CONSTRUCTIONS OF REJECTIVE CHAINS

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Abstract. In this note, we give some constructions of rejective chains. We show that if A is a locally hereditary algebra or Nakayama algebra with a heredity ideal, then the category $\mathsf{proj}A$ admits a total right rejective chain. Moreover, we construct a total right rejective chain of $\mathcal{F}(\Delta)$ over a quasi-hereditary algebra if $\mathcal{F}(\Delta)$ is representation-finite and multiplicity-free.

1. Preliminaries

Throughout this note, A is an artin algebra and J is the Jacobson radical of A. We denote by $\mathsf{mod}A$ the category of finitely generated right A-modules and by $\mathsf{proj}A$ the category of finitely generated projective right A-modules. For $M \in \mathsf{mod}A$, we write $\mathsf{add}M$ for the category of all direct summands of finite direct sums of copies of M. In the following, we assume that any subcategory of an additive category is full and closed under isomorphisms, direct sums and direct summands.

1.1. Rejective chains.

In this subsection, we recall the definition of rejective chains. In [5], Iyama introduced the notion of rejective chains and proved the finiteness theorem of representation dimensions of artin algebras conjectured by Auslander [1]. In this proof, rejective chains play a crucial role. We start this subsection with recalling the definition of right rejective subcategories.

Definition 1 ([5, 2.1(1)])**.** Let $\mathcal{C}$ be an additive category. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *right rejective subcategory* of $\mathcal{C}$ if, for any $X \in \mathcal{C}$, there exists a monic right $\mathcal{C}'$-approximation $f_X \in \mathcal{C}\,(Y, X)$ of X.

To define a total right rejective chain, we need the notion of cosemisimple subcategories. Let $\mathcal{J}_{\mathcal{C}}$ be the Jacobson radical of $\mathcal{C}$. For a subcategory $\mathcal{C}'$ of $\mathcal{C}$, we denote by $[\mathcal{C}']$ the ideal of $\mathcal{C}$ consisting of morphisms which factor through some object of $\mathcal{C}'$, and by $\mathcal{C}/[\mathcal{C}']$ the factor category (i.e., $ob(\mathcal{C}/[\mathcal{C}']) := ob(\mathcal{C})$ and $(\mathcal{C}/[\mathcal{C}'])(X,Y) := \mathcal{C}(X,Y)/[\mathcal{C}'](X,Y)$ for any $X, Y \in \mathcal{C}$). An additive category $\mathcal{C}$ is called a Krull–Schmidt category if any object of $\mathcal{C}$ is isomorphic to a finite direct sum of objects whose endomorphism rings are local.

Definition 2. Let $\mathcal{C}$ be a Krull–Schmidt category. A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is called a *cosemisimple* subcategory in $\mathcal{C}$ if $\mathcal{J}_{\mathcal{C}/[\mathcal{C}']} = 0$ holds.

We give a characterization of cosemisimple right rejective subcategories of $\mathsf{proj}A$ as follows.

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Proposition 3 ([6, Theorem 3.2(3)]). *Let A be a basic artin algebra and e an idempotent of A. Then $\mathsf{add}eA$ is a cosemisimple right rejective subcategory of $\mathsf{proj}A$ if and only if $(1-e)J \in \mathsf{add}eA$ as a right A-module.*

Now, we introduce the following key notion in this note.

Definition 4 ([5, 2.1(2)]). Let $\mathcal{C}$ be a Krull–Schmidt category. A chain

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

of subcategories of $\mathcal{C}$ is called a *total right rejective chain* (of length n) if the following conditions hold for $1 \leq i \leq n-1$:

(a) $\mathcal{C}_i$ is a right rejective subcategory of $\mathcal{C}$;
(b) $\mathcal{C}_i$ is a cosemisimple subcategory of $\mathcal{C}_{i-1}$.

Proposition 5 ([6, Theorem 3.3]). *Let A be an artin algebra. If $\mathsf{proj}A$ admits a total right rejective chain of length n, then the global dimension of A is at most n.*

1.2. **Rejective chains and strongly quasi-hereditary algebras.** In this subsection, we quickly review a relationship between rejective chains and strongly quasi-hereditary algebras. For more detail, we refer to [6, 8].

