TATE-HOCHSCHILD COHOMOLOGY AND EVENTUAL PERIODICITY FOR GORENSTEIN ALGEBRAS

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Abstract. This paper studies the eventual periodicity of an algebra by using the Tate-Hochschild cohomology ring. First, we deal with eventually periodic algebras and show that they are not necessarily Gorenstein algebras. Secondly, we characterize the eventual periodicity of a Gorenstein algebra as the existence of an invertible homogeneous element of the Tate-Hochschild cohomology ring of the algebra, which is our main result. Finally, we provide a construction method of eventually periodic Gorenstein algebras.

1. Introduction

The *Tate-Hochschild cohomology* of an algebra was introduced by Wang [19] based on the notion of *Tate cohomology* defined by Buchweitz [7]. It was proved in [19] that the Tate-Hochschild cohomology carries a structure of a graded commutative algebra. There are studies on the ring structure of the Tate-Hochschild cohomology, such as [9, 16, 17, 18]. Recently, Dotsenko, Gélinas and Tamaroff proved in [9, Corollary 6.4] that, for a monomial Gorenstein algebra Λ, the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ is isomorphic to $\widehat{\mathrm{HH}}^{\geq 0}(\Lambda)[\chi^{-1}]$, where $\widehat{\mathrm{HH}}^{\geq 0}(\Lambda)$ stands for the subring consisting of the non-negative part of $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ and χ is an invertible homogeneous element of positive degree. Moreover, the author also showed in [17, Corollary 3.4] that the same isomorphism holds for a periodic algebra. In both cases, the invertible element χ was obtained from the fact that any minimal projective resolution of the given algebra eventually becomes periodic.

In this paper, we first deal with eventually periodic algebras (i.e. algebras Λ with the n-th syzygy $\Omega^n_{\Lambda^e}(\Lambda)$ periodic for some $n \geq 0$). It will be revealed that eventually periodic algebras are not necessarily Gorenstein (see Example 4), although it is known that periodic algebras are all Gorenstein. Secondly, we give a necessity and sufficiency condition for a Gorenstein algebra to be eventually periodic, which is our main result (see Theorem 7). Finally, using tensor product of algebras, we provide one of the constructions of eventually periodic Gorenstein algebras.

This paper is organized as follows. In Section 2, we recall basic facts on Tate cohomology and Gorenstein algebras. In Section 3, we give examples of eventually periodic algebras and prove our main result. In Section 4, we establish a way to construct eventually periodic Gorenstein algebras.

The detailed version of this paper will be submitted for publication elsewhere.

2. Preliminaries

Throughout this paper, let k be an algebraically closed field. We write $\otimes_k$ as $\otimes$. By an algebra Λ, we mean a finite dimensional associative unital k-algebra. All modules are assumed to be finitely generated left modules. For an algebra Λ, we denote by Λ-mod the category of Λ-modules, by Λ-proj the category of projective Λ-modules, by $\mathrm{gl.dim}\,\Lambda$ the global dimension of Λ and by Λ^{e} the enveloping algebra $\Lambda \otimes \Lambda^{\mathrm{op}}$. Remark that we can identify Λ^{e}-modules with Λ-bimodules. For a Λ-module M, we denote by $\mathrm{inj.dim}_\Lambda M$ (resp. $\mathrm{proj.dim}_\Lambda M$) the injective (resp. projective) dimension of M. By a complex $X_\bullet$, we mean a chain complex

$$X_\bullet \;=\; \cdots \to X_{i+1} \xrightarrow{d^X_{i+1}} X_i \to \cdots .$$

For a complex $X_\bullet$ and an integer i, we denote by $\Omega_i(X_\bullet)$ the cokernel $\mathrm{Cok} d^X_{i+1}$ of the differential d^X_{i+1} and by $X_\bullet[i]$ the complex given by $(X_\bullet[i])_j = X_{j-i}$ and $d^{X[i]} = (-1)^i d^X$.

