras, full matrix ring.

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1. Introduction

Let k be a field. We say that k-subalgebras A and B of $\mathrm{M}_3(k)$ are equivalent (or $A \sim B$) if $P^{-1}AP = B$ for some $P \in \mathrm{GL}_3(k)$. If k is an algebraically closed field, then there are 26 equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ over k ([4]).

Definition 1 ([2, Definition 1.1], [3, Definition 3.1])**.** We say that a subsheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ is a *mold* of degree n on a scheme X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is a locally free sheaf. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf.

Proposition 2 ([2, Definition and Proposition 1.1], [3, Definition and Proposition 3.5])**.** *The following contravariant functor is representable by a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$:

$$\begin{array}{rcccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \ \mathcal{A} \mid \ \mathcal{A} \textit{ is a rank } d \textit{ mold of degree } n \textit{ on } X \ \right\}. \end{array}$$

We consider the moduli $\mathrm{Mold}_{3,d}$ of rank d molds of degree 3 over $\mathbb{Z}$. For $d = 1, 2, 3, 6, 7, 8, 9$, we have the following theorem:

Theorem 3 ([4])**.** *Let $n = 3$. If $d \leq 3$ or $d \geq 6$, then*

$$\begin{array}{rcl} \mathrm{Mold}_{3,1} & = & \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{3,2} & \cong & \mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,3} & = & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}}, \textit{ where the relative dimensions of} \\ & & \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}}, \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}}, \textit{ and } \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}} \textit{ over } \mathbb{Z} \textit{ are } 6, 4, \textit{ and } 4, \textit{ respectively,} \\ \mathrm{Mold}_{3,6} & \cong & \mathrm{Flag}_3 := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \textit{ for } i > j\}, \\ \mathrm{Mold}_{3,7} & \cong & \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{3,8} & = & \emptyset, \\ \mathrm{Mold}_{3,9} & = & \mathrm{Spec}\mathbb{Z}. \end{array}$$

The detailed version of this paper will be submitted for publication elsewhere.

The cases $d = 4, 5$ remain. In this paper, we describe the moduli $\mathrm{Mold}_{3,4}$ of rank 4 molds of degree 3. We introduce several rank 4 molds of degree 3 on a commutative ring R.

Definition 4 ([4])**.** For a commutative ring R, we define

(1) $(\mathrm{B}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(R) \right\}$,

(2) $\mathrm{N}_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(3) $\mathrm{S}_6(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(4) $\mathrm{S}_7(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(5) $\mathrm{S}_8(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$,

(6) $\mathrm{S}_9(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in R \right\}$.

There are 6 equivalence classes of 4-dimensional subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k: $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, and $\mathrm{S}_9(k)$.

The following theorem is our main result in this paper.

Theorem 5 (Theorem 19, [4])**.** *When $d = 4$, we have an irreducible decomposition*

$$\mathrm{Mold}_{3,4} = \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_8}$$

such that irreducible components are all connected components. The relative dimensions of $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}}$, $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$, and $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ over $\mathbb{Z}$ are 5, 2, and 2, respectively. Moreover, both $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$ and $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ are isomorphic to $\mathbb{P}^2_{\mathbb{Z}}$, and

$$\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} = \mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1} \cup \mathrm{Mold}_{3,4}^{\mathrm{S}_6} \cup \mathrm{Mold}_{3,4}^{\mathrm{S}_9} \cup \mathrm{Mold}_{3,4}^{\mathrm{N}_3}$$

is isomorphic to $\mathrm{Flag}_3 \times_{\mathbb{P}^2_{\mathbb{Z}}} \mathrm{Flag}_3 \times_{\mathbb{P}^2_{\mathbb{Z}}} \mathrm{Flag}_3 = \{(L_1 \subset W_2, L_1 \subset W_1, L_2 \subset W_1) \in \mathrm{Flag}_3 \times \mathrm{Flag}_3 \times \mathrm{Flag}_3\}$. *In particular,* $\mathrm{Mold}_{3,4}$ *is smooth over* $\mathbb{Z}$.

