THE STRUCTURE OF SALLY MODULES AND NORMAL HILBERT COEFFICIENTS

KAZUHO OZEKI

Abstract. The Sally module of an ideal is an important tool to interplay between Hilbert coefficients and the properties of the associated graded ring. In this talk we give new insights on the structure of the Sally module. We apply these results characterizing the almost minimal value of the first and the second normal Hilbert coefficients in an analytically unramified Cohen-Macaulay local ring.

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1. Introduction

This report is based on a joint work with S. K. Masuti, M. E. Rossi, and H. L. Truong.

Throughout this report, let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d > 0$ with infinite residue field $R/\mathfrak{m}$ and I an $\mathfrak{m}$-primary ideal of R. We say that, for an ideal J in R, the element $x \in R$ is integral over J, if there exist an integer $n > 0$ and elements $a_i \in J^i$ for $1 \leq i \leq n$ such that the equality

$$x^n + a_1 x^{n-1} + \cdots + a_i x^{n-i} + \cdots + a_n = 0$$

holds true. We set

$$\overline{J} = \{x \in R \mid x \text{ is integral over } J\}$$

and call it the integral closure of J. Consider the so called *normal filtration* $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ and we are interested in the corresponding Hilbert-Samuel polynomial. It is well-known that there are integers $\overline{\mathrm{e}}_i(I)$, called the *normal Hilbert coefficients* of I, such that for $n \gg 0$

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_0(I)\binom{n+d}{d} - \overline{\mathrm{e}}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d \overline{\mathrm{e}}_d(I).$$

Here $\ell_R(N)$ denotes, for an R-module N, the length of N. Since $R/\mathfrak{m}$ is infinite there exists a minimal reduction $J = (a_1, \ldots, a_d)$ of I and, under our assumptions, there exists an integer $r \geq 0$ such that $\overline{I^{n+1}} = J\overline{I^n}$ for all $n \geq r$. We set $\overline{\mathrm{r}}_J(I) := \min\{r \geq 0 \mid \overline{I^{n+1}} = J\overline{I^n} \text{ for all } n \geq r\}$ the normal reduction number of I with respect to J.

By [2, 3, 6, 8] it is known that

$$\overline{\mathrm{e}}_2(I) \geq \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) \geq \ell_R(\overline{I^2}/J\overline{I})$$

hold true and if either of the inequalities is an equality then $\overline{I^{n+1}} = J^{n-1}\overline{I^2}$ for every $n \geq 1$ (that is $\overline{\mathrm{r}}_J(I) \leq 2$). In this case the normal associated graded ring $\overline{G}(I)$ of I is

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Cohen-Macaulay (see also Corollary 4). We notice that $\ell_R(\overline{I^2}/J\overline{I})$ does not depend on a minimal reduction J of I (see for instance [14]).

Thus the ideals I with $\bar{\mathrm{e}}_1(I) = \bar{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I})$ and/or $\bar{\mathrm{e}}_2(I) = \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I})$ enjoy nice properties. In Section 2 we will introduce some auxiliary results on the structure of Sally module and explore the equality

$$\bar{\mathrm{e}}_1(I) = \bar{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1.$$

In Section 3 we will focus on the second normal Hilbert coefficient and investigate the equality

$$\bar{\mathrm{e}}_2(I) = \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1.$$

As the title outlines, an important tool in this report is the Sally module introduced by W. V. Vasconcelos [15]. The aim of this report was to define a module in between the associated graded ring and the Rees algebra taking care of important information coming from a minimal reduction. Actually, a more detailed information comes from the graded parts of a suitable filtration $\{C^{(\ell)}\}$ of the Sally module that was introduced by M. Vaz Pinto in [16]. In this report we introduce some important results on $\{C^{(2)}\}$ which will be key ingredients for proving the main result. Some of them are stated in a very general setting. Our hope is that these will be successfully applied to give new insights to problems related to the normal Hilbert coefficients, for instance [8].

