CHARACTERIZATION OF 4-DIMENSIONAL NON-THICK IRREDUCIBLE REPRESENTATIONS

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Abstract. We say that a group representation $\rho : G \to \mathrm{GL}(V)$ is *thick* if it has "enough transitivity" of the group action on the set of subspaces of V. If $\dim V \leq 3$, then ρ is thick if and only if it is irreducible. In the case $\dim V \geq 4$, if ρ is thick, then it is irreducible, but the converse is not true. In this paper, we give a characterization of 4-dimensional non-thick irreducible representations of an arbitrary group G.

Key Words: Thick representation, Dense representation, Characterization, 4-dimensional non-thick irreducible representation.

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1. Introduction

In [1], we described the moduli of 4-dimensional non-thick irreducible representations for the free group of rank 2. In this paper, we deal with characterization of 4-dimensional non-thick irreducible representations of an arbitrary group. Throughout this paper, V denotes a finite-dimensional vector space over a field k.

Definition 1 (*cf.* [2] and [3])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. We say that ρ is *m-thick* if for any subspaces V_1, V_2 of V with $\dim_k V_1 = m$ and $\dim_k V_2 = \dim_k V - m$ there exists $g \in G$ such that $(\rho(g)V_1) \cap V_2 = 0$. If ρ is m-thick for any $0 < m < \dim_k V$, then we say that ρ is *thick*.

Roughly speaking, m-thick representations $\rho : G \to \mathrm{GL}(V)$ have enough transitivity of the group action of G on the set of m-dimensional vector subspaces of V.

Definition 2 (*cf.* [2] and [3])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say that ρ is *m-dense* if the exterior representation

$$\wedge^m \rho : G \to \mathrm{GL}(\Lambda^m V)$$

is irreducible. Here, we define

$$(\wedge^m \rho)(g)(v_1 \wedge v_2 \wedge \cdots \wedge v_m) = \rho(g)v_1 \wedge \rho(g)v_2 \wedge \cdots \wedge \rho(g)v_m$$

for $g \in G, v_1, v_2, \ldots, v_m \in V$. If ρ is m-dense for any $0 < m < \dim_k V$, then we say that ρ is *dense*.

In [2], we obtained several results on thick representations and dense representations.

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Proposition 3 ([2, Proposition 2.6])**.** For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$m\text{-thick} \Longleftrightarrow (n-m)\text{-thick}.$$

Proposition 4 ([2, Proposition 2.6])**.** For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$m\text{-dense} \Longleftrightarrow (n-m)\text{-dense}$$

Proposition 5 ([2, Proposition 2.7])**.** Let $0 < m < n$. For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\begin{array}{ccccc} m\text{-dense} & \Longrightarrow & m\text{-thick} & & \\ & & \Downarrow & & \\ 1\text{-dense} & \Longleftrightarrow & 1\text{-thick} & \Longleftrightarrow & \text{irreducible.} \end{array}$$

Corollary 6 ([2, Corollary 2.8])**.** For a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\textit{dense} \Rightarrow \textit{thick} \Rightarrow \text{irreducible}$$

Corollary 7 ([2, Corollary 2.9])**.** Let $n \leq 3$. For an n-dimensional representation $\rho : G \to \mathrm{GL}(V)$,

$$\textit{dense} \Leftrightarrow \textit{thick} \Leftrightarrow \text{irreducile.}$$

Not all 4-dimensional irreducible representations are thick. In this paper, we describe a characterization of 4-dimensional non-thick irreducible representations for an arbitrary group G.

2. Motivation

In this section, we explain our motivation for considering thick reprsentations and dense representations.

Let $\mathrm{Rep}_n(G)$ be the *representation variety* of degree n for G. For simplicity, let us consider $\mathrm{Rep}_n(G)$ over an algebraically closed field Ω. We can regard

$$\mathrm{Rep}_n(G) = \{\rho \mid \rho : G \to \mathrm{GL}_n(\Omega)\}$$

as an affine algebraic scheme (variety) over Ω if G is a finitely generated group. Even if G is not finitely generated, $\mathrm{Rep}_n(G)$ can be defined, although it is not necessarily of finite type over Ω. The group scheme $\mathrm{PGL}_n(\Omega)$ acts on $\mathrm{Rep}_n(G)$ by

$$\rho \mapsto P^{-1}\rho P$$

for $\rho \in \mathrm{Rep}_n(G)$ and $P \in \mathrm{PGL}_n(\Omega)$. We define the $\mathrm{PGL}_n(\Omega)$-invariant open subscheme $\mathrm{Rep}_n(G)_{air}$ of $\mathrm{Rep}_n(G)$ by

$$\mathrm{Rep}_n(G)_{air} = \{\rho \in \mathrm{Rep}_n(G) \mid \rho \text{ is (absolutely) irreducible }\}.$$

