ON τ-TILTING FINITENESS OF GROUP ALGEBRAS

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ABSTRACT. In this note, we explore when a group algebra is τ-tilting finite. One classifies τ-tilting finite group algebras with 2 simple modules and studies τ-tilting finite blocks of group algebras when the characteristic of the base field is 2.

1. Introduction

T-tilting theory has been introduced by Adachi–Iyama–Reiten [1] and is now one of the most important subjects in representation theory of algebras. In the theory, support τ-tilting modules play a central role and are mutatable; that is, we can make a new support τ-tilting module from a given one by replacing a direct summand. Moreover, the set of support τ-tilting modules admits a poset structure whose Hasse quiver coincides with the mutation quiver.

In this note, we discuss τ-tilting theory for group algebras, and attack the problem on "the structure of a group algebra A vs. that of the poset $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$". Here, $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ stands for the set of basic support τ-tilting modules of A.

2. Preliminaries

Let A be a finite dimensional symmetric algebra over an algebraically closed field k. Thanks to Adachi–Iyama–Reiten [1], we know that the theory of support τ-tilting modules and that of 2-term tilting complexes coincide. In this note, we use the latter. We denote by $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)$ the perfect derived category of A.

Let us first recall the definition of tilting complexes.

Definition 1. (1) A perfect complex T is said to be *tilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)}(T, T[i]) = 0$ for any $i \neq 0$ and it generates $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)$.
(2) We say that T is *2-term* provided it concentrates on degree 0 and -1.
(3) The set of basic 2-term tilting complexes of A is denoted by $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$.
(4) We call A *τ-tilting finite* if $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is a finite set.

Remark 2. [1, Definition 0.3 and Theorem 3.2] A *support τ-tilting* module is defined to be the 0th cohomology of some 2-term tilting complex.

For perfect complexes T and U, we write $T \geq U$ if $\mathrm{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)}(T, U[i]) = 0$ for every $i > 0$. This actually gives a partial order on the set of basic tilting complexes [3, Theorem 2.11]. We utilize the 2-term version of this result.

Theorem 3. [1, Lemma 2.5] *The set $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is a partially orderd set by the relation $\geq$.*

The detailed version of this paper will be submitted for publication elsewhere.

The partial order is compatible with tilting mutation, which is provided as follows:

- Let T be a tilting complex with decomposition $T = X \oplus M$. Taking a (minimal) left $\mathsf{add}\, M$-approximation $f : X \to M'$ of X, we get the new complex $\mu_X^-(T) := Y \oplus M$, where Y is the mapping cone of f.

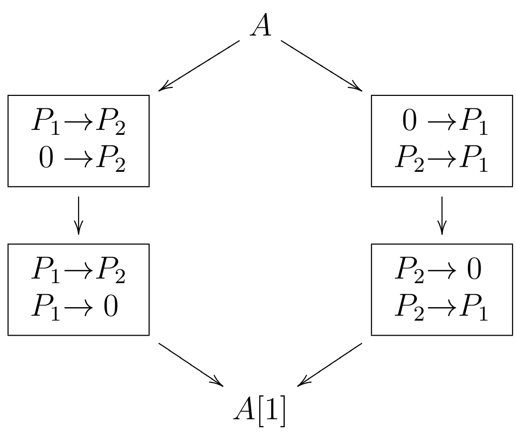
Then, $\mu_X^-(T)$ is also tilting [3, Theorem 2.31], called the *left mutation* of T with respect to X. Dually, we have right mutations $\mu_X^+(T)$. Note that $\mu_P^{\pm}(A)$ are nothing but Okuyama–Rickard complexes [5], which means that tilting mutations can be obtained by repeatedly taking Okuyama–Rickard complexes (via derived equivalences).

Remark 4. The assumption of A being symmetric plays a crucial role to get tilting complexes; that is, the operation above does not necessarily give a tilting complex in general. To take away the disadvantage, we need the notion of *silting* complexes; see [3] for the details. In this note, we will consider only symmetric algebras.

Let us introduce the *(2-)tilting quiver* of A. The vertices of the quiver are basic (2-term) tilting complexes and we draw an arrow $T \to U$ if $U \simeq \mu_X^-(T)$ for an indecomposable summand X of T. Then the following result indicates a relationship between the partial order and tilting mutation.

Theorem 5. [1, Corollary 2.34] *The Hasse quiver of the poset* $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ *coincides with the 2-tilting quiver of* A*.*

Example 6. Let G be the dihedral group of order 6 and $\operatorname{char} k = 3$. The group algebra $A := kG$ is presented by the quiver $1 \underset{x}{\overset{x}{\rightleftarrows}} 2$ with relations $x^3 = 0$. Let P_i denote the indecomposable projective module of A corresponding to the vertex i. Then, we have the 2-tilting quiver of A:



"hexagon"

We will observe that this is independent of the prime p of the order $2p$ and $\operatorname{char} k$ later.