2. Filtering the Sally module

In this section we study the Sally module associated to any I-admissible filtration $\mathcal{I}$. Following M. Vaz Pinto [16] we introduce a filtration of the Sally module, $C^{(\ell)}(\mathcal{I})$ for $\ell \geq 1$. This approach is extremely useful for relating the properties of the Hilbert coefficients and the graded module associated to an I-admissible filtration $\mathcal{I}$, as evidenced in [1], [12]. In [1] the authors analyzed the Sally module ($=C^{(1)}(\mathcal{I})$) of the normal filtration to study the equality $\bar{\mathrm{e}}_1(I) = \bar{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + 1$. In order to investigate the equality $\bar{\mathrm{e}}_1(I) = \bar{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$, in this section we prove some important properties of $C^{(2)}(\mathcal{I})$. These properties will play an important role in proving our main results.

We recall that $C^{(2)}(\mathcal{I})$ has been studied in [12] for I-adic filtration. For our purpose we need more deep results.

Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring (not necessarily analytically unramified) and I an $\mathfrak{m}$-primary ideal in R. Recall that a *a filtration* of ideals $\mathcal{I} := \{I_n\}_{n\in\mathbb{Z}}$ is a chain of ideals in R such that $R = I_0$ and $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{Z}$. We say that a filtration $\mathcal{I}$ is *I-admissible* if for all $m, n \in \mathbb{Z}$, $I_m \cdot I_n \subseteq I_{m+n}$, $I^n \subseteq I_n$ and there exists $k \in \mathbb{N}$ such that $I_n \subseteq I^{n-k}$ for all $n \in \mathbb{Z}$. It is well known that if R is analytically unramified, then $\{\overline{I^n}\}_{n\in\mathbb{Z}}$ is an I-admissible filtration.

For an I-admissible filtration $\mathcal{I} = \{I_n\}_{n\in\mathbb{Z}}$, let

$$\mathcal{R}(\mathcal{I}) = \sum_{i\geq 0} I^i t^i \subseteq R[t], \quad \mathcal{R}'(\mathcal{I}) = \sum_{i\in\mathbb{Z}} I^i t^i \subseteq R[t, t^{-1}], \quad \text{and} \quad G(\mathcal{I}) = \mathcal{R}'(\mathcal{I})/t^{-1}\mathcal{R}'(\mathcal{I})$$

denote, respectively, the Rees algebra, the extended Rees algebra, and the associated graded ring of $\mathcal{I}$ where t is an indeterminate over R. We set

$$\overline{\mathcal{R}}(I) := \sum_{n\geq 0} \overline{I^n}t^n \subseteq R[t], \quad \overline{\mathcal{R}'}(I) := \sum_{n\in\mathbb{Z}} \overline{I^n}t^n \subseteq R[t,t^{-1}], \quad \text{and}\, \overline{G}(I) := \overline{\mathcal{R}'}(I)/t^{-1}\overline{\mathcal{R}'}(I)$$

for the Rees algebra, the extended Rees algebra and the associated graded ring of $\{\overline{I^n}\}_{n\in\mathbb{Z}}$, respectively.

Since $R/\mathfrak{m}$ is infinite there exists a minimal reduction $J = (a_1, a_2, \ldots, a_d)$ of $\mathcal{I}$, that is there exists an integer $r \in \mathbb{Z}$ such that the equality $I_{n+1} = JI_n$ holds true for all $n \geq r$. Let

$$\mathrm{r_J}(\mathcal{I}) := \min\{\mathrm{r} \geq 0 \mid I_{n+1} = JI_n \text{ hold true for all } n \geq \mathrm{r}\ \}$$

be the *reduction number* of $\mathcal{I}$ with respect to J. We set

$$T := \mathcal{R}(J) := \mathcal{R}(\{J^n\}_{n\in\mathbb{Z}})$$

and $\mathcal{M} = \mathfrak{m}T + T_+$ denotes the graded maximal ideal of T. Then $\mathcal{R}(\mathcal{I})$ is a module finite extension of T. Hence there exist integers $\mathrm{e}_i(\mathcal{I})$, called as the *Hilbert coefficients* of $\mathcal{I}$ such that the equality

$$\ell_R(R/I_{n+1}) = \mathrm{e}_0(\mathcal{I})\binom{n+d}{d} - \mathrm{e}_1(\mathcal{I})\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_d(\mathcal{I})$$

holds true for all $n \gg 0$ (c.f. [11, Proposition 3.1]). This polynomial is called the *Hilbert-Samuel polynomial* of $\mathcal{I}$. For a graded T-module E and $\alpha \in \mathbb{Z}$ we denote by $E(\alpha)$ the graded T-module whose grading is given by $[E(\alpha)]_n = E_{\alpha+n}$ for all $n \in \mathbb{Z}$.