Remark 6 ([1]). We need to say the relation between $\mathrm{Mold}_{d,d}$ and the variety Alg_d of algebras defined by Gabriel in [1]. Let $V = ke_1 \oplus ke_2 \oplus \cdots \oplus ke_d$ be a d-dimensional vector space over a field k. For $\varphi \in \mathrm{Hom}_k(V \otimes_k V, V)$, put $\varphi(e_i \otimes e_j) = \sum_{l=1}^{n} c_{ij}^l e_l$. We say that φ *determines an algebra structure* on V with 1 if the multiplication $e_i \cdot e_j = c_{ij}^l e_l$ defines an algebra V over k with 1. Then we define the variety Alg_d of d-dimensional algebras in the sense of Gabriel by

$$\mathrm{Alg}_d = \left\{ \varphi \in \mathrm{Hom}_k(V \otimes_k V, V) \;\middle|\; \begin{array}{c} \varphi \textit{ determines an} \\ \text{algebra structure} \\ \text{on } V \text{ with } 1 \end{array} \right\} \subset \mathbb{A}_k^{d^3}.$$

Then we can define a morphism $\Psi_d : \mathrm{Alg}_d \to \mathrm{Mold}_{d,d}$ by

$$\varphi \mapsto \{\varphi(v \otimes -) \in \mathrm{End}_k(V) \cong \mathrm{M}_d(k) \mid v \in V\}.$$

If we could prove that $U_d = \{A \subset \mathrm{M}_d(k) \mid A \text{ is a } d\text{-dimensional tame algebra }\}$ is open in $\mathrm{Mold}_{d,d}$ for any d, then $\Psi_d^{-1}(U_d) = \{A \mid\ d\text{-dimensional tame algebra }\}$ would also be open in Alg_d, which gives an affirmative answer to "Tame type is open conjecture". Hence, we believe that $\mathrm{Mold}_{n,d}$ is an important geometric object. This is one of our motivations to investigate $\mathrm{Mold}_{n,d}$.

2. Several Tools

In this section, we introduce several tools for describing $\mathrm{Mold}_{3,4}$. Let A be an associative algebra over a commutative ring R. Assume that A is projective over R. Let $A^e = A \otimes_R A^{op}$ be the enveloping algebra of A. For an A-bimodule M over R, we can regard it as an A^e-module. We define the i-th Hochschild cohomology group $\mathrm{HH}^i(A, M)$ of A with coefficients in M as $\mathrm{Ext}^i_{A^e}(A, M)$.

Let $\mathcal{A}$ be the universal mold on $\mathrm{Mold}_{n,d}$. For $x \in \mathrm{Mold}_{n,d}$, denote by $\mathcal{A}(x) = \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} k(x) \subset \mathrm{M}_n(k(x))$ the mold corresponding to x, where $k(x)$ is the residue field of x. As applications of Hochschild cohomology to the moduli $\mathrm{Mold}_{n,d}$, we have the following tools.

Theorem 7 ([3, Theorem 1.1])**.** *For each point $x \in \mathrm{Mold}_{n,d}$,*

$$\dim_{k(x)} T_{\mathrm{Mold}_{n,d}/\mathbb{Z},x} = \dim_{k(x)} \mathrm{HH}^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim_{k(x)} N(\mathcal{A}(x)),$$

where $N(\mathcal{A}(x)) = \{b \in \mathrm{M}_n(k(x)) \mid [b, a] = ba - ab \in \mathcal{A}(x)$ for any $a \in \mathcal{A}(x)\}$.