2.1. **Tate cohomology rings.** In this subsection, we recall some facts on Tate cohomology rings and Tate-Hochschild cohomology rings. Let Λ be an algebra. Recall that the *singularity category* $\mathcal{D}_{\mathrm{sg}}(\Lambda)$ of Λ is defined to be the Verdier quotient of the bounded derived category $\mathcal{D}^{\mathrm{b}}(\Lambda\text{-mod})$ of Λ-mod by the bounded homotopy category $\mathcal{K}^{\mathrm{b}}(\Lambda\text{-proj})$ of Λ-proj. Let M and N be Λ-modules and i an integer. Following [7], we define the *i-th Tate cohomology group of M with coefficients in N* by

$$\widehat{\mathrm{Ext}}^i_\Lambda(M,N) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(\Lambda)}(M, N[i]),$$

where M and N are viewed as complexes concentrated in degree 0. We call $\widehat{\mathrm{Ext}}^i_{\Lambda^{\mathrm{e}}}(\Lambda,\Lambda)$ the *i-th Tate-Hochschild cohomology group* of Λ and denote it by $\widehat{\mathrm{HH}}^i(\Lambda)$.

Let $\mathcal{T}$ be a triangulated category with shift functor [1]. For an object X of $\mathcal{T}$, one can endow $\mathrm{End}^\bullet_{\mathcal{T}}(X) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, X[i])$ with a structure of a graded ring. The multiplication is given by the *Yoneda product*

$$\smile : \mathrm{Hom}_{\mathcal{T}}(X, X[i]) \otimes \mathrm{Hom}_{\mathcal{T}}(X, X[j]) \to \mathrm{Hom}_{\mathcal{T}}(X, X[i+j])$$

sending $\alpha\otimes\beta$ to $\alpha[j]\circ\beta$. If $\mathcal{T} = \mathcal{D}_{\mathrm{sg}}(\Lambda)$ and $X = M \in \Lambda$-mod, then we obtain a graded algebra $\widehat{\mathrm{Ext}}^\bullet_\Lambda(M,M) := \mathrm{End}^\bullet_{\mathcal{D}_{\mathrm{sg}}(\Lambda)}(M)$ and call it the *Tate cohomology ring* of M, which is called the *stabilized Yoneda Ext algebra* of M by Buchweitz [7]. It was proved by Wang [19] that the *Tate-Hochschild cohomology ring* $\widehat{\mathrm{HH}}^\bullet(\Lambda) := \widehat{\mathrm{Ext}}^\bullet_{\Lambda^{\mathrm{e}}}(\Lambda,\Lambda)$ of any algebra Λ is a graded commutative algebra.

2.2. **Singularity categories of Gorenstein algebras.** The aim of this subsection is to recall facts on the singularity category of a Gorenstein algebra from [7]. Let Λ be an algebra. Recall that the *stable category* Λ-$\underline{\mathrm{mod}}$ of Λ-modules is the category whose objects are the same as Λ-mod and morphisms are given by

$$\underline{\mathrm{Hom}}_\Lambda(M,N) := \mathrm{Hom}_\Lambda(M,N)/\mathcal{P}(M,N),$$

where $\mathcal{P}(M,N)$ is the space of morphisms factoring through a projective module. We denote by $[f]$ the element of $\underline{\mathrm{Hom}}_\Lambda(M,N)$ represented by a morphism $f : M \to N$. There exists a canonical functor $F : \Lambda\text{-}\underline{\mathrm{mod}} \to \mathcal{D}_{\mathrm{sg}}(\Lambda)$ making the following square commute:

$$\begin{tikzcd} \Lambda\text{-mod} \arrow[r] \arrow[d] & \mathcal{D}^{\mathrm{b}}(\Lambda\text{-mod}) \arrow[d] \\ \Lambda\text{-}\underline{\mathrm{mod}} \arrow[r, "F"] & \mathcal{D}_{\mathrm{sg}}(\Lambda) \end{tikzcd}$$