We fix a complete set of representatives of isomorphism classes of simple A-modules $\{S(i) \mid i \in I\}$. For $i \in I$, we denote by $P(i)$ the projective cover of $S(i)$. Let $\geq$ be a partial order on I. For each $i \in I$, we denote by $\Delta(i)$ the maximal factor module of $P(i)$ whose composition factors have the form $S(j)$ for some $j \leq i$. The module $\Delta(i)$ is called the *standard module* corresponding to $i \in I$. Let $\Delta := \{\Delta(i) \mid i \in I\}$ be the set of standard modules. We denote by $\mathcal{F}(\Delta)$ the full subcategory of $\mathsf{mod}A$ whose objects are the modules which have a Δ-filtration. For $M \in \mathcal{F}(\Delta)$, we denote by $(M : \Delta(i))$ the filtration multiplicity of $\Delta(i)$ in M. We write $\mathcal{F}(\Delta(\geq i))$ for the subcategory of $\mathcal{F}(\Delta)$ whose objects are the modules which filtered by standard modules $\Delta(j)$ with $j \geq i$.

In [3], Cline, Parshall and Scott introduced the notion of quasi-hereditary algebras to study highest weight categories in the representation theory of semisimple complex Lie algebras and algebraic group. It is well known that quasi-hereditary algebras have finite global dimension [4]. One of the advantages of right-strongly quasi-hereditary algebras is that they have better upper bound of global dimension than that of general quasi-hereditary algebras [7, §4].

We recall the definition of quasi-hereditary algebras and right-strongly quasi-hereditary algebras.

Definition 6 ([7, §4]). Let A be an artin algebra and $\geq$ a partial order on I. A pair $(A, \geq)$ (or simply A) is called a *quasi-hereditary* algebra if there exists a short exact sequence

$$0 \to K(i) \to P(i) \to \Delta(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $K(i) \in \mathcal{F}(\Delta(> i))$ for any $i \in I$;
(b) $\mathrm{End}_A(\Delta(i))$ is a division ring.

Moreover, a quasi-hereditary algebra $(A, \geq)$ (or simply A) is called a *right-strongly quasi-hereditary* algebra if $K(i) \in \mathsf{proj}A$, or equivalently, the right module $\Delta(i)$ has the projective dimension at most one.

The following proposition gives a connection between right-strongly quasi-hereditary algebras and total right rejective chains.

Proposition 7 ([8, Theorem 3.22])**.** *Let A be an artin algebra. Then the following conditions are equivalent.*

(i) *A is a right-strongly quasi-hereditary algebra.*
(ii) *$\mathsf{proj}A$ has a total right rejective chain.*

2. Main result

In this section, we give some constructions of total right rejective chains.

2.1. **Locally hereditary algebras.**

We start this subsection with recalling the definition of a locally hereditary algebra.

Definition 8. An artin algebra A is called a *locally hereditary* algebra if any right local ideal is projective, or equivalently, any morphism between indecomposable projective modules is a monomorphism.

We can easily check that hereditary algebras are locally hereditary algebras and it is known that canonical algebras are also locally hereditary algebras. We give a concrete example of a locally hereditary algebra.

Example 9. Let A be an algebra defined by the quiver

$$\begin{tikzcd}
 & 2 \arrow[r, "\beta"] & 5 \arrow[dr, "\varepsilon"] & \\
1 \arrow[ur, "\alpha"] \arrow[r, "\gamma"] \arrow[dr, "\alpha'"'] & 3 \arrow[ur, "\delta"'] \arrow[dr, "\delta'"] & & 7 \\
 & 4 \arrow[r, "\beta'"'] & 6 \arrow[ur, "\varepsilon'"'] &
\end{tikzcd}$$

with relations $\alpha\beta - \gamma\delta$, $\gamma\delta' - \alpha'\beta'$ and $\delta\varepsilon - \delta'\varepsilon'$. Then we can check that A is a locally hereditary algebra and the global dimension of A is three.

The aim of this subsection is to prove the following theorem.

Theorem 10. *Let A be an artin algebra. If A is a locally hereditary algebra, then the category $\mathsf{proj}A$ admits a total right rejective chain. In particular, any locally hereditary algebra is a right-strongly quasi-hereditary algebra.*

In the rest of this subsection, we give a proof of Theorem 10. We need the following lemma.