Following Vasconcelos [15], we consider

$$S_J(\mathcal{I}) := \frac{\mathcal{R}(\mathcal{I})_{\geq 1}t^{-1}}{I_1T} \cong \bigoplus_{n\geq 1} I_{n+1}/J^nI_1$$

the *Sally module* of $\mathcal{I}$ with respect to J. Notice that $S_J(\mathcal{I})$ is a finite T-module. In [16] Vaz Pinto introduced a filtration of the Sally module in the case $\mathcal{I} = \{I^n\}_{n\in\mathbb{Z}}$. Following this line, we extend the definition to any I-admissible filtration $\mathcal{I}$.

Definition 1. For each $\ell \geq 1$, consider the T-module

$$C_J^{(\ell)}(\mathcal{I}) := \frac{\mathcal{R}(\mathcal{I})_{\geq \ell}t^{-1}}{I_\ell T t^{\ell-1}} \cong \bigoplus_{n\geq \ell} I_{n+1}/J^{n-\ell+1}I_\ell.$$

Let $L_J^{(\ell)}(\mathcal{I}) = [C^{(\ell)}(\mathcal{I})]_\ell T$ be the T-submodule of $C^{(\ell)}(\mathcal{I})$. Then

$$L_J^{(\ell)}(\mathcal{I}) \cong \bigoplus_{n\geq \ell} J^{n-\ell}I_{\ell+1}/J^{n-\ell+1}I_\ell.$$

Hence for every $\ell \geq 1$ we have the following natural exact sequence of graded T-modules

$$0 \to L_J^{(\ell)}(\mathcal{I}) \to C_J^{(\ell)}(\mathcal{I}) \to C_J^{(\ell+1)}(\mathcal{I}) \to 0.$$

Throughout this section we set

$$S := S_J(\mathcal{I}),\ C^{(\ell)} := C_J^{(\ell)}(\mathcal{I}) \text{ and } L^{(\ell)} := L_J^{(\ell)}(\mathcal{I})$$

unless otherwise specified. Notice that $C^{(1)} = S$, and since $\mathcal{R}(\mathcal{I})$ is a finite graded T-module, $C^{(\ell)}$ and $L^{(\ell)}$ are finitely generated graded T-modules for every $\ell \geq 1$.

Let us begin with the following lemma.

Lemma 2. *Let $\ell \geq 1$ be an integer. Then the following assertions hold true.*

(1) $\mathfrak{m}^k C^{(\ell)} = (0)$ *for integers $k \gg 0$; hence* $\dim_T C^{(\ell)} \leq d$.

(2) $C^{(\ell)} = (0)$ *if and only if* $\mathrm{r}_J(\mathcal{I}) \leq \ell$.

In this report, the structure of the graded module $C^{(2)}$ plays an important role. We derive some basic properties of $C^{(2)}$ which we need.

In the following result we need that $J \cap I_2 = JI_1$ holds true. This condition is automatically satisfied if $\mathcal{I} = \{\mathfrak{m}^n\}_{n\in\mathbb{Z}}$ or if $\mathcal{I} = \{\overline{I^n}\}_{n\in\mathbb{Z}}$ (see [4, 7, 9]). We also notice that

$$\ell_R(I_2/JI_1) = \mathrm{e}_0(\mathcal{I}) + (d-1)\ell_R(R/I_1) - \ell_R(I_1/I_2)$$

holds true (see for instance [14, Corollary 2.1]), so that $\ell_R(I_2/JI_1)$ does not depend on a minimal reduction J of $\mathcal{I}$. We remark that the following Proposition 3 was proved in [12, Propositions 2.2, 2.8, and 2.9] in the case $\mathcal{I} = \{I^n\}_{n\in\mathbb{Z}}$.