Theorem 8 ([3, Theorem 1.2])**.** *Let $x \in \mathrm{Mold}_{n,d}$. If $\mathrm{HH}^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$, then the canonical morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ is smooth at x.*

For a rank d mold A of degree n on a locally noetherian scheme S, we can consider a $\mathrm{PGL}_{n,S}$-orbit $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ in $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$, where $\mathrm{PGL}_{n,S} = \mathrm{PGL}_n \otimes_{\mathbb{Z}} S$. For $x \in S$, put $A(x) = A \otimes_{\mathcal{O}_S} k(x)$, where $k(x)$ is the residue field of x. By using $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x))$, we have:

Theorem 9 ([3, Theorem 1.3])**.** *Assume that $\mathrm{HH}^1(A(x), \mathrm{M}_n(k(x))/A(x)) = 0$ for each $x \in S$. Then the $\mathrm{PGL}_{n,S}$-orbit $\{P^{-1}AP \mid P \in \mathrm{PGL}_{n,S}\}$ is open in $\mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$.*

These tools are useful for investigating $\mathrm{Mold}_{3,4}$. For each rank 4 molds of $\mathrm{M}_3(R)$ over a commutative ring R, we obtained the following table:

Table 1. Hochschild cohomology $\mathrm{HH}^*(A, \mathrm{M}_3(R)/A)$ for R-subalgebras A of $\mathrm{M}_3(R)$ (*cf.* [3, Table 2])

A	$d = \mathrm{rank} A$	$H^* = \mathrm{HH}^*(A, \mathrm{M}_3(R)/A)$	${}^tA$	$N(A)$	$\dim T_{\mathrm{Mold}_{3,d}/\mathbb{Z}, A}$
$(\mathrm{B}_2 \times \mathrm{D}_1)(R) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$	4	$H^i = 0$ for $i \geq 0$	$(\mathrm{B}_2 \times \mathrm{D}_1)(R)$	$(\mathrm{B}_2 \times \mathrm{D}_1)(R)$	5
$\mathrm{N}_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^2 & (i = 0) \\ R^{i+1} & (i \geq 1) \end{cases}$	$\mathrm{N}_3(R)$	$\mathrm{B}_3(R)$	5
$\mathrm{S}_6(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = R$ for $i \geq 0$	$\mathrm{S}_9(R)$	$\mathrm{S}_{13}(R)$	5
$\mathrm{S}_7(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^3 & (i = 0) \\ 0 & (i \geq 1) \end{cases}$	$\mathrm{S}_8(R)$	$\mathrm{P}_{2,1}(R)$	2
$\mathrm{S}_8(R) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = \begin{cases} R^3 & (i = 0) \\ 0 & (i \geq 1) \end{cases}$	$\mathrm{S}_7(R)$	$\mathrm{P}_{1,2}(R)$	2
$\mathrm{S}_9(R) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \right\}$	4	$H^i = R$ for $i \geq 0$	$\mathrm{S}_6(R)$	$\mathrm{S}_{14}(R)$	5

3. Description of $\mathrm{Mold}_{3,4}$

In this section, we describe $\mathrm{Mold}_{3,4}$. Let V be a free module of rank 3 over $\mathbb{Z}$. Fix a canonical basis $\{e_1, e_2, e_3\}$ of V over $\mathbb{Z}$. We define schemes $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and $\mathrm{Flag}(V)$ over $\mathbb{Z}$ as contravariant functors from the category of schemes to the category of sets in the following way:

$$\begin{aligned} \mathbb{P}^*(V)(X) &= \left\{ W \mid W \text{ is a rank 2 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \right\}, \\ \mathbb{P}_*(V)(X) &= \left\{ L \mid L \text{ is a rank 1 subbundle of } \mathcal{O}_X \otimes_{\mathbb{Z}} V \text{ on } X \right\}, \\ \mathrm{Flag}(V)(X) &= \left\{ (L, W) \in (\mathbb{P}_*(V) \times \mathbb{P}^*(V))(X) \mid L \subset W \right\} \end{aligned}$$

for a scheme X.

