HAPPEL'S FUNCTOR AND HOMOLOGICALLY WELL-GRADED IWANAGA-GORENSTEIN ALGEBRAS

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Abstract. In representation theory of algebras, derived category and stable category are two major classes of triangulated categories. It has been shown by many researchers that those different kinds of triangulated categories are related in various cases.

In this article, for a finitely graded Iwanaga-Gorenstein algebra A, we consider the functor $\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\nabla A) \to \underline{\mathsf{CM}}^{\mathbb{Z}} A$ from the bounded derived category of the category of modules over the Beilinson algebra ∇A of A to the stable category of $\mathbb{Z}$-graded Cohen-Macaulay A-modules. We study when it is fully faithful or gives an equivalence.

1. Introduction

This is a short summary of the paper [6]. In this article, we study some functor from the derived category to the stable category. Our starting point is the famous Happel's result. For a finite dimensional algebra Λ over a field K, one has the trivial extension $T(\Lambda) = \Lambda \oplus D\Lambda$ where $D = \mathrm{Hom}_K(-, K)$. $T(\Lambda)$ is a $\mathbb{Z}$-graded self-injective algebra, and so the stable category $\underline{\mathsf{mod}}^{\mathbb{Z}}\, T(\Lambda)$ of $\mathbb{Z}$-graded $T(\Lambda)$-modules has a canonical structure of triangulated category. In this setting, Happel showed the following theorem.

Theorem 1 ([3]). *There is a fully faithful triangle functor*

$$F : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \to \underline{\mathsf{mod}}^{\mathbb{Z}}\, T(\Lambda).$$

Moreover it gives an equivalence if and only if Λ is of finite global dimension.

The Happel's functor F can be generalized. Let us start from the stable category side. Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra with $A_\ell \neq 0$. Assume that A is *Iwanaga-Gorenstein*, which means that both $\operatorname{inj.dim} A_A$ and $\operatorname{inj.dim} {}_A A$ are finite. In this case, the category

$$\mathsf{CM}^{\mathbb{Z}} A = \{M \in \mathsf{mod}^{\mathbb{Z}} A \mid \mathrm{Ext}_A^{>0}(M, A) = 0\}$$

of *Cohen-Macaulay* $\mathbb{Z}$-graded A-modules is a Frobenius category, and so the stable category

$$\underline{\mathsf{CM}}^{\mathbb{Z}} A := \mathsf{CM}^{\mathbb{Z}} A/[\mathsf{proj}^{\mathbb{Z}} A]$$

has a canonical structure of triangulated category [3]. Note that if A is self-injective, then $\mathsf{CM}^{\mathbb{Z}} A = \mathsf{mod}^{\mathbb{Z}} A$, and so $\underline{\mathsf{CM}}^{\mathbb{Z}} A = \underline{\mathsf{mod}}^{\mathbb{Z}} A$ hold.

The detailed version of this paper will be submitted for publication elsewhere.

Next we explain the derived category side. In [2, 8], the *Beilinson algebra* ∇A of A is defined as a subalgebra

$$\nabla A := \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{\ell-2} & A_{\ell-1} \\ & A_0 & A_1 & \cdots & A_{\ell-3} & A_{\ell-2} \\ & & & \cdots & \cdots & \cdots \\ & & & & A_0 & A_1 \\ O & & & & & A_0 \end{pmatrix}$$

of the algebra $\mathrm{M}_\ell(A)$ of $\ell \times \ell$ matrices over A. We consider the bounded derived category

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A)$$

of the category $\mathsf{mod}\, \nabla A$ of ∇A-modules.

Then as we will show in Section2, there is always a triangle functor

$$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \to \underline{\mathsf{CM}}^{\mathbb{Z}}\, A.$$

This functor coincides with the functor F in the case $A = T(\Lambda)$. Hence $\mathcal{H}$ can be regarded as a generalization of F.

Theorem 1 asserts that there is a large class of finitely graded Iwanaga-Gorenstein algebras A such that $\mathcal{H}$ is fully faithful or gives an equivalence. It is natural to ask the following question.

Question. When $\mathcal{H}$ is fully faithful or gives an equivalence ?

The purpose of this article is to give an answer to this question. In the case A is self-injective, the answer had been given by [2, 5, 10]. It is recalled in Section 3. In [6], we gave the answer to the question for the general case. In Section 4, we state it and give examples.

Notations. Throughout this article, an algebra means a finite dimensional algebra over a filed K. For an algebra A, we always deal with finitely generated right A-modules.

When A is a $\mathbb{Z}$-graded algebra, we use the following notations. For $\mathbb{Z}$-graded A-modules M and N, we denote by

$$\mathrm{Hom}_A^{\mathbb{Z}}(M, N) := \{ f \in \mathrm{Hom}_A(M, N) \mid f(M_i) \subset N_i \text{ for all } i \in \mathbb{Z} \}$$

the morphism space from M to N in $\mathsf{mod}^{\mathbb{Z}}\, A$. For an integer k and a $\mathbb{Z}$-graded A-module $M = \bigoplus_{i \in \mathbb{Z}} M_i$, we denote by $M(k)$ the k-th degree shift of M, that is the $\mathbb{Z}$-graded A-module defined by $M(k) := M$ with $M(k)_i := M_{i+k}$ for all $i \in \mathbb{Z}$. The truncation $M_{\geq k}$ is a $\mathbb{Z}$-graded submodule of M defined by $(M_{\geq k})_i = M_i$ for $i \geq k$ and $(M_{\geq k})_i = 0$ for $i < k$.