Lemma 11. *If A is a non-semisimple locally hereditary algebra, then there exists a primitive idempotent $e \in A$ such that $0 \neq eJ \in \mathsf{add}(1-e)A$.*

Now, we are ready to prove Theorem 10.

Proof of Theorem 10. We may assume that A is basic and connected. We denote by n the number of simple modules.

(i) Assume that $n \geq 2$. Then there exists a primitive idempotent $e \in A$ such that $0 \neq eJ \in \mathsf{add}(1-e)A$ by Lemma 11. In particular, we have a simple module eA/eJ with the projective dimension exactly one. By Proposition 3, we obtain that the category $\mathsf{add}(1-e)A$ is a cosemisimple right rejective subcategory of $\mathsf{proj}A$. We show that if $\varphi : P \to Q$ is a monomorphism in $\mathsf{add}(1-e)A$, then it is also monic in $\mathsf{proj}A$. Let $\varphi : P \to Q$ be a monomorphism in $\mathsf{add}(1-e)A$. Then we have an exact sequence $0 \to \mathsf{Ker}\varphi \to P \to Q$. We obtain $\mathsf{Ker}\varphi \in \mathsf{mod}A/A(1-e)A$ by $\mathrm{Hom}_A(eA, \mathsf{Ker}\varphi) = 0$. Since $A/A(1-e)A$ is a simple algebra, we have $\mathsf{Ker}\varphi \cong (eA/eJ)^{\oplus l}$ for some $l \geq 0$. Suppose $l \geq 1$. Take a projective cover $eA \xrightarrow{\rho} eA/eJ$. Then we have that ρ is an isomorphism by the definition of locally hereditary algebras. This contradicts to the projective dimension of eA/eJ. Thus we have $l = 0$ and φ is a monomorphism in $\mathsf{proj}A$.

(ii) We construct a total right rejective chain of $\mathsf{proj}A$ by induction on n. If $n = 1$, then A is a simple algebra, and hence the assertion holds. We assume $n \geq 2$. By (i), $\mathsf{add}(1-e)A$ is a cosemisimple right rejective subcategory of $\mathsf{proj}A$ and we note that the subalgebra $(1-e)A(1-e)$ is a locally hereditary with $n-1$ simple modules. By induction hypothesis, there exists a total right rejective chain $\mathsf{proj}(1-e)A(1-e) \supset \mathsf{add}e_2A(1-e) \supset \cdots \supset \mathsf{add}e_nA(1-e) = 0$. Since $\mathsf{add}(1-e)A \simeq \mathsf{proj}(1-e)A(1-e)$, we obtain a chain of subcategories by composing it and $\mathsf{proj}A \supset \mathsf{add}(1-e)A$. By (i), we obtain that a monomorphism in $\mathsf{add}(1-e)A$ is also monic in $\mathsf{proj}A$. Thus we have a total right rejective chain of $\mathsf{proj}A$.

Moreover, by Proposition 7, we have that A is a right-strongly quasi-hereditary algebra since $\mathsf{proj}A$ admits a total right rejective chain. □

2.2. **Nakayama algebras with heredity ideals.** First, we recall the definitions of Nakayama algebras and heredity ideals.

Definition 12. Let A be an artin algebra.

(1) An algebra A is called a *Nakayama* algebra if every indecomposable projective module and every indecomposable injective module are uniserial.
(2) A two-sided ideal H is called a *heredity ideal* if it satisfies the following conditions:
   (a) H is an idempotent ideal, or equivalently, there exists an idempotent e of A such that $H = AeA$;
   (b) H is a projective A-module;
   (c) $HJH = 0$ holds.

The aim of this subsection is to prove the following theorem.

Theorem 13. *Let A be a Nakayama algebra with a heredity ideal AfA. Then there exists a total right rejective chain $\mathsf{proj}A \supset \mathsf{add}e_1A \supset \cdots \supset \mathsf{add}e_{n-1}A \supset 0$, where e_{n-1} is a primitive idempotent with $e_{n-1}A \in \mathsf{add}fA$.*

By combining Proposition 7 and Theorem 13, we have the following corollary which is a refinement of [2, Proposition 2.3] and [9, Proposition 3.1].

Corollary 14. *Let A be a Nakayama algebra. Then the following statements are equivalent.*

(1) *A is a right-strongly quasi-hereditary algebra.*
(2) *A is a quasi-hereditary algebra.*
(3) *There exists a heredity ideal of A.*

In the rest of this subsection, we give a proof of Theorem 13. We collect some results.

