The Grothendieck monoid of an extriangulated category.

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Overview

Categories	Monoids
· abelian cat · tri cat extriangulated (ET) cat. $\boxed{\mathcal{C}}$	The Grothendieck monoid $\boxed{M(\mathcal{C})}$
1. "ET quotient" $\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{N}$ $\longleftrightarrow$	monoid quotient $M(\mathcal{C}) \twoheadrightarrow M(\mathcal{C})/M(\mathcal{N})$
2. Intermediate subcats $\longleftrightarrow$	monoid localization

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  - Monoids (quotient, localization)
  - ET cat.
- Main result

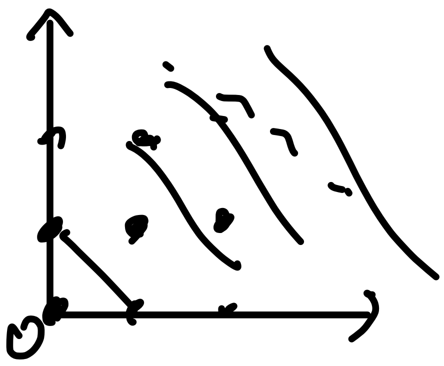
1. ET quotient v.s. monoid quotient
2. Intermediate subcats
   v.s. monoid localization

Monoid

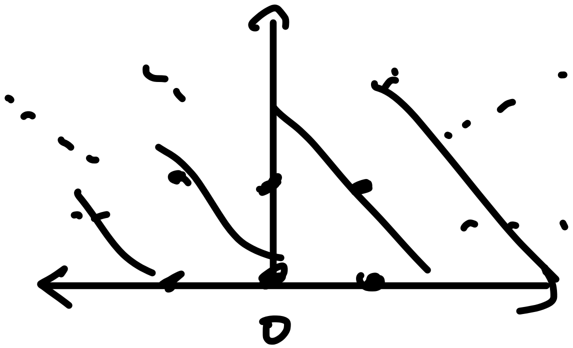
- Monoid $=$ commutative monoid $(M, +, 0)$
- monoid hom $f : M \longrightarrow M'$: preserves $+$ & 0

Example

- $\mathbb{Z}$ • abelian group, • $\mathbb{N} := \mathbb{Z}_{\geq 0}$
- M, N : monoid $\Rightarrow$ $M \oplus N$: monoid



$\mathbb{N} \oplus \mathbb{N}$



$\mathbb{Z} \oplus \mathbb{N}$

Monoid quotient

Def M : a monoid, $N \le M$: a submonoid.

Then a monoid quotient $M \overset{\pi}{\twoheadrightarrow} M/N$ is a monoid hom such that

(1) $\pi(N) = 0$

(2) $\forall$ $M \xrightarrow{\varphi} M'$: monoid hom with $\varphi(N)=0$,

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\pi"'] & M' \\ M/N \arrow[ur, dotted, "\exists! \overline{\varphi}"'] & \end{tikzcd}$$

Prop M/N exists!

($\because$ Define $x \equiv y$ for $x, y \in M$

$:\Longleftrightarrow \exists n_1, n_2 \in N, \ x+n_1 = y+n_2$ equal "modulo N"

Then $M \twoheadrightarrow M/\equiv$ gives a monoid quotient.)

Monoid quotient

Example

- M : abelian grp , $L \leq M$: subgroup

  $\rightsquigarrow \quad M/L = M/L$ (usual quotient grp)

- $\mathbb{N} \oplus \mathbb{N} / \mathbb{N} \oplus 0 \cong 0 \oplus \mathbb{N}$

- $\mathbb{N}/3\mathbb{N} = \{\bar{0}, \bar{1}, \bar{2}\} = \mathbb{Z}/3\mathbb{Z}$

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ \vdots & \vdots & \vdots \end{array}$$

Monoid localization ($\longleftrightarrow$ com. ring. localization)

Def M : a monoid, $S \leq M$: a submonoid.

Then a monoid localization $M \xrightarrow{\rho} M_S$

is a monoid hom s.t. $\quad (\exists a \ \rho(s) + a = 0)$

(1) $\forall s \in S$, $\rho(s) \in M_S$: invertible

(2) $\forall M \xrightarrow{\varphi} M'$: monoid hom s.t. $\forall_{s \in S}$ $\varphi(s)$: inv.