3. Main results

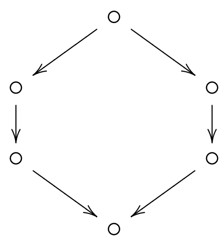
Let G be a finite group and $p := \operatorname{char} k$; then, the group algebra kG and its blocks (i.e., summands of kG as algebras) are symmetric algebras.

The first aim is to classify τ-tilting finite group algebras kG with 2 simple modules.

We say that G is *p-perfect* if it has no normal subgroup with index p. Here is the first main theorem.

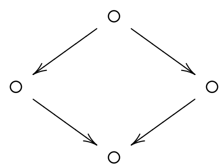
Theorem 7. *Assume that G is p-perfect.*

(1) *The following are equivalent:*
   (a) *the 2-tilting quiver is of the form:*



   (b) *it is the hexagon;*
   (c) $4 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < 8$*;*
   (d) *p is odd and G is isomorphic to the dihedral group of order $2p^n$.*

(2) *The following are equivalent:*
   (a) *the 2-tilting quiver is of the form:*



   (b) *it is the tetragon;*
   (c) $2 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < 6$*;*
   (d) *p is odd and G is isomorphic to the cyclic group of order 2.*

(3) *There is no group G such that kG has 2 simple modules and* $6 < |\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| < \infty$.

Remark 8. The assumption of G being p-perfect plays an important role:

(1) For a p-group P, the group algebras $k[G \times P]$ and kG admit the same poset structure for $\mathsf{s}\tau\text{-}\mathsf{tilt}(-)$ [4, 2].
(2) There exists a non-p-perfect group G such that kG has 2 simple modules and $|\,\mathsf{s}\tau\text{-}\mathsf{tilt}\, kG| = 8$. We will see its example later.

Let A be a block of kG with defect group D; the defect group of a block is a p-subgroup of G controlling the block, for example, D is cyclic (dihedral/semidihedral/quaternion and $p = 2$) iff A is representation-finite (representation-tame).

The second aim is to study the τ-tilting finiteness of A when $p = 2$. *In the rest of this note, assume that $p = 2$.*

The cyclic group of order n is denoted by C_n. Let us start with examples of τ-tilting finite 2-blocks.

Example 9 (See [4])**.** (1) Let G be the symmetric group of degree 4. Then kG is indecomposable, so $A = kG$ (D is isomorphic to the dihedral group of order 8), which has 2 simple modules. Moreover, A admits 8 support τ-tilting modules.

(2) Let G be the alternating group of degree 4. Then we have $A = kG$ with 3 simple modules (D is isomorphic to $C_2 \times C_2$), and there are 32 support τ-tilting modules.

(3) Let G be the alternating group of degree 5. Then $kG = A \oplus \mathrm{Mat}_4(k)$ and A has 3 simple modules (D is the same as (2)). Furthermore, $|\,\mathsf{s\tau\text{-}tilt}\,A| = 32$.

Denote by Λ the algebra A as in Example 9(2)(3). We now state the second main theorem of this note.

Theorem 10. *Assume that A is nonnilpotent ($\doteqdot$ nonlocal).*

(1) *Suppose that D is isomorphic to $C_{2^m} \times C_{2^n}$. Then the following are equivalent:*
  (a) *A is τ-tilting finite;*
  (b) *A is Morita equivalent to Λ;*
  (c) *$m = n = 1$.*

(2) *Suppose that D is isomorphic to $C_2 \times C_2 \times C_2$. Then A is τ-tilting finite if and only if it is Morita equivalen to $kC_2 \times \Lambda$. In the case, it admits the same poset structure for $\mathsf{s\tau\text{-}tilt}(-)$ as Λ.*

At the time of writing, we find no representation-infinite 2-block A such that $\mathsf{s\tau\text{-}tilt}\,A$ has a different poset structure as $\mathsf{s\tau\text{-}tilt}\,\Lambda$. We close this note by putting this question.

Question 11. Let $p = 2$ and A be a representation-infinite nonnilpotent block of a group algebra kG. Then does the following hold true?

- A is τ-tilting finite if and only if $\mathsf{s\tau\text{-}tilt}\,A \simeq \mathsf{s\tau\text{-}tilt}\,\Lambda$ as posets.

(Find a representation-infinite nonnilpotent 2-block A satisfying $\mathsf{s\tau\text{-}tilt}\,A \not\simeq \mathsf{s\tau\text{-}tilt}\,\Lambda$.)

References

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