Proposition 3. *Let $\mathcal{P} = \mathfrak{m}T$ and suppose that $J \cap I_2 = JI_1$. Then the following assertions hold true.*

(1) $\mathrm{Ass}_T\ (C^{(2)}) \subseteq \{\mathcal{P}\}$. *Hence* $\dim_T C^{(2)} = d$, *if* $C^{(2)} \neq (0)$.

(2) *For all $n \geq 0$,*

$$\begin{aligned}\ell_R(R/I_{n+1}) &= \mathrm{e}_0(\mathcal{I})\binom{n+d}{d} - \{\mathrm{e}_0(\mathcal{I}) - \ell_R(R/I_1) + \ell_R(I_2/JI_1)\}\binom{n+d-1}{d-1} \\ &\quad + \ell_R(I_2/JI_1)\binom{n+d-2}{d-2} - \ell_R([C^{(2)}]_n).\end{aligned}$$

(3) $\mathrm{e}_1(\mathcal{I}) = \mathrm{e}_0(\mathcal{I}) - \ell_R(R/I_1) + \ell_R(I_2/JI_1) + \ell_{T_\mathcal{P}}(C^{(2)}_\mathcal{P})$.

(4) *Suppose $C^{(2)} \neq (0)$ and let $c = \mathrm{depth}_T\, C^{(2)}$. Then* depth $G(\mathcal{I}) = c-1$, *if $c < d$. Moreover,* depth $G(\mathcal{I}) \geq d-1$ *if and only if $C^{(2)}$ is a Cohen-Macaulay T-module.*

Combining Proposition 3 (1), (3), (4), and Lemma 2 (2), and using the Valabrega-Valla criterion (c.f.[14, Theorem 1.1]) we obtain the following result that was proven by Elias and Valla [2, Theorem 2.1] in the case $\mathcal{I} = \{\mathfrak{m}^n\}_{n\in\mathbb{N}}$ and by Guerrieri and Rossi [3, Theorem 2.2 and Proof of Proposition 2.3] for any I-admissible filtration.

Corollary 4. *Suppose that $J \cap I_2 = JI_1$. Then we have*

$$\mathrm{e}_1(\mathcal{I}) \geq \mathrm{e}_0(\mathcal{I}) - \ell_R(R/I_1) + \ell_R(I_2/JI_1).$$

The equality $\mathrm{e}_1(\mathcal{I}) = \mathrm{e}_0(\mathcal{I}) - \ell_R(R/I_1) + \ell_R(I_2/JI_1)$ holds true if and only if $\mathrm{r_J}(\mathcal{I}) \leq 2$. When this is the case, the following assertions hold true:

(*i*) *If $d \geq 2$, then $\mathrm{e}_2(\mathcal{I}) = \ell_R(I_2/JI_1) = \mathrm{e}_1(\mathcal{I}) - \mathrm{e}_0(\mathcal{I}) + \ell_R(R/I_1)$ and $\mathrm{e}_i(\mathcal{I}) = 0$ for all $3 \leq i \leq d$, and*

(ii) $G(\mathcal{I})$ *is a Cohen-Macaulay ring, and so is* $\mathcal{R}(\mathcal{I})$ *if* $d \geq 3$.

We now prove an important property of $C^{(2)}$ in Proposition 5 which plays a crucial role in the proof of the main result.

Proposition 5. *Let $d \geq 3$ and $0 \leq n \leq d-1$. Suppose that $J \cap I_2 = JI_1$ and $\mathcal{R}(\mathcal{I})$ satisfies Serre's property (S_n) as a T-module. Then $C^{(2)}$ also satisfies Serre's property (S_n) (as T-module).*

We set $\overline{C} := C_J^{(2)}(\{\overline{I^n}\}_{n\in\mathbb{Z}})$ and $B := T/\mathfrak{m}T \cong (R/\mathfrak{m})[X_1, X_2, \cdots, X_d]$ the polynomial ring with d indeterminates over the field $R/\mathfrak{m}$. Then thanks to Proposition 5, the module $\overline{C}$ satisfies Serre's property (S_2).

The first main result of this report is stated as follows.