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\rho"'] & M' \\ M_S \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

monoid hom.

Prop M_S exists!

($M \times S \big/ (m_1, s_1) \sim (m_2, s_2)$

$\Leftrightarrow \exists s \in S, \ m_1 + s_2 + s = m_2 + s_1 + s$

"$m - s$"

$\square$)

Monoid localization

Example

- M : abelian group

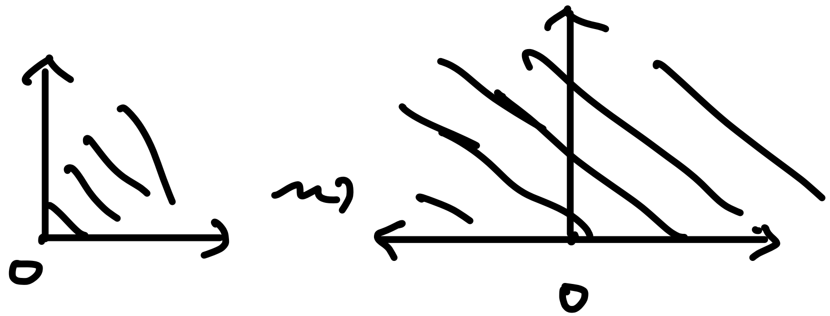
  $\Rightarrow \forall S, \quad M_S = M.$

- $M \xrightarrow{\rho} M_M =: \mathrm{gp}M$: group completion of M

  (universal abelian group from M)

- $\mathrm{gp}\,\mathbb{N} = \mathbb{N}_{\mathbb{N}} = \mathbb{Z}$

- $\mathbb{N} \oplus \mathbb{N}_{\mathbb{N} \oplus 0} = \mathbb{Z} \oplus \mathbb{N}$



Extriangulated category (ET cat).

short exact seq

Similar!
- $\mathcal{C}$: abelian (exact) cat $\rightsquigarrow$ $\{0 \to X \to Y \to Z \to 0\}$
- $\mathcal{C}$: tri cat $\rightsquigarrow$ $\{X \to Y \to Z \to X[1]\}$

triangle

Def [Nakaoka-Palu 2019]

$\mathcal{C}$: ET cat.

" :$\Longleftrightarrow$ " $\mathcal{C}$: additive cat, together with

$\{X \xrightarrow{f} Y \xrightarrow{g} Z\}$: the class of conflations

Example

$\mathcal{C}$	conflations
abelian (exact) cat	short exact seq
tri. cat	triangle
ext-closed sub of tri cat	$C_1 \to C_2 \to C_3 \to$: tri, $C_i \in \mathcal{C}$

Grothendieck monoid of ET cat

Def $\mathcal{C}$: ET cat.

The <u>Grothendieck monoid</u> $M(\mathcal{C})$ is a <u>monoid</u>

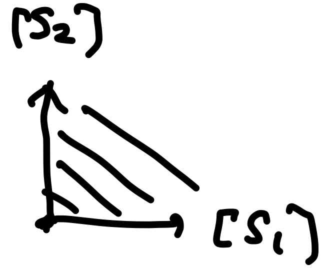
with
- generators : $[X]$ for $X \in \mathcal{C}$
- relations : $\forall\, X \to Y \to Z$: confl,
  $[Y] = [X] + [Z]$

Similarly, we have the <u>Grothendieck group</u> $K_0(\mathcal{C})$.

Prop

$$K_0(\mathcal{C}) \cong \mathrm{gp}\, M(\mathcal{C}) \quad : \text{group compl.}$$

Grothendieck monoid



Example

- Λ : f.d. alg with $|\Lambda| = n$.
  Then $\underline{\dim} : M(\mathrm{mod}\,\Lambda) \xrightarrow{\sim} \mathbb{N}^n$

- $\mathcal{T}$: tri cat. $\rightsquigarrow$ $M(\mathcal{T}) \xrightarrow{\sim} K_0(\mathcal{T})$!

$\forall X \in \mathcal{T}$, $X \to 0 \to X[1] \to$: tri

$\rightsquigarrow$ $[X] + [X[1]] = 0$ in $M(\mathcal{T})$

$\therefore$ $M(\mathcal{T})$ is an abelian grp

$\therefore$ $K_0(\mathcal{T}) = \mathrm{gp}\, M(\mathcal{T}) = M(\mathcal{T})$

Grothendieck monoid : "categorification"

- $\mathcal{A}$: abelian cat.