where the two vertical functors are the canonical ones, and the upper horizontal functor is the one sending a module M to the complex M concentrated in degree 0. Further, the functor F satisfies $F \circ \Omega_\Lambda \cong [-1] \circ F$, where Ω_Λ is the syzygy functor on $\Lambda\text{-}\underline{\mathrm{mod}}$ (i.e. the functor sending a module M to the kernel of a projective cover of M). On the other hand, let $\underline{\mathrm{APC}}(\Lambda)$ be the homotopy category of acyclic complexes of projective Λ-modules. Then taking the cokernel $\Omega_0(X_\bullet) = \mathrm{Cok} d_1^X$ of the differential d_1^X for a complex $X_\bullet$ defines a functor $\Omega_0 : \underline{\mathrm{APC}}(\Lambda) \to \Lambda\text{-}\underline{\mathrm{mod}}$ satisfying $\Omega_0 \circ [-1] \cong \Omega_\Lambda \circ \Omega_0$.

Recall that an algebra Λ is *Gorenstein* if $\mathrm{inj.dim}_\Lambda \Lambda < \infty$ and $\mathrm{inj.dim}_{\Lambda^{\mathrm{e}}} \Lambda < \infty$. Since the two dimensions coincide (see [20, Lemma A]), we call a Gorenstein algebra Λ with $\mathrm{inj.dim}_\Lambda \Lambda = d$ a *d-Gorenstein algebra*. In the rest of this subsection, let Λ denote a Gorenstein algebra. We call a Λ-module M *Cohen-Macaulay* if $\mathrm{Ext}^i_\Lambda(M, \Lambda) = 0$ for all $i > 0$. It is clear that projective Λ-modules are Cohen-Macaulay. We denote by $\mathrm{CM}(\Lambda)$ the category of Cohen-Macaulay Λ-modules. It is well-known that $\mathrm{CM}(\Lambda)$ is a Frobenius exact category whose projective objects are precisely projective Λ-modules, so that the stable category $\underline{\mathrm{CM}}(\Lambda)$ carries a structure of a triangulated category (see [7, 12]). In particular, the syzygy functor Ω_Λ on $\Lambda\text{-}\underline{\mathrm{mod}}$ gives rise to the inverse of the shift functor Σ on $\underline{\mathrm{CM}}(\Lambda)$. We end this subsection with the following result due to Buchweitz.

Theorem 1 ([7, Theorem 4.4.1])**.** *Let Λ be a Gorenstein algebra. Then there exist equivalences of triangulated categories*

$$\underline{\mathrm{APC}}(\Lambda) \xrightarrow{\Omega_0} \underline{\mathrm{CM}}(\Lambda) \xrightarrow{\iota_\Lambda} \mathcal{D}_{\mathrm{sg}}(\Lambda),$$

where the equivalence ι_Λ is given by the restriction of $F : \Lambda\text{-}\underline{\mathrm{mod}} \to \mathcal{D}_{\mathrm{sg}}(\Lambda)$ to $\underline{\mathrm{CM}}(\Lambda)$.

2.3. **Tate cohomology over Gorenstein algebras.** This subsection is devoted to recalling another description of Tate cohomology over a Gorenstein algebra. Throughout, let Λ denote a d-Gorenstein algebra unless otherwise stated. Thanks to Theorem 1, we can associate to any Λ-module M an object $T_\bullet = T_\bullet^M$ in $\underline{\mathrm{APC}}(\Lambda)$, uniquely determined up to isomorphism, satisfying that $\Omega_0(T_\bullet) \cong M$ in $\mathcal{D}_{\mathrm{sg}}(\Lambda)$. Thus the triangle equivalence $\iota_\Lambda : \underline{\mathrm{CM}}(\Lambda) \to \mathcal{D}_{\mathrm{sg}}(\Lambda)$ induces an isomorphism

$$\widehat{\mathrm{Ext}}^i_\Lambda(M, M) \cong \underline{\mathrm{Hom}}_\Lambda(\Omega_0(T_\bullet), \Sigma^i \Omega_0(T_\bullet))$$

for all $i \in \mathbb{Z}$. We identify $\widehat{\mathrm{Ext}}^\bullet_\Lambda(M, M)$ with $\mathrm{End}^\bullet_{\underline{\mathrm{CM}}(\Lambda)}(\Omega_0(T_\bullet))$ via this isomorphism.