We also define the character variety $\mathrm{Ch}_n(G)_{air}$ of n-dimensional irreducible representations for G by

$$\begin{aligned} \mathrm{Ch}_n(G)_{air} &= \mathrm{Rep}_n(G)_{air}/\mathrm{PGL}_n(\Omega) \\ &= \{[\rho] \mid \text{ eq. classes of } n\text{-dim. irreducible representations }\}. \end{aligned}$$

Theorem 8 ([2, Theorem 3.9 and Proposition 3.11])**.** *The representation variety* $\mathrm{Rep}_n(G)_{air}$ *has open subschemes*

$$\begin{aligned}\mathrm{Rep}_n(G)_{thick} &= \{\rho \mid \rho \textit{ is (absolutely) thick }\} \\ \mathrm{Rep}_n(G)_{dense} &= \{\rho \mid \rho \textit{ is (absolutely) dense }\}.\end{aligned}$$

Let us define

$$\begin{aligned}\mathrm{Ch}_n(G)_{thick} &= \mathrm{Rep}_n(G)_{thick}/\mathrm{PGL}_n(\Omega) \\ \mathrm{Ch}_n(G)_{dense} &= \mathrm{Rep}_n(G)_{dense}/\mathrm{PGL}_n(\Omega).\end{aligned}$$

Then we have the following diagram:

$$\begin{array}{ccccc}\mathrm{Rep}_n(G)_{dense} & \subseteq & \mathrm{Rep}_n(G)_{thick} & \subseteq & \mathrm{Rep}_n(G)_{air} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Ch}_n(G)_{dense} & \subseteq & \mathrm{Ch}_n(G)_{thick} & \subseteq & \mathrm{Ch}_n(G)_{air}.\end{array}$$

Let us consider the morphism

$$\begin{array}{cccc}\wedge^m : & \mathrm{Rep}_n(G)_{dense} & \to & \mathrm{Rep}_{\binom{n}{m}}(G)_{air} \\ & \rho & \mapsto & \wedge^m\rho.\end{array}$$

Theorem 9 ([4])**.** *Let* $2 \le m \le n-2$*. Then*

$$\mathrm{Im}\wedge^m \subseteq \mathrm{Rep}_{\binom{n}{m}}(G)_{non-thick} := \mathrm{Rep}_{\binom{n}{m}}(G)_{air} \setminus \mathrm{Rep}_{\binom{n}{m}}(G)_{thick}.$$

Roughly speaking, any exterior representations can not become thick representations.

Any exterior representations are contained in $\mathrm{Rep}_n(G)_{non-thick}$, while $\mathrm{Rep}_n(G)_{thick}$ is open in $\mathrm{Rep}_n(G)_{air}$. Then how can we describe $\mathrm{Rep}_n(G)_{non-thick}$? Not only the exterior representations but also the tensor products of two representations can not become thick representations.

Theorem 10 ([4])**.** Let $\rho : G \to \mathrm{GL}(V)$ and $\tau : G \to \mathrm{GL}(W)$ be finite-dimensional representations of G over k. If $\dim_k V \ge 2$ and $\dim_k W \ge 2$, then $\rho\otimes\tau : G \to \mathrm{GL}(V\otimes_k W)$ is not 2-thick. In particular, $\rho\otimes\tau$ is not thick.

Moreover, we obtain:

Theorem 11 ([4])**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation over k. If $2 \le m \le n-2$, then $\wedge^m\rho : G \to \mathrm{GL}(\Lambda^m V)$ is not 3-thick. In particular, $\wedge^m\rho$ is not thick.

Theorem 12 ([4])**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation over k. If $n \ge 3$ and $m \ge 2$, then the m-th symmetric tensor $S^m(\rho) : G \to \mathrm{GL}(S^m(V))$ is not 3-thick. In particular, $S^m(\rho)$ is not thick.

By these theorems, we can construct many non-thick representations. It is difficult to investigate which representations are thick for finite groups and discrete groups. However, we have already classified (finite-dimensional) thick representations over $\mathbb{C}$ of connected complex simple Lie groups.