  $D^b(\mathcal{A})$: the bounded derived cat

  $\rightsquigarrow$ $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$: natural inclusion

$$\begin{array}{ccc}
\overset{\rightsquigarrow}{M(-)} \quad M(\mathcal{A}) & \longrightarrow & M(D^b(\mathcal{A})) = K_0(D^b(\mathcal{A})) \\
 & \searrow^{\text{group compl.}} & \cong \\
 & & K_0(\mathcal{A}) \\
 & & \| \\
 & & {}^{gp}M(\mathcal{A})
\end{array}$$

In this sense, $\mathcal{A} \hookrightarrow D^b(\mathcal{A})$ "categorifies"
group compl. $M(\mathcal{A}) \longrightarrow K_0(\mathcal{A})$.

ET quotient [Nakaoka-Ogawa-Sakai 2021]

Def $\mathcal{C}$: ET cat, $\mathcal{N} \subseteq \mathcal{C}$: ext-closed sub.

$\rightsquigarrow$ an ET quotient $\mathcal{C} \xrightarrow{Q} \mathcal{C}/\mathcal{N}$ is an ET functor Q

which is universal satisfying $Q(\mathcal{N}) = 0$.

Thm [NOS]

There's a sufficient condition $(*)$ s.t.

$\mathcal{C}/\mathcal{N}$ exists. $\lrcorner$

Example $(*)$ is satisfied for :

- $\mathcal{A}$: abelian cat, $\mathcal{X} \subseteq \mathcal{A}$: Serre subcat,

  $\rightsquigarrow$ $\mathcal{A}/\mathcal{X}$: Serre quotient

- $\mathcal{T}$: tri cat, $\mathcal{N} \subseteq \mathcal{T}$: thick subcat,

  $\rightsquigarrow$ $\mathcal{T}/\mathcal{N}$: Verdier quotient

ET quotients v.s. monoid quotients

Assume $\mathcal{E}$: ET cat. $\mathcal{N} \leq \mathcal{E}$: satisfies $(*)$

$$\mathcal{N} \overset{\iota}{\hookrightarrow} \mathcal{E} \xrightarrow{Q} \mathcal{E}/\mathcal{N}$$

$$\overset{M(-)}{\rightsquigarrow} \quad M(\mathcal{N}) \xrightarrow{M(\iota)} M(\mathcal{E}) \xrightarrow{M(Q)} M(\mathcal{E}/\mathcal{N})$$

Thm 1 [E-Saito]

(1)
$$\begin{tikzcd} M(\mathcal{E}) \arrow[r, "M(Q)"] \arrow[dr, two heads] & M(\mathcal{E}/\mathcal{N}) \arrow[d, phantom, "\wr\wr"] \\ & M(\mathcal{E})/\mathrm{Im}\, M(\iota) \end{tikzcd}$$
induces

(2) This can be applied to

(i) Serre quotient of abelian cat

(ii) Verdier quotient of tri cat.

(iii) $\mathcal{E}$: Frobenius exact cat, $\mathcal{N} := \{\text{projs}\}$

$\rightsquigarrow$ $\mathcal{E}/\mathcal{N} = \underline{\mathcal{E}}$: stable cat.

ET quotients v.s. monoid quotients.

Example • Λ : f.d. alg with 2 simples

$$M(-)\left\{\begin{array}{ccccc} \mathrm{Filt}\, S_1 & \hookrightarrow & \mathrm{mod}\,\Lambda & \twoheadrightarrow & \mathrm{mod}\,\Lambda / \mathrm{Filt}\, S_1 \\ \mathbb{N} & \xrightarrow[{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}]{} & \mathbb{N}\oplus\mathbb{N} & \twoheadrightarrow & \dfrac{\mathbb{N}\oplus\mathbb{N}}{\mathbb{N}\oplus 0} \simeq \mathbb{N}. \end{array}\right.$$