Recall that, for an algebra Λ, the *Gorenstein dimension* $\mathrm{G\text{-}dim}_\Lambda M$ of a Λ-module M is defined by the shortest length of a resolution of M by Λ-modules X with $X \cong X^{**}$ and $\mathrm{Ext}^i_\Lambda(X, \Lambda) = 0 = \mathrm{Ext}^i_{\Lambda^{\mathrm{e}}}(X^*, \Lambda)$ for all $i > 0$, where we set $(-)^* := \mathrm{Hom}_\Lambda(-, \Lambda)$ (see [1] for its original definition). The next proposition is easily obtained from the results in [3] applied to the case of Gorenstein algebras: (1), (2) and (3) follow from [3, Theorems 3.1 and 3.2], [3, Lemma 2.4 and Theorem 3.1] and [3, Theorem 5.2], respectively.

Proposition 2. *Let M be a module over a d-Gorenstein algebra Λ. Then the following statements hold.*

(1) *The Gorenstein dimension* $\operatorname{G-dim}_\Lambda M$ *of M satisfies* $\operatorname{G-dim}_\Lambda M \leq d$ *and is equal to the smallest integer $r \geq 0$ for which $\Omega_\Lambda^r(M)$ is Cohen-Macaulay.*

(2) *There exists a diagram $T_\bullet \xrightarrow{\theta} P_\bullet \xrightarrow{\varepsilon} M$ satisfying the following conditions:*
   (i) *$T_\bullet \in \underline{\mathrm{APC}}(\Lambda)$ and $P_\bullet \xrightarrow{\varepsilon} M$ is a projective resolution of M.*
   (ii) *$\theta : T_\bullet \to P_\bullet$ is a chain map with θ_i an isomorphism for any $i \gg 0$.*

(3) $\operatorname{Ext}_\Lambda^i(M, M) \cong \widehat{\operatorname{Ext}}_\Lambda^i(M, M)$ *for all* $i > \operatorname{G-dim}_\Lambda M$.

We call such a diagram as in Proposition 2 (2) a *complete resolution* of M (see [3] for its definition in a general setting). A complete resolution is unique in the sense of [3, Lemma 5.3] (when it exists).

3. Tate-Hochschild cohomology for eventually periodic Gorenstein algebras

In this section, we first define eventually periodic algebras and provide examples of them. We then prove our main result.

3.1. **Eventually periodic algebras.** As mentioned above, let us first define the eventual periodicity of algebras and provide examples of eventually periodic algebras.

Definition 3. Let Λ be an algebra. A Λ-module M is called *periodic* if $\Omega_\Lambda^p(M) \cong M$ in Λ-mod for some $p > 0$. The smallest such p is said to be the *period* of M. We say that $M \in \Lambda$-mod is *eventually periodic* if $\Omega_\Lambda^n(M)$ is periodic for some $n \geq 0$. An algebra Λ is called *periodic* (resp. *eventually periodic*) if $\Lambda \in \Lambda^{\mathrm{e}}$-mod is periodic (resp. eventually periodic).

From the definition, periodic algebras are eventually periodic algebras. Periodic algebras have been studied for a long time (see [10]). We know from [11, Lemma 1.5] that periodic algebras are self-injective algebras (i.e. 0-Gorenstein algebras). On the other hand, it follows from the proof of [9, Corollary 6.4] that monomial Gorenstein algebras are eventually periodic algebras. It also follows from the formula $\operatorname{gl.dim}\Lambda = \operatorname{proj.dim}_{\Lambda^{\mathrm{e}}}\Lambda$ (see [13, Section 1.5]) that algebras of finite global dimension are eventually periodic algebras. As will be seen in Example 4 below, not all eventually periodic algebras are Gorenstein algebras.