Let G be a connected semi-simple Lie group over $\mathbb{C}$. Let $\mathfrak{g}, \mathfrak{h}, \Delta^+ (\subset \mathfrak{h}^*)$ be the Lie algebra of G, a Cartan subalgebra, the set of positive roots, respectively. For a finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$, denote by $W(V)$ the set of weights of V. We can regard $W(V)$ as a partially ordered set by using Δ^+. We say that $\rho : G \to \mathrm{GL}(V)$ is *weight multiplicity-free* if the dimension of the φ-eigenspace is 1 for any $\varphi \in W(V)$.

Theorem 13 ([3])**.** For a connected semi-simple Lie group G over $\mathbb{C}$, a finite-dimensional irreducible representation $\rho : G \to \mathrm{GL}(V)$ over $\mathbb{C}$ is thick if and only if ρ is weight multiplicity-free and $W(V)$ is a totally ordered set.

By the classification of weight multiplicity-free irreducible representations by Howe and Panyushev, we have the following theorem:

Theorem 14 ([3, Theorem 3.6])**.** *The classification of thick representations of connected simple Lie groups:*

(1) *the trivial* 1*-dimensional representation for any groups*
(2) A_n $(n \geq 1)$
  - *the standard representation V for A_n $(n \geq 1)$ with highest weight ω_1*
  - *the dual representation V^* of V for A_n $(n \geq 1)$ with highest weight ω_n*
  - *the symmetric tensor $S^m(V)$ $(m \geq 2)$ of V for A_1 with highest weight $m\omega_1$*

(3) B_n $(n \geq 2)$
  - *the standard representation V for B_n $(n \geq 2)$ with highest weight ω_1*
  - *the spin representation for B_2 with highest weight ω_2*

(4) C_n $(n \geq 3)$
  - *the standard representation V for C_n $(n \geq 3)$ with highest weight ω_1*

(5) G_2
  - *the* 7*-dimensional representation V for G_2 with highest weight ω_1.*

For constructing non-thick representations, the following lemma is useful.

Lemma 15. *Let $\rho : G \to \mathrm{GL}(V)$ be a non-thick representation. For a group homomorphism $\phi : G' \to G$, $\rho \circ \phi : G' \to \mathrm{GL}(V)$ is a non-thick representation.*

By this lemma, we can construct a non-thick representation $\rho \circ \phi : G' \overset{\phi}{\to} G \overset{\rho}{\to} \mathrm{GL}(V)$, where $\rho : G \to \mathrm{GL}(V)$ is a non-thick representation of a connected complex simple Lie group G which is not listed in Theorem 14.

Anyway, what is $\mathrm{Rep}_n(G)_{non-thick}$? In this paper, we would like to investigate a characterization of 4-dimensional non-thick irreducible representations to describe $\mathrm{Rep}_4(G)_{non-thick}$ and $\mathrm{Ch}_4(G)_{non-thick}$.

3. Four-dimensional non-thick irreducible representations

Let us give an example of a non-thick irreducible representation. We denote by S_n the symmetric group of degree n. We regard $\mathrm{GL}(V)^n$ and S_n as subgroups of $\mathrm{GL}(V^{\oplus n})$ by $(A_1, \ldots, A_n) \cdot (v_1, \ldots, v_n) = (A_1 v_1, \ldots, A_n v_n)$ and $\sigma \cdot (v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$ for $(A_1, \ldots, A_n) \in \mathrm{GL}(V)^n, \sigma \in S_n$ and $(v_1, \ldots, v_n) \in V^{\oplus n}$, respectively. Then the semidirect product $\mathrm{GL}(V)^n \rtimes S_n$ is defined as a subgroup of $\mathrm{GL}(V^{\oplus n})$. The inclusion $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ gives a representation of $\mathrm{GL}(V)^n \rtimes S_n$.

Theorem 16 ([4])**.** Let $n \geq 2$ and $\dim_k V \geq 2$. The representation $\rho_{V,n} : \mathrm{GL}(V)^n \rtimes S_n \to \mathrm{GL}(V^{\oplus n})$ is a non-thick irreducible representation. More precisely, $\rho_{V,n}$ is neither n-thick nor $\dim_k V$-thick.

Let us consider 4-dimensional non-thick irreducible representations. Our main theorem is the following:

Theorem 17 ([4])**.** Let $\rho : G \to \mathrm{GL}(V)$ be a 4-dimensional non-thick irreducible representation of a group G over a field k. Then ρ is equivalent to one of the following two cases:

(1) the composition $G \xrightarrow{\phi} \mathrm{GL}(W)^2 \rtimes S_2 \xrightarrow{\rho_{W,2}} \mathrm{GL}(W \oplus W)$ with $\dim_k W = 2$, where $\phi : G \to \mathrm{GL}(W)^2 \rtimes S_2$ is a group homomorphism

(2) a representation $\rho' : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ with $\dim_k V_1 = \dim_k V_2 = 2$ which is equivalent to $\tau_1 \otimes \tau_2$ as projective representations, where $\tau_i : G \to \mathrm{PGL}(V_i)$ is a projective representation for $i = 1, 2$.

