

Structure theorem for flat cotorsion modules
over Noether algebras arXiv: 2108.03153

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Aim Classify all flat cotorsion modules for
Noether algebras in terms of prime ideals
(generalizing [Enochs 1984] for comm noeth rings).

§1 Flat cotorsion modules

$A : \text{ring} . \quad \text{Mod } A := \{\text{right } A\text{-modules}\}.$

Def $M \in \text{Mod } A$: flat

$\Leftrightarrow M \underset{A}{\otimes} - : \text{Mod } A^{\text{op}} \rightarrow \text{Mod } \mathbb{Z}$: exact.

$\text{Flat } A := \{\text{flat modules}\} \subset \text{Mod } A$.

$M \in \text{Mod } A$: cotorsion

$\Leftrightarrow \text{Ext}_A^1(\text{Flat } A, M) = 0$.

$\text{Cot } A := \{\text{cotorsion modules}\} \subset \text{Mod } A$.

$\text{FCot } A := \text{Flat } A \cap \text{Cot } A$.

Def \mathcal{A} : abelian cat. $\mathcal{X}, \mathcal{Y} \subset_{\text{full}} \mathcal{A}$.

$(\mathcal{X}, \mathcal{Y})$: cotorsion pair

$$\Leftrightarrow \begin{cases} \mathcal{X} = \{M \mid \text{Ext}^1(M, Y) = 0\} \\ Y = \{M \mid \text{Ext}^1(X, M) = 0\} \end{cases}$$

Moreover

hereditary: $\Leftrightarrow \text{Ext}^{>0}(X, Y) = 0$.

complete: $\Leftrightarrow \forall M \in \mathcal{A}$,

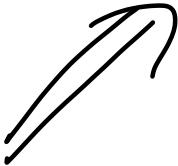
$$0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0 \quad (x, x' \in X)$$

$$0 \rightarrow M \rightarrow Y' \rightarrow X' \rightarrow 0 \quad (y, y' \in Y).$$

exact

Ex $(\text{Rj}A, \text{Mod}A)$, $(\text{Mod}A, \text{Inj}A)$.

Fact $(\text{Flat } A, \text{Cot } A)$ is a complete hereditary
cotorsion pair
in $\text{Mod } A$.



Flat Cover Conjecture
solved by [Bican - El Bashir - Enochs 2001]

cf [Nakamura - Thompson 2020]

§2 Structure theorem.

R : comm noeth ring.

A : Noether R -algebra.

i.e. $R \subset Z(A) \subset A$ & A is fin.gen. as an R -mod.

subring \nwarrow center

$\text{Spec } A := \{ \text{prime (two-sided) ideals of } A \}$.

$\downarrow \text{-1} \uparrow$ [Gabriel 1962]

$$\begin{array}{ccc} \{\text{indec injectives in } \text{Mod } A\} & \xrightarrow{\cong} & \mathbb{P} \\ \downarrow & \nearrow & \downarrow \\ E_A(\frac{A}{P}) & \cong & I_A(P)^{n_P} \\ \downarrow & & \downarrow \\ I_A(P) & & \end{array}$$

$\text{Spec } A \rightarrow \text{Spec } R$

$$P \longmapsto P \cap R$$

Thm [Kanda-Nakamura; Enochs when $A=R$]

$M \in \text{Mod } A$; flat cotorsion

$$\Leftrightarrow M \cong \prod_{P \in \text{Spec } A} \text{Hom}_R(I_{A^{\text{op}}}(P), E_R\left(\frac{R}{P \cap R}\right)^{\oplus B_P})$$

set

(The cardinality of B_P is uniquely determined)

Cor { indec flcot in $\text{Mod } A$ } $\xrightarrow[\cong]{1-1}$ Spec A

$$T_A(P) := \text{Hom}_R(I_{A^{\text{op}}}(P), E_R\left(\frac{R}{P \cap R}\right)) \hookrightarrow P$$

($\exists b_{ij}$ by [Herzog 1993])

Thm [KN]

$\forall p \in \text{Spec } R$,

$\widehat{} \leftarrow p\text{-adic completion}$

$$\widehat{A}_p \cong \bigoplus_{\substack{P \in \text{Spec } A \\ P \cap R = p}} T_A(P)^{n_P} \quad \text{in Mod } A$$

Ex $A = \begin{pmatrix} R & \\ R & R \end{pmatrix} = R(- \rightarrow -).$

$$\widehat{A}_p = \begin{pmatrix} \widehat{R}_p & \\ \widehat{R}_p & \widehat{R}_p \end{pmatrix} = \begin{pmatrix} (\widehat{R}_p, 0) \\ \bigoplus \\ (\widehat{R}_p, \widehat{R}_p) \end{pmatrix}.$$

$$\{\text{indec flst in Mod } A\}_{\leq} = \{(\widehat{R}_p, 0), (\widehat{R}_p, \widehat{R}_p) \mid p \in \text{Spec } R\},$$

§3 Ziegler Spectrum.

$\mathcal{Zg}_A := \text{Index pure-injectives in } \text{Mod } A \rangle \not\models \rightarrow N$

\downarrow closed \cup closed
 (Index flots) $\not\models$ (Index inj's) $\not\models$
 {Index injectives in $\text{Func}^{\mathbb{D}}(\text{mod } A^{\text{op}}, \text{Mod } \mathbb{Z}) \rangle \not\models \rightarrow N$ }
 $\not\models$

Fact (Elementary duality)

{open subsets of $\mathcal{Zg}_A \rangle \cong$ {open subsets of $\mathcal{Zg}_{A^{\text{op}}} \rangle$ }
 poset

Thm $\mathcal{Zg}_A \cong \mathcal{Zg}_{A^{\text{op}}}$

flots $\xrightarrow{\exists!}$ inj's that is compatible w. elementary duality

[Hertzog].

This is given by $T_A(P) \mapsto I_{A^P}(P)$ [KN].