Lemma 15. *Let A be a basic Nakayama algebra with a heredity ideal AfA. Then the following statements hold.*

(1) *There exists a primitive idempotent $e \in A$ such that $eJ \in \mathsf{add}fA$.*
(2) [9, Lemma 1.4] *Any non-zero morphism $\varphi : fA \to P$ with $P \in \mathsf{proj}A$ is a monomorphism.*

Now, we are ready to prove Theorem 13.

Proof of Theorem 13. We may assume that A is basic and n is the number of simple modules. If $n = 1$, then this is clear. For $n \geq 2$ we proceed by induction. It follows from [9, Lemma 1.2] that there exists a primitive idempotent e_{n-1} such that $e_{n-1}A$ is a direct summand of fA and $Ae_{n-1}A$ is also a heredity ideal of A. By induction on n, we show that there exists a total right rejective chain $\mathsf{proj}A \supset \mathsf{add}e_1A \supset \cdots \supset \mathsf{add}e_{n-1}A \supset 0$ such that if $\varphi : P \to Q$ is a minimal right $\mathsf{add}e_jA$-approximation of $Q \in \mathsf{proj}A$, then $P \in \mathsf{add}e_{n-1}A$. By Lemma 15(1), there exists a primitive idempotent e such that $eJ \in \mathsf{add}e_{n-1}A$. Let $e_1 := 1-e$. By Proposition 3, we have that $\mathsf{add}e_1A$ is a cosemisimple right rejective subcategory of $\mathsf{proj}A$ since $(1-e_1)J \in \mathsf{add}e_{n-1}A \subset \mathsf{add}e_1A$. We note that e_1Ae_1 is a Nakayama algebra with a heredity ideal $e_1Ae_{n-1}Ae_1$ by [4, Statement10]. By induction hypothesis, there exists a total right rejective chain $\mathsf{proj}e_1Ae_1 \supset \mathsf{add}e_2Ae_1 \supset \cdots \supset \mathsf{add}e_{n-1}Ae_1 \supset 0$ and if $\varphi : P \to Q$ is a minimal right $\mathsf{add}e_jAe_1$-approximation of $Q \in \mathsf{proj}e_1Ae_1$, then $P \in \mathsf{add}e_{n-1}Ae_j$. By composing it and $\mathsf{proj}A \supset \mathsf{add}e_1A$, we have a chain of subcategories $\mathsf{proj}A \supset \mathsf{add}e_1A \supset \cdots \supset \mathsf{add}e_{n-1}A \supset 0$. Since $e_1J \in \mathsf{add}e_{n-1}A$ by Lemma 15(1), $e_1J \to e_1A$ is a minimal right $\mathsf{add}e_jA$-approximation for all $1 \leq j \leq n-1$. Thus we have that if $\varphi : P \to Q$ is a minimal right $\mathsf{add}e_jA$-approximation, then $P \in \mathsf{add}e_{n-1}A$. By Lemma 15(2), each minimal right $\mathsf{add}e_jA$-approximation is a monomorphism. Therefore we obtain a desired total right rejective chain. $\square$

2.3. **Good module category $\mathcal{F}(\Delta)$.** Let A be a quasi-hereditary algebra. For a Krull–Schmidt category $\mathcal{C}$, we denote by $\mathsf{ind}\mathcal{C}$ the set of isoclasses of indecomposable objects in $\mathcal{C}$. Put $\mathsf{F}_i := \{X \in \mathsf{ind}\mathcal{F}(\Delta(\geq i)) \mid (X : \Delta(i)) \neq 0\}$. Throughout this subsection, we assume that $\mathcal{F}(\Delta)$ is representation-finite (i.e., there exist finitely many indecomposable objects up to isomorphism) and multiplicity-free (i.e., $(X : \Delta(i)) = 1$ for all $X \in \mathsf{F}_i$). In this subsection, we give a construction of a total right rejective chain of $\mathcal{F}(\Delta)$ over a quasi-hereditary algebra A.