Remark 18. In Theorem 17, the case (2) can not be replaced with

(2)' a representation $\tau_1 \otimes \tau_2 : G \to \mathrm{GL}(V_1 \otimes_k V_2)$ with $\dim_k V_1 = \dim_k V_2 = 2$, where $\tau_i : G \to \mathrm{GL}(V_i)$ is a linear representation for $i = 1, 2$.

Indeed, there exists an example of (2) which does not satisfy (2)'. Let us give such an example. Suppose that $\dim_k V_1 = \dim_k V_2 = 2$. Let us consider the fiber product G of ψ and π:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}(V_1 \otimes V_2) \\ \downarrow_{(\rho_1, \rho_2)} & & \downarrow_{\pi} \\ \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) & \xrightarrow{\psi} & \mathrm{PGL}(V_1 \otimes V_2) \\ (g_1, g_2) & \mapsto & g_1 \otimes g_2. \end{array}$$

In other words,

$$G = \{(g_1, g_2, A) \in \mathrm{PGL}(V_1) \times \mathrm{PGL}(V_2) \times \mathrm{GL}(V_1 \otimes V_2) \mid \pi(A) = g_1 \otimes g_2\}.$$

If $\mathrm{ch}\, k \neq 2$, then $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ is a 4-dimensional non-thick irreducible representation such that the projective representation $\rho_i : G \to \mathrm{PGL}(V_i)$ can not be lifted as a linear representation $(i = 1, 2)$.

Remark 19. In [1, Propositions 18 and 21], we claimed that any 4-dimensional non-thick irreducible representation is equivalent to (1) or (2)' in Theorem 17 instead of (2). However, this is not true, as seen in Remark 18. On the other hand, any projective representation can be lifted as a linear representation for free groups.

Before giving an outline of the proof of Theorem 17, we need some definitions.

Definition 20 (perfect pairing)**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation. For $0 < m < n$, the perfect pairing $\Lambda^m V \otimes_k \Lambda^{n-m} V \xrightarrow{\wedge} \Lambda^n V$ is defined by

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_n) \\ \mapsto v_1 \wedge v_2 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge v_{m+2} \wedge \cdots \wedge v_n.$$

The perfect pairing $\wedge$ is G-equivariant. For a subspace $W \subseteq \Lambda^m V$, we put

$$W^{\perp} := \{w' \in \Lambda^{n-m} V \mid w \wedge w' = 0 \text{ for any } w \in W\} \subseteq \Lambda^{n-m} V$$

Note that $(W^{\perp})^{\perp} = W$.

Definition 21 ([2])**.** We say that $W \subseteq \Lambda^m V$ is *realizable* if there exist linearly independent vectors $v_1, v_2, \ldots, v_m \in V$ such that $0 \neq v_1 \wedge v_2 \wedge \cdots \wedge v_m \in W$.

The following proposition gives a characterization of m-thickness and m-denseness.

Proposition 22 ([2])**.** Let $\rho : G \to \mathrm{GL}(V)$ be an n-dimensional representation. Then

(1) ρ is <u>not m-dense</u> if and only if there exist non-trivial G-invariant subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^{\perp} = W_2$.
(2) ρ is <u>not m-thick</u> if and only if there exist non-trivial G-invariant <u>realizable</u> subspaces $W_1 \subseteq \Lambda^m V$ and $W_2 \subseteq \Lambda^{n-m} V$ such that $W_1^{\perp} = W_2$.

The following proposition is trivial, but useful.

Proposition 23. *For a* 4*-dimensional irreducible representation* $\rho : G \to \mathrm{GL}(V)$*, the following statements are equivalent:*

(1) ρ *is thick.*
(2) ρ *is* 2*-thick.*

Proof. Since ρ is irreducible, it is 1-thick by Proposition 5. It is also 3-thick by Proposition 3. Hence, ρ is thick if and only if it is 2-thick by the definition. □

Thereby, any 4-dimensional non-thick irreducible representation $\rho : G \to \mathrm{GL}(V)$ is not 2-thick. By Proposition 22, the 6-dimensional vector space $\Lambda^2 V$ has non-trivial G-invariant realizable subspaces W_1 and W_2 such that $W_1^{\perp} = W_2$. Using [2, Corollary 4.5], we obtain $\dim_k W_1 \geq 2$ and $\dim_k W_2 \geq 2$. Since $\dim_k W_1 + \dim_k W_2 = 6$, there are only two types:

(1) $\Lambda^2 V$ has a 2-dimensional G-invariant realizable subspace.
(2) $\Lambda^2 V$ has a 3-dimensional G-invariant realizable subspace.