Theorem 6. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d > 0$ and I an $\mathfrak{m}$-primary ideal in R, Then following statements are equivalent:*

(1) $\overline{\mathrm{e}}_1(I) = \overline{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$;
(2) $\overline{C} \simeq B(-m)$ *as graded T-modules for some* $m \geq 2$;
(3) $\ell_R(\overline{I^{m+1}}/J\overline{I^m}) = 1$ *and* $\overline{I^{n+1}} = J\overline{I^n}$ *for all* $2 \leq n \leq m-1$ *and* $n \geq m+1$ *for some* $m \geq 2$.

In this case, the following assertions follow:

(i) $\overline{\mathrm{r}}_J(I) = m+1$.
(ii) $\overline{\mathrm{e}}_2(I) = \ell_R(\overline{I^2}/J\overline{I}) + m$ *and* $\overline{\mathrm{e}}_i(I) = \binom{m}{i-1}$ *for* $3 \leq i \leq d$.
(iii) depth $\overline{G}(I) \geq d-1$.
(iv) $\overline{G}(I)$ *is Cohen-Macaulay if and only if* $\overline{I^3} \nsubseteq J$. *In this case, we have* $m = 2$.

3. The structure of the Sally module when $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$

In this section let us introduce the structure theorem of the Sally module with the equality $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$. We set $\overline{C} = \overline{C}_J(I) = C_J^{(2)}(\{\overline{I^n}\}_{n\in\mathbb{Z}})$.

In the following theorem we recall few results on the vanishing of local cohomology modules from [8] (see also [4]). From now onwards we set $M' = (t^{-1}, \mathfrak{m}, It)\mathcal{R}'(I)$ for the graded maximal ideal of $\mathcal{R}'(I) := \mathcal{R}'(\{I^n\}_{n\in\mathbb{Z}})$ and $N' = It\mathcal{R}'(I)$.

Theorem 7. ([8, Proposition 13]) *Suppose that $d \geq 2$. Then we have the following.*

(1) $[\mathrm{H}^i_{N'}(\overline{\mathcal{R}'}(I))]_n = (0)$ *for all* $n \gg 0$ *and all* $i \geq 0$;
(2) $\mathrm{H}^0_{M'}(\overline{\mathcal{R}'}(I)) = \mathrm{H}^1_{M'}(\overline{\mathcal{R}'}(I)) = (0)$;
(3) $[\mathrm{H}^2_{M'}(\overline{\mathcal{R}'}(I))]_n = (0)$ *for* $n \leq 0$;
(4) $\mathrm{H}^i_{M'}(\overline{\mathcal{R}'}(I)) \cong \mathrm{H}^i_{N'}(\overline{\mathcal{R}'}(I))$ *for* $0 \leq i \leq d-1$.

To prove the main result of this section we use induction on the dimension d. One of the main difficulties in applying the induction on d for the normal filtration is that the image of a normal ideal going modulo a superficial element need not be normal. Thanks to [8, Theorem 1] (see also [4]) we may choose $a_1 \in I$ such that $\overline{I(R/(a_1))} = \overline{I}(R/(a_1))$, and $\overline{I^n(R/(a_1))} = \overline{I^n}(R/(a_1))$ for all $n \gg 0$. In particular, a_1t is $\overline{G}(I)$-regular. From now onwards we set $S = R/(a_1)$.

We remark that the following result works like the Sally's machine [14, Lemma 1.4], but is not a consequence of it.

Proposition 8. *Assume $d \geq 3$ and* depth $\overline{\mathrm{G}}(IS) \geq 2$. *Then we have* depth $\overline{\mathrm{G}}(I) =$ depth $\overline{\mathrm{G}}(IS) + 1$.

The following result plays a key role for our proof of Theorem 10. Thanks to Proposition 8, we need only to show the case where $d \leq 3$ for the proof of Theorem 9.

Theorem 9. *Suppose $d \geq 2$. Assume $\bar{\mathrm{e}}_2(I) = \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ and $\bar{\mathrm{e}}_3(I) \neq 0$ (if $d \geq 3$), then $\ell_R(\overline{I^3}/J\overline{I^2}) = 1$ and $\overline{I^{n+1}} = J\overline{I^n}$ for all $n \geq 3$. We then furthermore have* depth$\overline{\mathrm{G}}(I) \geq d - 1$.

