Numerical torsion pairs and canonical decompositions for elements in the Grothendieck group

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Motivation

Let A be a fin. dim. algebra over an alg. closed field K.

- $\mathcal{T} \subset \operatorname{mod} A$: a torsion class $:\Longleftrightarrow$ $\mathcal{T}$: closed under quot. and ext.
- $\mathcal{T} \subset \operatorname{mod} A$: a functorially finite torsion class $:\Longleftrightarrow$
  $\mathcal{T} = \operatorname{Fac} M := \{X \mid M^{\oplus m} \to X\text{: surj.}\}$ for some $M \in \operatorname{mod} A$.
  They are related to τ-tilting theory [AIR].

We deal with the following things in this talk:

- Numerical torsion classes $\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta$ for $\theta \in K_0(\operatorname{proj} A)$.
  - Defined via stability conditions [King, BKT].
  - They contain all func. fin. torsion classes [Yurikusa, BST].
- Canonical decompositions $\theta = \bigoplus_{i=1}^{m} \theta_i$ in $K_0(\operatorname{proj} A)$.
  - Comes from decompositions of 2-term complexes in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ into indecomposable direct summands [Derksen–Fei].

How are they related?

Setting

- A: a fin. dim. algebra over an alg. closed field K.
- proj A: the category of fin. gen. projective A-modules.
- $P_1, P_2, \ldots, P_n$: the non-iso. indec. proj. modules.
- $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$: the homotopy cat. of bounded complexes over proj A.
- mod A: the category of fin. gen. A-modules.
- $S_1, S_2, \ldots, S_n$: the non-iso. simple modules
  (we may assume there exists a surj. $P_i \to S_i$).
- $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$: the derived cat. of bounded complexes over mod A.
- $K_0(C)$: the Grothendieck group of C.
- $K_0(C)_{\mathbb{R}} := K_0(C) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.

The Euler form

$K_0(\operatorname{proj} A)$ and $K_0(\operatorname{mod} A)$ are free abelian groups.

Proposition (see [Happel])

(1) $K_0(\operatorname{proj} A) = K_0(\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[P_i]$.

(2) $K_0(\operatorname{mod} A) = K_0(\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)) = \bigoplus_{i=1}^{n} \mathbb{Z}[S_i]$.

(3) $\langle [P_i], [S_j] \rangle = \delta_{i,j}$, where

$$\langle \cdot, \cdot \rangle \colon K_0(\operatorname{proj} A) \times K_0(\operatorname{mod} A) \longrightarrow \mathbb{Z}$$

is the Euler form.

Via the Euler form, each $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ induces the $\mathbb{R}$-linear form

$$\theta := \langle \theta, \cdot \rangle \colon K_0(\operatorname{mod} A)_{\mathbb{R}} \longrightarrow \mathbb{R}.$$

Numerical torsion pairs

Definition [Baumann–Kamnitzer–Tingley] (cf. [King])

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.
We define numerical torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},\\
\mathcal{F}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},\\
\mathcal{T}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},\\
\overline{\mathcal{F}}_\theta &:= \{M \in \operatorname{mod} A \mid \theta(L) \geq 0 \text{ for any submodule } L \text{ of } M\}.
\end{aligned}$$

Definition

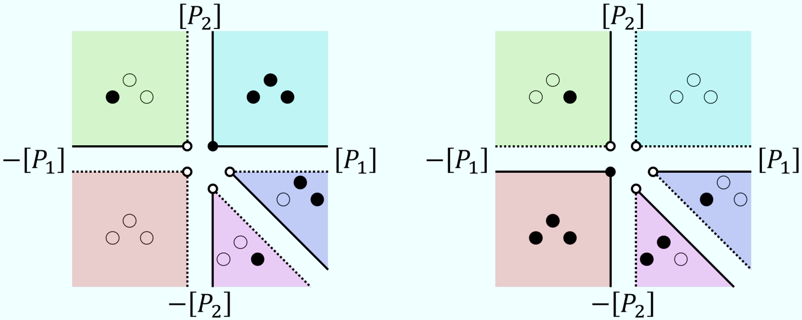
$\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ are TF equivalent $:\Longleftrightarrow$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'}), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'})$$

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{smallmatrix} & P_1 & \\ S_2 & & S_1 \end{smallmatrix}$ are the indec. A-modules.

Then, $\overline{\mathcal{T}}_\theta$ and $\overline{\mathcal{F}}_\theta$ are given as follows.