Let us discuss Case (1):

Proposition 24 ([4])**.** Assume that $\Lambda^2 V$ has a 2-dimensional G-invariant realizable subspace. Then there exists a basis $e_1, e_2, e_3, e_4 \in V$ such that

$$W := \langle e_1 \wedge e_2, e_3 \wedge e_4 \rangle \subset \Lambda^2 V$$

is G-invariant. Furthermore, with respect to this basis, we can write

$$\rho(g) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

for any $g \in G$, where $A_1, A_2 \in \mathrm{GL}_2(k)$.

In this case, we can define a group homomorphism $\phi : G \to \mathrm{GL}(W)^2 \rtimes S_2$ such that ρ is equivalent to $\rho_{W,2} \circ \phi$.

Let us discuss Case (2):

Proposition 25 ([4])**.** Assume that $\Lambda^2 V$ has a 3-dimensional G-invariant realizable subspace and that $\Lambda^2 V$ has no 2-dimensional G-invariant realizable subspace. Then there exists a basis $e_1, e_2, e_3, e_4 \in V$ such that

$$W := \langle e_1 \wedge e_2, e_3 \wedge e_4, (e_1 + e_3) \wedge (e_2 + e_4) \rangle \subset \Lambda^2 V$$

is G-invariant. Furthermore, with respect to this basis, we can write

$$\rho(g) = \begin{pmatrix} aA' & bA' \\ cA' & dA' \end{pmatrix}$$

for any $g \in G$, where $a, b, c, d \in k$ with $ad - bc \neq 0$ and $A' \in \mathrm{GL}_2(k)$.

In this case, ρ can be decomposed as the tensor product of two projective representations.

Summarizing the two cases, we can prove Theorem 17.

4. Appendix

In this appendix, we deal with $\mathrm{Rep}_4(F_2)_{thick}$ for the free group F_2 of rank 2 over an algebraically closed field Ω.

Example 26. Let F_m be a free group of rank m with $m \geq 2$. The representation variety $\mathrm{Rep}_n(F_m)$ is an irreducible smooth variety of dimension mn^2. The character variety $\mathrm{Ch}_n(F_m)_{air}$ is an irreducible smooth variety of dimension $(m-1)n^2 + 1$.

For $n = 4$ and $m = 2$, $\dim \mathrm{Rep}_4(F_2) = \dim \mathrm{Rep}_4(F_2)_{air} = 32$.
Let $F_2 = \langle \alpha, \beta \rangle$. In Case (1), by conjugating ρ by $\mathrm{PGL}_4(\Omega)$ we obtain

$$\rho(\alpha) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix}$$

$$\rho(\beta) = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}.$$

Roughly speaking,

$$\dim \mathrm{Rep}_4(F_2)_{Case(1)} \leq 4 + 4 + 4 + 4 + \dim \mathrm{PGL}_4(\Omega) = 31 < \dim \mathrm{Rep}_4(F_2)_{air}.$$

In Case (2), note that any projective representation can be lifted as a linear representation for the free group F_2. Set

$$(\mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2))^o = \{(\rho_1, \rho_2) \in \mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2) \mid \rho_1 \otimes \rho_2 \text{ is irreducible } \}.$$

Then we obtain a surjective morphism

$$\begin{array}{rcl} (\mathrm{Rep}_2(F_2) \times \mathrm{Rep}_2(F_2))^o \times \mathrm{PGL}_4(\Omega) & \to & \mathrm{Rep}_4(F_2)_{Case(2)} \\ (\rho_1, \rho_2, P) & \mapsto & P^{-1}(\rho_1 \otimes \rho_2)P. \end{array}$$

Roughly speaking,

$$\dim \mathrm{Rep}_4(F_2)_{Case(2)} \leq 2 \dim \mathrm{Rep}_2(F_2) + \dim \mathrm{PGL}_2(\Omega) = 2 \times 8 + 15 = 31.$$

Since $\dim \mathrm{Rep}_4(F_2)_{air} = 32 > \dim \mathrm{Rep}_4(F_2)_{non-thick}$, we obtain $\mathrm{Rep}_4(F_2)_{thick} \neq \emptyset$.

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