Example 4. (1) Let Λ_1 be the algebra given by a quiver with relation

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2 \qquad \alpha\beta\alpha = 0,$$

Then Λ_1 is a monomial algebra that is not Gorenstein. Using Bardzell's result [4], we see that Λ_1 is an eventually periodic algebra having $\Omega^2_{\Lambda_1^{\mathrm{e}}}(\Lambda_1)$ as its first periodic syzygy.

(2) Let Λ_2 be the algebra given by a quiver with relation

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2 \qquad \alpha^2 = 0$$

Then the algebra Λ_2 is monomial 1-Gorenstein and hence eventually periodic. As in (1), one sees that $\Omega^2_{\Lambda_2^{\mathrm{e}}}(\Lambda_2)$ is the first periodic syzygy of Λ_2.

We note that the algebras in [8, Example 4.3] are eventually periodic algebras.

3.2. **Main Result.** This subsection is devoted to showing our main result. We prove it after two propositions below. Before the first one, we prepare some terminology. Recall that we write $\Omega_i(X_\bullet) = \mathrm{Cok} d^X_{i+1}$ for a complex $X_\bullet$ and $i \in \mathbb{Z}$. For a module M over a Gorenstein algebra Λ, its complete resolution $T_\bullet \to P_\bullet \to M$ is called *periodic* if there exists an integer $p > 0$ such that $\Omega_i(T_\bullet) \cong \Omega_{i+p}(T_\bullet)$ in Λ-mod for all $i \in \mathbb{Z}$. We call the least such p the *period* of the complete resolution. We now prove that eventually periodic modules over a Gorenstein algebra have periodic complete resolutions.

Proposition 5. *Let Λ be a Gorenstein algebra and M a Λ-module. If there exists an integer $n \geq 0$ such that $\Omega^n_\Lambda(M)$ is periodic of period p, then M admits a periodic complete resolution of period p. Further, the period of the periodic complete resolution is independent of the choice of periodic syzygies.*

Using the proposition, we are able to characterize eventually periodic modules by means of Tate cohomology rings. Recall that the Yoneda product of the Tate cohomology ring $\widehat{\mathrm{Ext}}^{\bullet}_\Lambda(M, M)$ is denoted by $\smile$.

Proposition 6. *Let Λ be a Gorenstein algebra and M a Λ-module. Then the following are equivalent.*

(1) *M is eventually periodic.*
(2) *The Tate cohomology ring $\widehat{\mathrm{Ext}}^{\bullet}_\Lambda(M, M)$ has an invertible homogeneous element of positive degree.*

Proposition 6 enables us to obtain the main result of this paper.

Theorem 7. *Let Λ be a Gorenstein algebra. Then the following are equivalent.*

(1) *Λ is an eventually periodic algebra.*
(2) *The Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ has an invertible homogeneous element of positive degree.*

In this case, there exists an isomorphism $\widehat{\mathrm{HH}}^{\bullet}(\Lambda) \cong \widehat{\mathrm{HH}}^{\geq 0}(\Lambda)[\chi^{-1}]$ of graded algebras, where the degree of an invertible homogeneous element χ equals the period of the periodic syzygy $\Omega^n_{\Lambda^{\mathrm{e}}}(\Lambda)$ of Λ for some $n \geq 0$.

Proof. We know from [2, Proposition 2.2] that if Λ is a Gorenstein algebra, then so is the enveloping algebra Λ^{e}. Hence the former statement follows from Proposition 6 applied to $\Lambda \in \Lambda^{\mathrm{e}}$-mod. On the other hand, suppose that the Gorenstein algebra Λ satisfies that $\Omega^n_{\Lambda^{\mathrm{e}}}(\Lambda)$ is periodic for some $n \geq 0$. By the proof of Proposition 6, there exists an invertible homogeneous element $\chi \in \widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ whose degree equals the period of the periodic Λ^{e}-module $\Omega^n_{\Lambda^{\mathrm{e}}}(\Lambda)$. Then the graded commutativity of $\widehat{\mathrm{HH}}^{\bullet}(\Lambda)$ yields the desired isomorphism of graded algebras (cf. the proof of [17, Corollary 3.4]). $\square$

We end this subsection with the following three remarks.