Now we give a complete structure of the Sally module and we describe the Hilbert series of the associated graded ring in the case $\bar{\mathrm{e}}_2(I) = \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ and $\bar{\mathrm{e}}_3(I) \neq 0$.

Theorem 10. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ and I an $\mathfrak{m}$-primary ideal in R. Suppose that $d \geq 2$. Then following statements are equivalent:*

(1) $\bar{\mathrm{e}}_2(I) = \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ *and, if* $d \geq 3$, $\bar{\mathrm{e}}_3(I) \neq 0$,
(2) $\bar{\mathrm{e}}_2(I) = \ell_R(\overline{I^2}/J\overline{I}) + 2$,
(3) $\overline{C}_J(I) \cong B(-2)$ *as graded T-modules, and*
(4) $\ell_R(\overline{I^3}/J\overline{I^2}) = 1$ *and* $\overline{I^{n+1}} = J\overline{I^n}$ *for all* $n \geq 3$.

In this case, the following assertions follow:

(*i*) $\bar{\mathrm{e}}_1(I) = \bar{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 1$.
(*ii*) $\bar{\mathrm{e}}_3(I) = 1$ *if* $d \geq 3$, *and* $\bar{\mathrm{e}}_i(I) = 0$ *for* $4 \leq i \leq d$.
(*iii*) depth $\overline{G}(I) \geq d - 1$, *and $\overline{G}(I)$ is Cohen-Macaulay if and only if* $\overline{I^3} \nsubseteq J$.

By the above result we notice that if $d \geq 3$ and $\bar{\mathrm{e}}_2(I) \leq \bar{\mathrm{e}}_1(I) - \bar{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$, then $\bar{\mathrm{e}}_3(I) \leq 1$.

The following example, due to Huckaba and Huneke [5, Theorem 3.12], shows that if I is normal, $\mathrm{e}_2(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_R(R/I) + 1$ and $\mathrm{e}_3(I) \neq 0$, then $G(I)$ need not be Cohen-Macaulay and hence Theorem 10 is sharp.

Example 11. Let K be a field of characteristic $\neq 3$ and set $R = K[\![X, Y, Z]\!]$ the formal power series ring over K, where X, Y, Z are indeterminates. Let $N = (X^4, X(Y^3 + Z^3), Y(Y^3 + Z^3), Z(Y^3 + Z^3))$ and set $I = N + \mathfrak{m}^5$, where $\mathfrak{m}$ is the maximal ideal of R. Then the ideal I is a normal $\mathfrak{m}$-primary ideal whose associated graded ring $G(I)$ has depth $d - 1 = 2$. Moreover,

$$HS_{G(I)}(t) = \frac{31 + 43t + t^2 + t^3}{(1-t)^3},$$

and hence $\ell_R(R/I) = 31,\ \mathrm{e}_0(I) = 76,\ \mathrm{e}_1(I) = 48,\ \mathrm{e}_2(I) = 4,\ \mathrm{e}_3(I) = 1$. Thus $\mathrm{e}_2(I) = \mathrm{e}_1(I) - \mathrm{e}_0(I) + \ell_R(R/I) + 1$. For the computations see [1, Example 3.2].

4. The structure of the Sally module when $\overline{\mathrm{e}}_3(I) = 0$

In the rest of this report let us consider the case where $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ and $\overline{\mathrm{e}}_3(I) = 0$ in three dimensional case.

This case faces the difficult problem stated by Itoh in [8] on the vanishing of $\overline{\mathrm{e}}_3(I)$ which asserts that if $\overline{\mathrm{e}}_3(I) = 0$ and R is Gorenstein, then $\overline{G}(I)$ is Cohen-Macaulay or equivalently $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I})$. Hence for the class of ideals verifying Itoh's conjecture the assumptions of this section doesn't occur. This is the case for instance when $\overline{I} = \mathfrak{m}$ and R is Gorenstein, see [8, Theorem 3(2)] (more generally, R satisfying $\ell_R(\overline{I^2}/J\overline{I}) \geq \mathrm{type}(R) - 2$, see [1]). If R is not Gorenstein or R is Gorenstein and $\overline{I} \neq \mathfrak{m}$, our analysis can be useful for proving or disproving Itoh's conjecture, also because the doubt of the validity of Itoh's conjecture is growing among the experts.