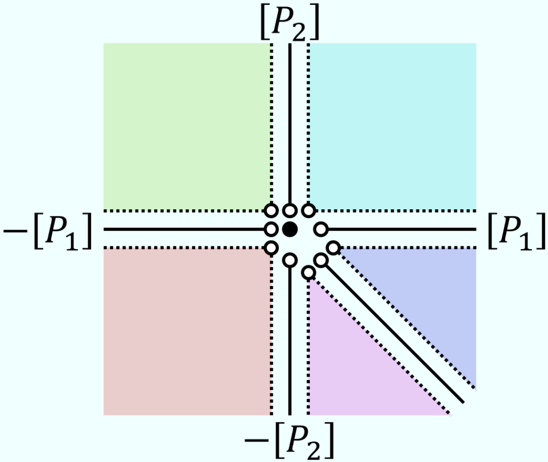


(●: belong, ○: not belong)

Example of TF equiv. classes

Let $A = K(1 \to 2)$, $\begin{smallmatrix} & P_1 & \\ S_2 & & S_1 \end{smallmatrix}$ are the indec. A-modules.

There are exactly 11 TF equivalence classes.



Presentation spaces

Definition [Derksen–Fei]

Let $\theta \in K_0(\operatorname{proj} A)$.

(1) Take $P_+, P_- \in \operatorname{proj} A$ (unique up to iso.) such that $\theta = [P_+] - [P_-]$ and $\operatorname{add} P_+ \cap \operatorname{add} P_- = \{0\}$.

(2) $\operatorname{Hom}(\theta) := \operatorname{Hom}_A(P_-, P_+)$: the presentation space of θ.

(3) For each $f \in \operatorname{Hom}(\theta)$, set $P_f := (P_- \xrightarrow{f} P_+) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ (the terms except -1st and 0th ones vanish).

$\operatorname{Hom}(\theta)$ is an affine variety.

Direct sums in $K_0(\mathrm{proj}\, A)$

Definition [DF]

Let $\theta, \theta_1, \theta_2 \in K_0(\mathrm{proj}\, A)$.

(1) We say that the direct sum $\theta_1 \oplus \theta_2$ holds in $K_0(\mathrm{proj}\, A)$ if any general $f \in \mathrm{Hom}(\theta_1 + \theta_2)$ admits $f_i \in \mathrm{Hom}(\theta_i)$ $(i = 1, 2)$ such that $P_f \cong P_{f_1} \oplus P_{f_2}$ in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$.

(2) $\theta = \theta_1 \oplus \theta_2 :\Longleftrightarrow \theta = \theta_1 + \theta_2$ and $\theta_1 \oplus \theta_2$ hold.

Proposition [DF]

$\theta_1 \oplus \theta_2 \iff \exists (f_1, f_2) \in \mathrm{Hom}(\theta_1) \times \mathrm{Hom}(\theta_2)$,

$$\mathrm{Hom}(P_{f_1}, P_{f_2}[1]) = 0, \quad \mathrm{Hom}(P_{f_2}, P_{f_1}[1]) = 0.$$

In this case, the above hold for any general pair (f_1, f_2).

Canonical decompositions

Definition

θ: indecomposable in $K_0(\operatorname{proj} A)$:$\Longleftrightarrow$
for any general $f \in \operatorname{Hom}(\theta)$, $P_f \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ is indec.

Theorem [DF, Plamondon]

Any $\theta \in K_0(\operatorname{proj} A)$ admits a decomposition unique up to reordering

$$\theta = \bigoplus_{i=1}^{m} \theta_i \quad (\theta_i\text{: indecomposable}).$$

We call it the canonical decomposition of θ.

Presilting complexes and canon. decomp.

Definition

Let $U \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ be a 2-term complex.
U: presilting $:\Longleftrightarrow$ $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(U, U[1]) = 0$.

Proposition [Plamondon, Demonet–Iyama–Jasso]

Let $U = \bigoplus_{i=1}^{m} U_i$ be a 2-term presilting complex with U_i indec.

(1) Any general $f \in \operatorname{Hom}([U])$ satisfies $P_f \cong U$ in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

(2) $[U] = \bigoplus_{i=1}^{m} [U_i]$ is the canon. decomp. of $[U]$ in $K_0(\operatorname{proj} A)$.