Remark 10. If we consider the case over a field k, then $\mathbb{P}^*(V)$, $\mathbb{P}_*(V)$, and Flag over k are regarded as

$$\begin{aligned} \mathbb{P}^*(V) &= \{W \subset V \mid W \text{ is a 2-dimensional subspace of } V\}, \\ \mathbb{P}_*(V) &= \{L \subset V \mid L \text{ is a 1-dimensional subspace of } V\}, \\ \mathrm{Flag}(V) &= \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid 0 \subset L \subset W \subset V\}, \end{aligned}$$

respectively.

Let us consider rank 4 molds

$$(\mathrm{B}_2 \times \mathrm{D}_1)(\mathbb{Z}) = \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(\mathbb{Z}) \right\},$$

$$\mathrm{S}_7(\mathbb{Z}) = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z} \right\},$$
$$\mathrm{S}_8(\mathbb{Z}) = \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z} \right\}$$

over $\mathbb{Z}$. Let $A = \mathrm{B}_2 \times \mathrm{D}_1$, S_7, or S_8. Then $\mathrm{HH}^1(A(k), \mathrm{M}_3(k)/A(k)) = 0$ for any field k by Table 1. The image of the morphism $\phi_A : \mathrm{PGL}_3 \to \mathrm{Mold}_{3,4}$ defined by $P \mapsto P^{-1}A(\mathbb{Z})P$ is open by Theorem 9.

Definition 11 ([4])**.** We define open subschemes of $\mathrm{Mold}_{3,4}$ by

$$\begin{aligned} \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} &= \mathrm{Im}\,\phi_{\mathrm{B}_2\times\mathrm{D}_1}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_7} &= \mathrm{Im}\,\phi_{\mathrm{S}_7}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_8} &= \mathrm{Im}\,\phi_{\mathrm{S}_8}. \end{aligned}$$

Remark 12. Let $A = \mathrm{B}_2 \times \mathrm{D}_1$, S_7, or S_8. Then $\mathrm{HH}^2(A(k), \mathrm{M}_3(k)/A(k)) = 0$ for any field k by Table 1. By [3], the canonical morphism $\mathrm{Mold}_{3,4}^{A} \to \mathbb{Z}$ is smooth.

Theorem 13 ([4])**.** *The subschemes* $\mathrm{Mold}_{3,4}^{\mathrm{S}_7}$ *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8}$ *are open and closed in* $\mathrm{Mold}_{3,4}$*. Moreover,* $\mathrm{Mold}_{3,4}^{\mathrm{S}_7} \cong \mathbb{P}^*(V)$ *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8} \cong \mathbb{P}_*(V)$*.*

Outline of proof. For simplicity, here we only consider the case over a field k. For $W \in \mathbb{P}^*(V)$, set

$$A_W = \{f \in \mathrm{End}_k(V) \cong \mathrm{M}_3(k) \mid f(W) \subseteq W \text{ and } f|W \text{ is scalar } \} \subset \mathrm{M}_3(k).$$

Let us define a morphism

$$\begin{array}{cccc} \psi_{\mathrm{S}_7} : & \mathbb{P}^*(V) & \to & \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \\ & W & \mapsto & A_W. \end{array}$$

We can verify that ψ_{S_7} is an isomorphism.

For $L \in \mathbb{P}_*(V)$, set

$$A_L = \{f \in \mathrm{End}_k(V) \cong \mathrm{M}_3(k) \mid f(L) \subseteq L \text{ and } f : V/L \to V/L \text{ is scalar } \}.$$

Let us define a morphism

$$\begin{array}{cccc} \psi_{\mathrm{S}_8} : & \mathbb{P}_*(V) & \to & \mathrm{Mold}_{3,4}^{\mathrm{S}_8} \\ & L & \mapsto & A_L. \end{array}$$

