VANISHING OF EXT MODULES OVER COHEN–MACAULAY RINGS

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Abstract. In this article, we give criteria for projectivity of modules in terms of vanishing of Ext modules. One of the applications shows that the Auslander–Reiten conjecture holds for Cohen–Macaulay normal rings.

Key Words: (Auslander) transpose, Auslander–Reiten conjecture, canonical module, Cohen–Macaulay ring, Ext module, maximal Cohen–Macaulay module, syzygy, tensor product, Tor module.

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1. Introduction

We refer the reader to [10] (`arXiv:2106.08583`) for details on the contents of this article.

Vanishing of Ext modules over a commutative ring is an actively studied subject in commutative algebra. A lot of criteria for a given module to be projective have been described in terms of vanishing of Ext modules so far; see [1, 2, 3, 5, 6, 8, 11, 12] for instance.

In this article, we consider vanishing of Ext modules over a (commutative) Cohen–Macaulay ring. The following theorem is the first main theorem of this article.

Theorem 1. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 2$, and let $1 \leq n \leq d-1$ be an integer. Let M be a finitely generated R-module which locally has finite projective dimension in codimension n. Let N be a maximal Cohen–Macaulay R-module with $\operatorname{Supp} N = \operatorname{Spec} R$. Then M is free provided that the following hold:*

(1) $\operatorname{Ext}_R^i(M,N) = 0$ *for all* $1 \leq i \leq d$.
(2) $\operatorname{Ext}_R^j(M^*, \operatorname{Hom}_R(M,N)) = 0$ *for all* $n \leq j \leq d-1$.

As a application of Theorem 1, the following theorem is proved in Section 3.

Theorem 2. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 2$, and let $1 \leq n \leq d-1$ be an integer. Let M be a finitely generated R-module which locally has finite projective dimension in codimension n. Then M is free provided that the following hold:*

(1) $\operatorname{Ext}_R^i(M,R) = 0$ *for all* $1 \leq i \leq 2d+1$.
(2) $\operatorname{Ext}_R^j(M,M) = 0$ *for all* $n \leq j \leq d-1$.

The above theorems yield various applications. Among other things, the following corollary is obtained.

The detailed version [10] of this article has been submitted for publication elsewhere.

Corollary 3. *Let R be a Cohen–Macaulay ring. If $R_{\mathfrak{p}}$ satisfies the Auslander–Reiten conjecture for every $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} \leq 1$, then $R_{\mathfrak{p}}$ satisfies that conjecture for every $\mathfrak{p} \in \operatorname{Spec} R$. In particular, the Auslander–Reiten conjecture holds true for an arbitrary Cohen–Macaulay normal ring.*

The Auslander–Reiten conjecture asserts that every finitely generated module M over a noetherian ring R such that $\operatorname{Ext}_R^{>0}(M, M \oplus R) = 0$ is projective. This conjecture is known to hold true if R is a locally excellent Cohen–Macaulay normal ring containing the field of rational numbers [8], or if R is a Gorenstein normal ring [1], or if R is a local Cohen–Macaulay normal ring and M is a maximal Cohen–Macaulay module such that $\operatorname{Ext}_R(M, M)$ is free [6]. The above corollary gives a common generalization of these three facts. Other applications of our theorems recover and refine a lot of results in the literature; see Remark 7 and 9 for details.

We close the section by stating our convention adopted throughout the remainder of this article.

Convention. Let R be a commutative noetherian ring. All modules are assumed to be finitely generated. Call an R-module M *maximal Cohen–Macaulay* if $\operatorname{depth} M_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp} M$, so that the zero module is thought of as maximal Cohen–Macaulay. Denote by $\operatorname{mod} R$ the category of (finitely generated) R-modules. For an integer $n \geq 0$, we denote by $\mathrm{X}^n(R)$ the set of prime ideals of R with height at most n. Set $(-)^* = \operatorname{Hom}_R(-, R)$. Whenever R is a Cohen–Macaulay local ring with a canonical module ω, put $(-)^\dagger = \operatorname{Hom}_R(-, \omega)$. The set of nonnegative integers is denoted by $\mathbb{N}$.

2. Comments on Theorem 1

We begin with proving our key proposition. For R-modules X and Y, we define homomorphisms $\beta_{X,Y} : X \otimes_R \operatorname{Hom}_R(X, Y) \to Y$ by $\beta_{X,Y}(x \otimes f) = f(x)$ for $x \in X$ and $f \in \operatorname{Hom}(X, Y)$.

Lemma 4. *Let X and Y be R-modules. Then $\beta_{Y, Y \otimes X}$ is surjective.*

Let R be a local ring with residue field k. For an R-module M we denote by $\mu(M)$ the the *minimal number of generators* of M, i.e., $\mu(M) = \dim_k(M \otimes_R k)$. The next proposition plays a key role in the proof of the first main result of this article.

Proposition 5. *Let R be a local ring and M, N be R-modules. If $N \neq 0$ and $\operatorname{Tor}_1^R(\operatorname{Tr} M, M \otimes_R N) = 0$, then M is free.*

Proof. We may assume that $M \neq 0$. Set $L = M \otimes N$ and $I = \operatorname{Im} \beta_{M,R}$. By induction on $\mu(M)$, it suffices to show $I = R$. Lemma 4 shows the surjectivity of $\beta_{M,L}$. As $\operatorname{Tor}_1(\operatorname{Tr} M, L) = 0$, the map $\varpi : M^* \otimes L \to \operatorname{Hom}(M, L)$ given by $\varpi(f \otimes z)(x) = f(x)z$ for $f \in M^*$, $z \in L$ and $x \in M$ is surjective. Since the composition

$$M \otimes M^* \otimes L \xrightarrow{M \otimes \varpi} M \otimes \operatorname{Hom}(M, L) \xrightarrow{\beta_{M,L}} L$$

coincides with the map $\beta_{M,R} \otimes L : M \otimes M^* \otimes L \to R \otimes L \cong L$, $\beta_{M,R} \otimes L$ is surjective. Therefore $R/I \otimes_R L = 0$, and Nakayama's lemma implies $I = R$. $\square$

In the proof of Theorem 1, we derive the vanishing of Tor modules similar to the assumption of Proposition 5. Hence we can obtain the conclusion of Theorem 1.