Cor If $\mathcal{N} \subseteq \mathcal{E}$ satisfies (*), then

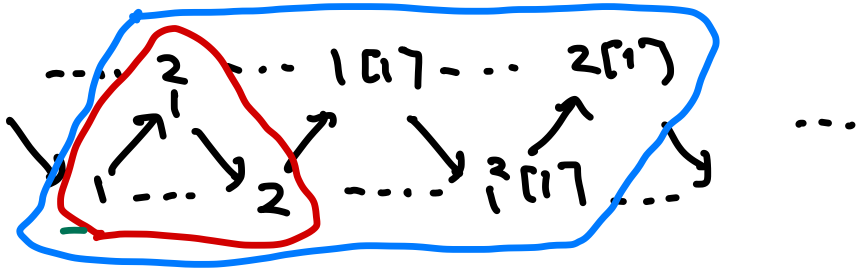
$$K_0(\mathcal{N}) \longrightarrow K_0(\mathcal{E}) \longrightarrow K_0(\mathcal{E}/\mathcal{N}) \longrightarrow 0$$

: exact seq of abelian groups

Intermediate subcat : Example

$Q: 1 \leftarrow 2$, $\mathcal{A} = \operatorname{mod} kQ$

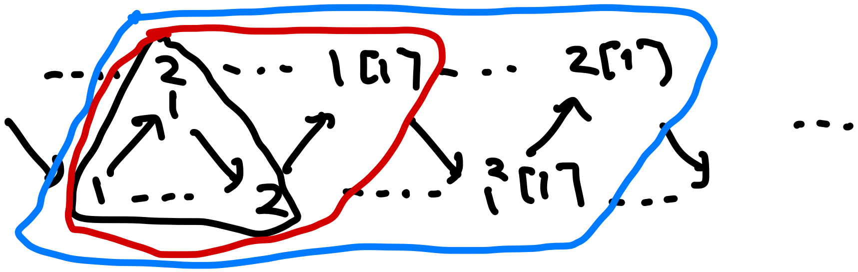
$D^b(\mathcal{A})$: ⋯

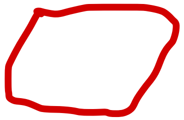
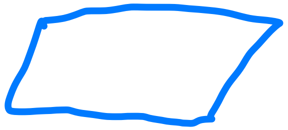


$\mathcal{C}$	$M(\mathcal{C})$	invertible elems.
$= \mathcal{A}$	$\mathbb{N} \oplus \mathbb{N}$	$0 \oplus 0$
$\cap$	$\downarrow$	$\cap$
	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

Intermediate subcat : Example

$Q: 1 \leftarrow 2$, $\mathcal{A} = \operatorname{mod} kQ$

$D^b(\mathcal{A})$: 

$\mathcal{C}$	$M(\mathcal{C})$	invertible elems.
$\triangle = \mathcal{A}$	$\mathbb{N} \oplus \mathbb{N}$	$0 \oplus 0$
$\cap$	$\downarrow$	$\cap$
	$\mathbb{Z} \oplus \mathbb{N}$	$\mathbb{Z} \oplus 0$
$\cap$	$\downarrow$	$\cap$
	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

intermediate subcat

Intermediate subcat

Def $\mathcal{A}$: abelian cat, $D^b(\mathcal{A})$: the bounded derived cat.

$\mathcal{C} \subset D^b(\mathcal{A})$ is an intermediate subcat

$:\Longleftrightarrow$ (1) $\mathcal{A} \subseteq \mathcal{C} \subseteq \mathcal{A}[1] * \mathcal{A} \quad (= \{X \mid H^{\neq 0,-1}(X) = 0\})$

(2) $\mathcal{C} \subset D^b(\mathcal{A})$: closed under direct summands and extensions ($\rightsquigarrow$ ET cat)

Thm $\mathcal{C} \subset D^b(\mathcal{A})$: an intermediate subcat ($:\Longleftrightarrow$ closed under ext & sub)

$\Longleftrightarrow$ $\exists\, \mathcal{F} \subseteq \mathcal{A}$: a torsion-free class

s.t. $\mathcal{C} = \mathcal{F}[1] * \mathcal{A} \quad (= \{X \in D^b(\mathcal{A}) \mid H^{\neq 0,-1}(X) = 0,\ H^{-1}(X) \in \mathcal{F}\})$

In $M(\mathcal{F}[1] * \mathcal{A})$, $\forall F \in \mathcal{F}$, $[F]$ is invertible

($F \to 0 \to F[1]$: confl. $\rightsquigarrow$ $[F] + [F[1]] = 0$.)