We can verify that ψ_{S_8} is an isomorphism. □

Definition 14. We define

$$\begin{aligned} \mathrm{Q}(V) &= \mathrm{Flag}(V) \times_{\mathbb{P}_*(V)} \mathrm{Flag}(V) \times_{\mathbb{P}^*(V)} \mathrm{Flag}(V) \\ &= \{(L_1, W_2; L_1, W_1; L_2, W_1) \mid \dim_k L_i = 1, \dim_k W_i = 2\} \\ &= \{(L_1, L_2, W_1, W_2) \mid L_1 \subset W_1, L_1 \subset W_2, L_2 \subset W_1\}. \end{aligned}$$

Let us define the projection $\pi : \mathrm{Q}(V) \to \mathrm{Flag}(V)$ by

$$(L_1, L_2, W_1, W_2) \mapsto (L_1, W_1).$$

We can verify that π is a fiber bundle with fiber $\mathbb{P}^1 \times \mathbb{P}^1$.

For $(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V)$, set

$$A_{(L_1,L_2,W_1,W_2)} = \left\{ f \in \mathrm{M}_3(k) \;\middle|\; \begin{array}{c} f(L_i) \subset L_i, f(W_i) \subset W_i \ (i=1,2), \text{ and} \\ L_2 \cong W_1/L_1 \cong V/W_2 \text{ as } k[f]\text{-modules} \end{array} \right\}.$$

Let us define a morphism

$$\begin{array}{cccc} \psi_{\mathrm{B}_2\times\mathrm{D}_1} & : & \mathrm{Q}(V) & \to & \mathrm{Mold}_{3,4} \\ & & (L_1, L_2, W_1, W_2) & \mapsto & A_{(L_1,L_2,W_1,W_2)}. \end{array}$$

Theorem 15 ([4])**.** *The image of* $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ *is open and closed in* $\mathrm{Mold}_{3,4}$*. Moreover,* $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ *gives an isomorphism between* $\mathrm{Q}(V)$ *and the closure* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$ *of* $\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}$.

Outline of proof. It can be verified that $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ is a monomorphism. By a long discussion, we can also prove that $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ is formally étale. Hence, $\psi_{\mathrm{B}_2\times\mathrm{D}_1}$ gives an isomorphism between $\mathrm{Q}(V)$ and an open subscheme of $\mathrm{Mold}_{3,4}$ which coincides with $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$. □

Definition 16 ([4])**.** Let $A = \mathrm{N}_3$, S_6, or S_9. We define

$$\mathrm{Mold}_{3,4}^{A} = \{x \in \mathrm{Mold}_{3,4} \mid \mathcal{A}(x) \otimes_{k(x)} \overline{k(x)} \sim A(\overline{k(x)})\},$$

where $\overline{k(x)}$ is an algebraic closure of $k(x)$.

We can also prove the following theorems.

Theorem 17 ([4])**.** *For the closure* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$ *of* $\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}$*, we obtain*

$$\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} = \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_6} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_9} \coprod \mathrm{Mold}_{3,4}^{\mathrm{N}_3}.$$

Theorem 18 ([4])**.** *By the isomorphism* $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}} \cong \mathrm{Q}(V)$*, we have*

$$\begin{array}{rcl} \mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 \neq L_2, W_1 \neq W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_6} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 = L_2, W_1 \neq W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{S}_9} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 \neq L_2, W_1 = W_2\}, \\ \mathrm{Mold}_{3,4}^{\mathrm{N}_3} & = & \{(L_1, L_2, W_1, W_2) \in \mathrm{Q}(V) \mid L_1 = L_2, W_1 = W_2\}. \end{array}$$

By using Theorem 18, let us describe a deformation of 4-dimensional subalgebras of M_3. We define a 2-dimensional closed subscheme $\mathrm{Q}^{st}(V)$ of $\mathrm{Q}(V) \cong \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2\times\mathrm{D}_1}}$.