Let $X \in \mathsf{F}_i$. Then there uniquely exists a short exact sequence $0 \to X' \xrightarrow{\iota_X} X \xrightarrow{\rho_X} \Delta(i) \to 0$ such that $X' \in \mathcal{F}(\Delta(\geq i+1))$. Let $f : X \to Y$ be a morphism with $X, Y \in \mathsf{F}_i$. Since $\mathrm{Hom}_A(M, \Delta(i)) = 0$ holds for all $M \in \mathcal{F}(\Delta(\geq i+1))$, there exist morphisms which make the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "\iota_X"] \arrow[d, "f'"] & X \arrow[r, "\rho_X"] \arrow[d, "f"] & \Delta(i) \arrow[r] \arrow[d, "\bar{f}"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "\iota_Y"] & Y \arrow[r, "\rho_Y"] & \Delta(i) \arrow[r] & 0.
\end{tikzcd}$$

Definition 16. Let $X, Y \in \mathsf{F}_i$. We write $X \geq Y$ if there exists a morphism $f : X \to Y$ such that $\bar{f} \neq 0$.

Lemma 17. $(\mathsf{F}_i, \geq)$ *is a partially ordered set.*

Proof. Since the reflexivity is clear, we check the transitivity and the antisymmetry. Let $X, Y, Z \in \mathsf{F}_i$ with $X \geq Y, Y \geq Z$. Since $X \geq Y$ and $Y \geq Z$, there exist $f : X \to Y$ and $g : Y \to Z$ such that $\bar{f} \neq 0$ and $\bar{g} \neq 0$. Since $\mathrm{End}_A(\Delta(i))$ is a division ring, $\bar{f}$ and $\bar{g}$ are isomorphisms. Thus we have $\overline{gf} = \bar{g}\bar{f} \neq 0$.

Let $X, Y \in \mathsf{F}_i$ with $X \geq Y, Y \geq X$. We show that $X \cong Y$. Since $X \geq Y$ and $Y \geq X$, there exist $f : X \to Y$ and $g : Y \to X$ such that $\bar{f} \neq 0$ and $\bar{g} \neq 0$. Then $\bar{f}$ and $\bar{g}$ are isomorphisms. Suppose that gf is not an isomorphism. Then there exists $n \geq 1$ such that $(gf)^n = 0$ because X is indecomposable. Since ρ_X is surjective, we have that $(\bar{g}\bar{f})^n = 0$ and this is a contradiction. □

By Lemma 17, we enumerate the objects in F_i as follows: $X_i^1, X_i^2, \ldots, X_i^{m_i}$ such that $X_i^k < X_i^{k'}$ implies that $k < k'$, where m_i is the number of elements in F_i. We define the subset F_i^k of $\mathsf{ind}\mathcal{F}(\Delta)$ as

$$\mathsf{F}_i^k := \{X_i^1, \ldots, X_i^k\} \cup \mathsf{ind}\mathcal{F}(\Delta(\geq i+1)).$$

We put $\mathcal{F}_i^k := \mathsf{add} \bigoplus_{X \in \mathsf{F}_i^k} X$.

The following proposition is a main result of this subsection.

Proposition 18. *Let A be a quasi-hereditary algebra with n simple modules. Assume that $\mathcal{F}(\Delta)$ has an additive generator M and is multiplicity-free. Then the category $\mathcal{F}(\Delta)$ admits the following total right rejective chain*

$$\mathcal{F}(\Delta) = \mathcal{F}_1^{m_1} \supset \cdots \supset \mathcal{F}_1^1 \supset \mathcal{F}_2^{m_2} \supset \cdots \supset \mathcal{F}_n^{m_n} \supset \cdots \supset \mathcal{F}_n^1 \supset 0.$$

In particular, the endomorphism algebra $\mathrm{End}_A(M)$ is a right-strongly quasi-hereditary algebra.

Remark 19. In [10], Xi proved that the endomorphism algebra $\mathrm{End}_A(M)$ is a quasi-hereditary algebra, where M is an additive generator of $\mathcal{F}(\Delta)$ over a quasi-hereditary algebra A. If $\mathcal{F}(\Delta)$ is multiplicity-free, then Xi's quasi-hereditary structure coincides with our right-strongly quasi-hereditary structure. However, if $\mathcal{F}(\Delta)$ is not multiplicity-free, then $\mathrm{End}_A(M)$ is not necessarily right-strongly quasi-hereditary with respect to the Xi's quasi-hereditary structure in general.

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