In Theorem 14 we prove that if $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ and $\overline{\mathrm{e}}_3(I) = 0$, then $\overline{G}(I^\ell)$ is Cohen-Macaulay for all $\ell \geq 2$. For this purpose we need the following proposition which is a consequence of Serre's formula and it seems to be well known. We set, for $\ell \in \mathbb{Z}$, $\mathcal{I}^{(\ell)} = \{I_{n\ell}\}_{n\in\mathbb{Z}}$, and $\mathrm{a}_i(\mathrm{G}(\mathcal{I})) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_M(\mathrm{G}(\mathcal{I}))]_n \neq (0)\}$ for $i \in \mathbb{Z}$.

Proposition 12. *Let $\ell > \max\{\mathrm{a}_i(\mathrm{G}(\mathcal{I})) \,|\, 0 \leq i \leq d\}$ be an integer. Then we have $\ell_R(R/I_{\ell(n+1)}) = \sum_{i=0}^{d}(-1)^i \mathrm{e}_i(\mathcal{I}^{(\ell)})\binom{n+d-i}{d-i}$ for all $n \geq 0$. In particular, the equality $\ell_R(R/I_\ell) = \sum_{i=0}^{d}(-1)^i\mathrm{e}_i(\mathcal{I}^{(\ell)})$ holds true for all $n \geq 0$.*

As a consequence of Proposition 12 we obtain a result of Rees [13, Theorem 2.6] (see also [7, Theorem 4.5]) in dimension two which states that: $\overline{\mathrm{e}}_2(I) = 0$ if and only if $\overline{\mathrm{e}}_1(I^\ell) = \mathrm{e}_0(I^\ell) - \ell_R(R/\overline{I^\ell})$ for all $\ell \geq 1$. In particular, by [14, Theorem 2.9], $\overline{\mathrm{G}}(I^\ell)$ is Cohen-Macaulay for all $\ell \geq 1$.

Analogously we obtain the following result on vanishing of $\overline{\mathrm{e}}_3(I)$ in dimension three as a consequence of Proposition 12. Notice that next result for normal ideals can be also obtained as a consequence of [10, Corollary 5.3.]

Corollary 13. *Let $d = 3$, then the following conditions are equivalent.*

1. $\overline{\mathrm{e}}_3(I) = 0$,
2. $\overline{\mathrm{e}}_1(I^\ell) = 2\overline{\mathrm{e}}_0(I^\ell) + \ell_R(R/\overline{I^\ell}) - \ell_R(\overline{I^\ell}/\overline{I^{2\ell}})$ *for some (equiv. all)* $\ell > \max\{\mathrm{a}_i(\overline{\mathrm{G}}(I)) \mid 1 \leq i \leq 3\}$, *and*
3. $\overline{\mathrm{e}}_2(I^\ell) = \overline{\mathrm{e}}_1(I^\ell) - \overline{\mathrm{e}}_0(I^\ell) + \ell_R(R/\overline{I^\ell})$ *for some (equiv. all)* $\ell > \max\{\mathrm{a}_i(\overline{\mathrm{G}}(I)) \,|\, 1 \leq i \leq 3\}$.

In particular, $\overline{\mathrm{G}}(I^\ell)$ is Cohen-Macaulay for all $\ell > \max\{\mathrm{a}_i(\overline{\mathrm{G}}(I)) \mid 1 \leq i \leq 3\}$ if any of the above conditions are satisfied.

The last assertion of Corollary 13 is a consequence of [8, Theorem 2(2)]. Now we are ready to introduce the main theorem of this section.