The result below is a direct corollary of Theorem 1.

Corollary 6. *Let R be a Cohen–Macaulay local ring of dimension $d \geq 2$ with a canonical module ω. Let $1 \leq n \leq d-1$. Let M be a maximal Cohen–Macaulay R-module which is locally of finite projective dimension on $\mathrm{X}^n(R)$. If $\mathrm{Ext}_R^i(M^*, M^\dagger) = 0$ for every $n \leq i \leq d-1$, then M is free.*

Remark 7. Corollary 6 deduces [2, Theorem 1.4] (a bit weaker version of [12, Corollary 3.9(1)]). Indeed, let R, M be as in [2, Theorem 1.4]. Similarly as in the beginning of [2, Proof of Theorem 1.4] M is reflexive and locally free on $\mathrm{X}^n(R)$. Then Corollary 6 applies to M^* to yield that M is free.

3. Comments on Theorem 2

Let $m, n \in \mathbb{N} \cup \{\infty\}$. By $\mathcal{G}_{m,n}$ we denote the full subcategory of $\operatorname{mod} R$ consisting of modules M such that $\mathrm{Ext}_R^i(M, R) = 0$ for all $1 \leq i \leq m$ and $\mathrm{Ext}_R^j(\operatorname{Tr} M, R) = 0$ for all $1 \leq j \leq n$. Applying Theorem 1, we give the proof of Theorem 2.

Proof of Theorem 2. Put $L = \Omega^{d+2} M$. We apply Theorem 1 to L^*. It suffices to show $\mathrm{Ext}_R^i(L^*, R) = 0$ for all $1 \leq i \leq d$ and $\mathrm{Ext}_R^j(L^{**}, L^{**}) = 0$ for all $n \leq j \leq d-1$, and only the former is proved here. Since $M \in \mathcal{G}_{2d+1,0}$, we get $L \in \mathcal{G}_{d-1,d+2}$ and $\Omega^2 \operatorname{Tr} L \in \mathcal{G}_{d,d+1} \subseteq \mathcal{G}_{d,0}$ by [9, Proposition 1.1.1], while $L^* \cong \Omega^2 \operatorname{Tr} L$ up to free summands. Thus $\mathrm{Ext}^i(L^*, R) = 0$ for all $1 \leq i \leq d$. □

Remark 8. The condition (1) of Theorem 2 can be extended to the following condition:

(1') $M \in \mathcal{G}_{2d-t+1,t}$ for some integer $0 \leq t \leq d+2$.

Note that (1) is the situation where we let $t = 0$ in (1').

Here we recall a celebrated long-standing conjecture due to Auslander and Reiten [4].

Conjecture (Auslander–Reiten)**.** *Every R-module M such that $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$ is projective.*

The corollary below is deduced from Theorem 2, which gives positive answers to the Auslander–Reiten conjecture.

Corollary 9. (1) *Let R be a Cohen–Macaulay ring. Let M be an R-module with $\mathrm{Ext}_R^{>0}(M, M \oplus R) = 0$. If M is locally of finite projective dimension on $\mathrm{X}^1(R)$, then M is a projective R-module.*

(2) *Let R be a Cohen–Macaulay ring. Suppose that R locally satisfies the Auslander–Reiten conjecture on $\mathrm{X}^1(R)$. Then R locally satisfies the Auslander–Reiten conjecture on $\operatorname{Spec} R$. In particular, R satisfies the Auslander–Reiten conjecture.*

(3) *The Auslander–Reiten conjecture holds true for every Cohen–Macaulay normal ring R.*

Finally, we compare our results obtained in this section with ones in the literature.

Remark 10. (1) Let R be a Cohen–Macaulay local ring of dimension $d > 0$ with a canonical module ω. By Remark 8 and [7, Theorem 3.6], R is Gorenstein if it is locally Gorenstein on $\mathrm{X}^1(R)$ and $\mathrm{Ext}_R^i(\omega, R) = 0$ for $1 \leq i \leq 2d-1$. This is a weak version of [5, Theorem 2.1], which asserts a generically Gorenstein local ring R of dimension d with a dualizing complex D is Gorenstein if $\mathrm{Ext}_R^i(D,R)=0$ for $1\le i\le d$.

(2) Theorem 2 highly refines [12, Corollary 1.5] in the case of a Cohen–Macaulay ring. More precisely, if R is a Cohen–Macaulay ring, then [12, Corollary 1.5] asserts the same as Theorem 2 under the additional assumptions that M is locally free on $\mathrm{X}^n(R)$, that $\mathrm{Ext}_R^{>0}(M,R)=0$, and that either $\mathrm{Ext}_R^i(\mathrm{Hom}_R(M,M),R)=0$ for all $n<i\le\dim R$ or $\mathrm{Hom}_R(M,M)$ has finite G-dimension.

(3) Corollary 9(2) extends [1, Theorem 3] from Gorenstein rings to Cohen–Macaulay rings.

(4) By virtue of [8, Theorem 0.1], the Auslander–Reiten conjecture is known to hold true for a locally excellent Cohen–Macaulay normal ring R that contains $\mathbb{Q}$. Corollary 9(3) removes from this statement the assumptions that R is locally excellent and that R contains $\mathbb{Q}$.

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