Remark 8. From the definition of singularity categories, an algebra Λ has finite projective dimension as a Λ^{e}-module if and only if its Tate-Hochschild cohomology ring is the zero ring (cf. [7, Section 1]). Thus Theorem 7 provide a new result if and only if a given Gorenstein algebra has infinite global dimension.

Remark 9. Applying Theorem 7 to monomial Gorenstein algebras and to periodic algebras, one obtains [9, Corollary 6.4] and [17, Corollary 3.4], respectively.

4. Construction of eventually periodic Gorenstein algebras

In this section, we aim at establishing a way of constructing eventually periodic Gorenstein algebras. Before that, we prepare for two propositions which will be used latter.

Proposition 10. *Any periodic Λ-module M over a d-Gorenstein algebra Λ is Cohen-Macaulay.*

The second implies that, for an eventually periodic Gorenstein algebra Λ, the smallest integer $n \geq 0$ satisfying that $\Omega^n_{\Lambda^{\mathrm{e}}}(\Lambda)$ is periodic has a lower bound.

Proposition 11. *Let Λ be a d-Gorenstein algebra. Assume that there exists an integer $n \geq 0$ such that $\Omega^n_{\Lambda^{\mathrm{e}}}(\Lambda)$ is periodic. Then the least such integer n satisfies $n \geq d$. In particular, an equality holds if and only if there exists a simple Λ-module S such that $\mathrm{Ext}^n_{\Lambda}(S, \Lambda) \neq 0$.*

Now, we recall some facts on projective resolutions for tensor algebras. Let Λ and Γ be algebras and $P_{\bullet} \xrightarrow{\varepsilon_{\Lambda}} \Lambda$ and $Q_{\bullet} \xrightarrow{\varepsilon_{\Gamma}} \Gamma$ projective resolutions as bimodules. Then the tensor product $P_{\bullet} \otimes Q_{\bullet} \xrightarrow{\varepsilon_{\Lambda} \otimes \varepsilon_{\Gamma}} \Lambda \otimes \Gamma$ is a projective resolution of the tensor algebra $\Lambda \otimes \Gamma$ over $(\Lambda \otimes \Gamma)^{\mathrm{e}}$ (see [15, Section X.7]). Here, we identify $(\Lambda \otimes \Gamma)^{\mathrm{e}}$ with $\Lambda^{\mathrm{e}} \otimes \Gamma^{\mathrm{e}}$. It also follows from [6, Lemma 6.2] that if both $P_{\bullet} \to \Lambda$ and $Q_{\bullet} \to \Gamma$ are minimal, then so is $P_{\bullet} \otimes Q_{\bullet} \to \Lambda \otimes \Gamma$.

From now on, we assume that Λ is a periodic algebra of period p and that Γ is an algebra of finite global dimension n. Set $A := \Lambda \otimes \Gamma$. Since periodic algebras are self-injective algebras, it follows from [6, Lemma 6.1] that we have $\mathrm{inj.dim}\, A = \mathrm{inj.dim}\, \Lambda + \mathrm{inj.dim}\, \Gamma = 0 + n = n$ as one-sided modules. Thus A is an n-Gorenstein algebra. Note that the same lemma also implies that the enveloping algebra A^{e} is a $(2n)$-Gorenstein algebra. The following result shows that the algebra A has an eventually periodic minimal projective resolution.

Proposition 12. *Let Λ and Γ be as above. Then $A = \Lambda \otimes \Gamma$ is an eventually periodic n-Gorenstein algebra having $\Omega^n_{A^{\mathrm{e}}}(A)$ as its first periodic syzygy.*

Remark 13. Proposition 2 enables us to get $\text{G-dim}_{A^{\mathrm{e}}} A \leq 2n = \mathrm{inj.dim}_{A^{\mathrm{e}}} A^{\mathrm{e}}$ and hence $\mathrm{HH}^i(A) \cong \widehat{\mathrm{HH}}^i(A)$ for all $i > 2n$. On the other hand, the i-th syzygy $\Omega^i_{A^{\mathrm{e}}}(A)$ of A is Cohen-Macaulay for any $i \geq n$ by Propositions 10 and 12. Again, Proposition 2 yields that $\text{G-dim}_{A^{\mathrm{e}}} A \leq n$. One of the advantages of this observation is that there exists an isomorphism $\mathrm{HH}^i(A) \cong \widehat{\mathrm{HH}}^i(A)$ for all $i > n$.