Theorem 14. *Let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension three and let I be an $\mathfrak{m}$-primary ideal in R. Suppose $d = 3$. Then the following conditions are equivalent.*

1. $\overline{\mathrm{e}}_2(I) = \overline{\mathrm{e}}_1(I) - \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I}) + 1$ *and* $\overline{\mathrm{e}}_3(I) = 0$*;*

(2) *there exists an exact sequence*

$$0 \to B(-3) \to B(-2)^{\oplus 3} \to \overline{C} \to 0$$

of graded T-modules.

When this is the case, the following assertions hold true:

(*i*) $\mathfrak{m}\overline{C} = (0)$ *and* $\operatorname{rank}_B \overline{C} = 2$, *and* $\operatorname{depth}_T \overline{C} = 2$,

(*ii*) $\mathfrak{m}\overline{I^3} \subseteq J\overline{I^2}$, $\ell_R(\overline{I^3}/J\overline{I^2}) = 3$, $\overline{I^{n+1}} = J\overline{I^n}$ *for all* $n \geq 3$,

(*iii*) $\overline{\mathrm{e}}_1(I) = \overline{\mathrm{e}}_0(I) - \ell_R(R/\overline{I}) + \ell_R(\overline{I^2}/J\overline{I}) + 2$ *and* $\overline{\mathrm{e}}_2(I) = \ell_R(\overline{I^2}/J\overline{I}) + 3$,

(*iv*) $\operatorname{depth} \overline{\mathrm{G}}(I) = 1$, *and* $\mathrm{H}^1_{M'}(\overline{\mathrm{G}}(I)) = [\mathrm{H}^1_{M'}(\overline{\mathrm{G}}(I))]_0$, $\ell_R([\mathrm{H}^1_{M'}(\overline{\mathrm{G}}(I))]_0) = 1$, $\mathrm{a}_2(\overline{\mathrm{G}}(I)) = 1$, *and* $\mathrm{a}_3(\overline{\mathrm{G}}(I)) \leq -1$,

(*v*) $\overline{\mathrm{G}}(I^\ell)$ *is Cohen-Macaulay for all* $\ell \geq 2$.

References

[1] A. Corso, C. Polini and M. E. Rossi, *Bounds on the normal Hilbert coefficients*, Proc. Amer. Math. Soc. **144** (2016), 1919–1930.

[2] J. Elias and G. Valla, *Rigid Hilbert functions*, J. Pure and Appl. Algebra **71** (1991) 19–41.

[3] A. Guerrieri and M. E. Rossi, *Hilbert coefficients of Hilbert filtrations*, J. Algebra **199** (1998) 40–61.

[4] J. Hong and B. Ulrich, *Specialization and integral closure*, J. Lond. Math. Soc. (2) **90** (2014), 861–878.

[5] S. Huckaba and C. Huneke, *Normal ideals in regular rings*, J. Reine Angew. Math. **510** (1999), 63–82.

[6] S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. London Math. Soc. (2) **56** (1997), 64–76.

[7] C. Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987) 293–318.

[8] S. Itoh, *Coefficients of normal Hilbert polynomials*, J. Algebra **150** (1992), 101–117.

[9] S. Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), 390–401.

[10] J. Lipman and B. Teissier, *Pseudo-rational local rings and a theorem of Brianҫn-Skoda about integral closure of ideals*, Michgan Math. J. **28** (1981), 97–116.

[11] T. Marley, *Hilbert functions of ideals in Cohen-Macaulay rings*, Ph. D. Thesis, Purdue University, (1989).

[12] K. Ozeki and M. E. Rossi, *The structure of the Sally module of integrally closed ideals*, Nagoya Math. J. **227** (2017), 49–76.

[13] D. Rees, *Hilbert functions and pseudo-rational local rings of dimension two*, J. London Math. Soc. **24** (1981), 467–479.

[14] M. E. Rossi and G. Valla *Hilbert functions of filtered modules*, UMI Lecture Notes 9, Springer (2010).

[15] W. V. Vasconcelos, *Hilbert Functions, Analytic Spread, and Koszul Homology*, Contemporary Mathematics, Vol **159** (1994) 410–422.

[16] M. Vaz Pinto, *Hilbert functions and Sally modules*, J. Algebra, **192** (1996) 504–523.

Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
1677-1 Yoshida Yamaguchi 753-8512 JAPAN

Email address: `mail@address`