Intermediate subcat v.s. monoid localization

$\mathcal{A}$: abelian cat, $\mathcal{F} \subseteq \mathcal{A}$: torsion-free class.

$\mathcal{A} \hookrightarrow \mathcal{F}[1] * \mathcal{A}$: inclusion

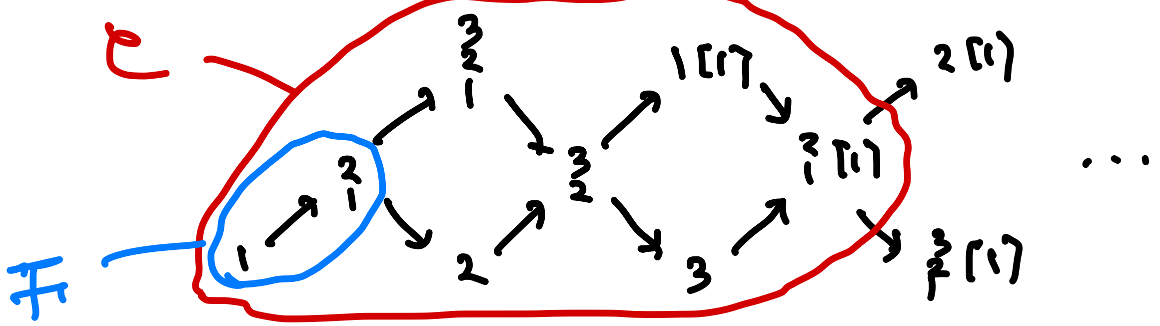
Thm 2 [E-Saito]

$$\begin{tikzcd} M(\mathcal{A}) \arrow[r] \arrow[dr] & M(\mathcal{F}[1]*\mathcal{A}) \arrow[d, phantom, "\wr\|"] \\ & M(\mathcal{A})_{M_{\mathcal{F}}} \end{tikzcd}$$

induces

$(M_{\mathcal{F}} := \{[F] \mid F \in \mathcal{F}\})$

Example $\mathcal{A} = \mathrm{mod}\, k(1 \leftarrow 2 \leftarrow 3)$



$\mathcal{F} = \mathrm{add}\{1, \begin{smallmatrix}2\\1\end{smallmatrix}\}$

$\mathcal{C} = \mathcal{F}[1] * \mathcal{A}$

Then $M(\mathcal{C}) = \mathbb{N} \oplus \mathbb{N} \oplus \mathbb{N}_{\langle (1,0,0), (1,1,0) \rangle} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{N}.$

Intermediate subcat v.s. monoid localization

Rem $\mathcal{A}$: abelian cat.

Then any monoid localization of $M(\mathcal{A})$ comes from intermediate subcat.

$X \subseteq M(\mathcal{A})$: any submonoid $\rightsquigarrow M(\mathcal{A})_X \cong M(\mathcal{A})_{\langle X \rangle_{face}}$,

where $\langle X \rangle_{face}$: the smallest face of $M(\mathcal{A})$ containing X

$\updownarrow$ Saito's talk

$\exists\, \mathcal{S} \subseteq \mathcal{A}$: Serre subcat ($\rightsquigarrow$ torsion-free)

$\therefore\ M(\mathcal{A})_X = M(\mathcal{A})_{\langle X \rangle_{face}} \cong M(\mathcal{S}[1] * \mathcal{A})$ $\square$

Summary

Categories		Monoids
· abelian cat · tri cat extriangulated (ET) cat. $\boxed{\mathcal{C}}$		The Grothendieck monoid $\boxed{M(\mathcal{C})}$
1. ET quotient $\mathcal{C} \twoheadrightarrow \mathcal{C}/\mathcal{N}$	$\longleftrightarrow$	monoid quotient $M(\mathcal{C}) \twoheadrightarrow M(\mathcal{C})/M(\mathcal{N})$
2. Intermediate subcats $\mathcal{A} \hookrightarrow \mathcal{F}[1] * \mathcal{A} \subset D^b(\mathcal{A})$	$\longleftrightarrow$	monoid localization $M(\mathcal{A}) \longrightarrow M(\mathcal{A})_{M_{\mathcal{F}}}$