For simplicity, let us consider the case over a field k. Set $L_1^{st} = ke_1$ and $W_1^{st} = ke_1 \oplus ke_2$. Put $* = (L_1^{st}, W_1^{st}) \in \mathrm{Flag}(V)$. Then we have the following fiber product:

$$\begin{array}{ccc} \mathrm{Q}^{st}(V) & \to & \mathrm{Q}(V) \\ \downarrow & & \downarrow \\ * & \to & \mathrm{Flag}(V). \end{array}$$

Note that $\mathrm{Q}^{st}(V) \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$.

Let $L_2(s_1) = \left\langle \begin{bmatrix} 1 \\ -s_1 \\ 0 \end{bmatrix} \right\rangle$ and $W_2(s_2) = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ s_2 \end{bmatrix} \right\rangle$. Then

$$\{(s_1, s_2) \in \mathbb{A}^2_k\} \cong (\mathbb{P}^1_k \setminus \{\infty\}) \times (\mathbb{P}^1_k \setminus \{\infty\})$$

gives an affine open subscheme of $\mathrm{Q}^{st}(V)$ by considering $(L_1^{st}, L_2(s_1), W_1^{st}, W_2(s_2))$. We write

$$A(s_1, s_2) = \left\{ \begin{bmatrix} a + s_1 b & b & c \\ 0 & a & d \\ 0 & 0 & a + s_2 d \end{bmatrix} \middle| \, a, b, c, d \in k \right\}$$

for $\psi_{\mathrm{B}_2 \times \mathrm{D}_1}(s_1, s_2) \in \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}}$.

Note that

$$\begin{aligned} A(s_1, s_2) &: \quad \mathrm{B}_2 \times \mathrm{D}_1 \text{ type} \quad && \text{if } s_1 \neq 0, s_2 \neq 0, \\ A(0, s_2) &: \quad \mathrm{S}_6 \text{ type} && \text{if } s_2 \neq 0, \\ A(s_1, 0) &: \quad \mathrm{S}_9 \text{ type} && \text{if } s_1 \neq 0, \\ A(0, 0) &: \quad \mathrm{N}_3 \text{ type.} && \end{aligned}$$

Summarizing the discussions above, we obtain the main theorem.

Theorem 19 ([4]). *We have an irreducible decomposition*

$$\mathrm{Mold}_{3,4} = \overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_7} \coprod \mathrm{Mold}_{3,4}^{\mathrm{S}_8},$$

whose irreducible components are all connected components. Moreover, $\overline{\mathrm{Mold}_{3,4}^{\mathrm{B}_2 \times \mathrm{D}_1}} \cong \mathrm{Q}(V)$, $\mathrm{Mold}_{3,4}^{\mathrm{S}_7} \cong \mathbb{P}^2_{\mathbb{Z}}$, *and* $\mathrm{Mold}_{3,4}^{\mathrm{S}_8} \cong \mathbb{P}^2_{\mathbb{Z}}$ *over* $\mathbb{Z}$.

By considering the PGL_3-orbits in $\mathrm{Mold}_{3,4}$ over a field k, we have:

Corollary 20 ([4]). *Let k be an arbitrary field. Then there exist* 6 *equivalence classes of* 4*-dimensional subalgebras of* $\mathrm{M}_3(k)$ *over k:* $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, *and* $\mathrm{S}_9(k)$.

Remark 21. Let S be a 4-dimensional subalgebra of $\mathrm{M}_3(k)$ over a field k. Let A be one of $(\mathrm{B}_2 \times \mathrm{D}_1)(k)$, $\mathrm{N}_3(k)$, $\mathrm{S}_6(k)$, $\mathrm{S}_7(k)$, $\mathrm{S}_8(k)$, or $\mathrm{S}_9(k)$. If $S \otimes_k K$ is equivalent to $A \otimes_k K$ for an extension field K of k, then S is equivalent to A over k by Corollary 20.

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