Remark 14. It follows from Theorem 7 and the proof of Proposition 12 that the Tate-Hochschild cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A)$ of A is of the form $\widehat{\mathrm{HH}}^{\geq 0}(A)[\chi^{-1}]$, where the degree of χ divides the period p of Λ. We hope to address the degree of χ in a future paper.

We end this section with the following two examples. Note that the tensor algebra A in Example 16 can be found in [6, Example 6.3].

Example 15. For an integer $n \geq 0$, let Γ_n be the algebra given by a quiver with relations

$$0 \xrightarrow{\alpha_0} 1 \longrightarrow \cdots \longrightarrow n-1 \xrightarrow{\alpha_{n-1}} n \qquad \alpha_{i+1}\alpha_i = 0 \text{ for } i = 0, \ldots, n-2.$$

Then we have $\operatorname{gl.dim} \Gamma_n = n$. By Proposition 12, any periodic algebra Λ gives us an eventually periodic n-Gorenstein algebra $A = \Lambda \otimes \Gamma_n$ with $\Omega^n_{A^e}(A)$ the first periodic syzygy of A.

Example 16. Let $\Lambda = k[x]/(x^2)$ and let Γ be the algebra Γ_1 defined in Example 15. Thanks to Bardzell's minimal projective resolution, we see that Λ is a periodic algebra whose period is equal to 1 if $\operatorname{char} k = 2$ and to 2 otherwise. On the other hand, the tensor algebra $A = \Lambda \otimes \Gamma$ is given by the following quiver with relations

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 \arrow[loop right, "\gamma"] \end{tikzcd} \qquad \alpha^2 = 0 = \gamma^2 \quad \text{and} \quad \beta\alpha = \gamma\beta.$$

Thus we see that A is a (non-monomial) eventually periodic Gorenstein algebra whose first periodic syzygy is $\Omega^1_{A^e}(A)$. Now, we compute $\dim_k \widehat{\mathrm{HH}}^i(A)$ for all $i \in \mathbb{Z}$. It follows from [13, Section 1.6] that the Hochschild cohomology ring $\mathrm{HH}^\bullet(\Gamma)$ is of the form

$$\mathrm{HH}^\bullet(\Gamma) = k.$$

According to [5, Section 5], the Hochschild cohomology ring $\mathrm{HH}^\bullet(\Lambda)$ is as follows:

$$\mathrm{HH}^\bullet(\Lambda) = \begin{cases} k[a_0,\, a_1]/(a_0^2) & \text{if } \operatorname{char} k = 2; \\ k[a_0,\, a_1,\, a_2]/(a_0^2,\, a_1^2,\, a_0a_1,\, a_0a_2) & \text{if } \operatorname{char} k \neq 2, \end{cases}$$

where the index i of a homogeneous element a_i denotes the degree of a_i. On the other hand, by [14, Lemma 3.1], there exists an isomorphism of graded algebras

$$\mathrm{HH}^\bullet(A) \cong \mathrm{HH}^\bullet(\Lambda) \otimes \mathrm{HH}^\bullet(\Gamma) = \mathrm{HH}^\bullet(\Lambda).$$

It follows from Remark 13 that $\mathrm{HH}^i(A) \cong \widehat{\mathrm{HH}}^i(A)$ for all $i > 1$. Since $\widehat{\mathrm{HH}}^*(A) \cong \widehat{\mathrm{HH}}^{*+p}(A)$ with the period p of Λ by Remark 14, we have, for any integer i,

$$\dim_k \widehat{\mathrm{HH}}^i(A) = \begin{cases} 2 & \text{if } \operatorname{char} k = 2; \\ 1 & \text{if } \operatorname{char} k \neq 2. \end{cases}$$

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