Presilting and func. fin. torsion pairs

Let U be a 2-term presilting complex. We set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^{\perp}H^{-1}(\nu U), \mathsf{Sub}\, H^{-1}(\nu U)),$$
$$(\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\mathsf{Fac}\, H^0(U), H^0(U)^{\perp}).$$

Then, $\mathcal{T}_U \subset \overline{\mathcal{T}}_U$ and $\mathcal{F}_U \subset \overline{\mathcal{F}}_U$.

Theorem [Smalø, Auslander–Smalø, Adachi–Iyama–Reiten]

Let U be a 2-term presilting complex.

(1) $(\overline{\mathcal{T}}_U, \mathcal{F}_U)$, $(\mathcal{T}_U, \overline{\mathcal{F}}_U)$ are func. fin. torsion pairs.

(2) All func. fin. torsion(-free) classes are obtained in this way.

Presilting and TF equiv. classes

Theorem (⇒): [Yurikusa, Brüstle–Smith–Treffinger], (⇐): [A]

Let $U = \bigoplus_{i=1}^{m} U_i$ be a 2-term presilting complex with U_i indec.
For $\eta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$,

$$\eta \in \sum_{i=1}^{m} \mathbb{R}_{>0}[U_i] \iff \overline{\mathcal{T}}_\eta = \overline{\mathcal{T}}_U, \quad \overline{\mathcal{F}}_\eta = \overline{\mathcal{F}}_U.$$

In this case,

$$\overline{\mathcal{T}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{T}}_{U_i}, \quad \overline{\mathcal{F}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{F}}_{U_i}.$$

Thus, $\sum_{i=1}^{m} \mathbb{R}_{>0}[U_i]$ is a TF equiv. class.

Our results 1

Theorem 1 [AI] (with Demonet)

Let $\theta = \bigoplus_{i=1}^{m} \theta_i$ in $K_0(\mathrm{proj}\, A)$. Then,

$$\eta \in \sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i \Longrightarrow \overline{\mathcal{T}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{T}}_{\theta_i}, \quad \overline{\mathcal{F}}_\eta = \bigcap_{i=1}^{m} \overline{\mathcal{F}}_{\theta_i}.$$

Thus, for any i, $\mathcal{T}_{\theta_i} \subset \mathcal{T}_\eta \subset \overline{\mathcal{T}}_\eta \subset \overline{\mathcal{T}}_{\theta_i}$, $\mathcal{F}_{\theta_i} \subset \mathcal{F}_\eta \subset \overline{\mathcal{F}}_\eta \subset \overline{\mathcal{F}}_{\theta_i}$.

We can recover the following sign-coherence.

Proposition [Plamondon]

Let $\theta \oplus \theta'$ in $K_0(\mathrm{proj}\, A)$, $\theta = \sum_{i=1}^{n} a_i[P_i]$ and $\theta' = \sum_{i=1}^{n} a'_i[P_i]$.
Then, $a_i a'_i \geq 0$ for all i.

$\because$ If $a_i > 0$ and $a'_i < 0$, then $S_i \in \mathcal{T}_\theta \cap \mathcal{F}_{\theta'} \subset \mathcal{T}_{\theta+\theta'} \cap \mathcal{F}_{\theta+\theta'} = \{0\}$.

Our results 2

By Theorem 1, if $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$, then $\sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$ is contained in some TF equiv. class.
Is it really a TF equiv. class?

Theorem 2 [AI]

Assume that

- A is a hereditary algebra; or
- $\theta \oplus \theta$ holds in $K_0(\operatorname{proj} A)$ for any $\theta \in K_0(\operatorname{proj} A)$.

If $\theta = \bigoplus_{i=1}^{m} \theta_i$ is a canon. decomp. in $K_0(\operatorname{proj} A)$, then $\sum_{i=1}^{m} \mathbb{R}_{>0}\theta_i$ is a TF equiv. class in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

In above, TF equiv. classes “generalize” canonical decompositions.

E-tame algebras

We say A is E-tame if $\theta \oplus \theta$ holds in $K_0(\mathrm{proj}\, A)$ for any $\theta \in K_0(\mathrm{proj}\, A)$.

Theorem [Geiss–Labardini-Fragoso–Schröer]

Let A be representation-finite or tame.
Then, A is E-tame.

The above theorem depends on the strong results of [Crawley-Boevey] on 1-parameter families of modules over tame algebras.
See also [Plamondon–Yurikusa].

Thank you for your attention.

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