2. Definition of Happel's functor

Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra with $A_\ell \neq 0$. In this section, we give a precise definition of the functor $\mathcal{H}$. It is defined as a composite of several functors below.

(I) Let us consider an abelian full subcategory

$$\mathsf{mod}^{[0,\ell)}\, A := \left\{ M \in \mathsf{mod}^{\mathbb{Z}}\, A \;\middle|\; M_i = 0 \text{ for all } i \notin [0, \ell) \right\}$$

of $\mathsf{mod}^{\mathbb{Z}} A$. This abelian category has a canonical projective generator

$$T = \bigoplus_{i=1}^{\ell} (A/A_{\geq i})(i-\ell).$$

It holds that $\operatorname{End}_A^{\mathbb{Z}}(T) \simeq \nabla A$. Therefore by Morita theory, there is an equivalence

$$p = \operatorname{Hom}_A^{\mathbb{Z}}(T, -) : \mathsf{mod}^{[0,\ell)} A \xrightarrow{\simeq} \mathsf{mod}\, \nabla A.$$

This induces a triangle equivalence

$$p^{-1} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \xrightarrow{\simeq} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{[0,\ell)} A),$$

and we have a fully faithful triangle functor

$$\mathcal{P} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \xrightarrow{p^{-1}} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{[0,\ell)} A) \hookrightarrow \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A).$$

(II) The derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A)$ contains the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}} A)$ of bounded complexes of $\mathbb{Z}$-graded projective A-modules as a thick subcategory. Buchweitz [1] introduced the *stable derived category* as the Verdier quotient

$$\mathsf{Sing}^{\mathbb{Z}}(A) := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A) / \mathsf{K}^{\mathrm{b}}(\mathsf{proj}^{\mathbb{Z}} A).$$

It is also called the *singularity category* in [9]. We denote by

$$\mathcal{Q} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A) \to \mathsf{Sing}^{\mathbb{Z}}(A)$$

the quotient functor.

(III) Assume that A is Iwanaga-Gorenstein. Then the singularity category is related to the stable category of Cohen-Macaulay modules. Buchweitz [1] showed that there is a natural triangle equivalence

$$\beta : \underline{\mathsf{CM}}^{\mathbb{Z}} A \xrightarrow{\simeq} \mathsf{Sing}^{\mathbb{Z}}(A).$$

Now we are ready to define the functor $\mathcal{H}$.

Definition 2. Assume that A is Iwanaga-Gorenstein. Then we define $\mathcal{H}$ as the composite

$$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \xrightarrow{\mathcal{P}} \mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A) \xrightarrow{\mathcal{Q}} \mathsf{Sing}^{\mathbb{Z}}(A) \xrightarrow{\beta^{-1}} \underline{\mathsf{CM}}^{\mathbb{Z}} A.$$

We call $\mathcal{H}$ *Happel's functor*.

3. Known Results: the case A is self-injective

Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra with $A_\ell \neq 0$. In this section, we recall the answer to the question for the case A is self-injective. The following condition plays a key role.

Definition 3. We call A *right strictly well-graded* if it satisfies

$$\operatorname{Hom}_A^{\mathbb{Z}}(A_0, A(i)) = 0$$

for all $i \neq \ell$. It is obvious that this condition is equivalent to $\operatorname{Soc}(A_A) \subset A_\ell$.

If A is right and left strictly well-graded, then A is called strictly well-graded.

In [2], Chen called A *right well-graded* if for each nonzero idempotent $e \in A_0$, $eA_\ell \neq 0$ holds. It is easy to see that A is right well-graded if it is right strictly well-graded, but the converse does not hold. Note that if A is self-injective, then it is right well-graded if and only if it is right strictly well-graded.

The following proposition was observed in [2].

Proposition 4 ([2])**.** *The following conditions are equivalent.*

(a) *A is right strictly well-graded self-injective.*
(b) *A is left strictly well-graded self-injective.*
(c) *A is graded Frobenius, i.e. $A(\ell) \simeq DA$ in $\mathsf{mod}^{\mathbb{Z}} A$.*

Now we state the answer to the question for the case A is self-injective.

Theorem 5 ([2, 5, 10])**.** *Assume that A is self-injective. Then the following assertions hold.*

(1) *$\mathcal{H}$ is fully faithful if and only if A is strictly well-graded.*
(2) *$\mathcal{H}$ is an equivalence if and only if A is strictly well-graded and A_0 is of finite global dimension.*

We mention that if part was shown in [2, Corollary 1.2.] and [5, Theorem 4.22.(4)], and only if part can be proved by using the argument in [10, Section 3.2.].

It is natural to ask whether strictly well-gradedness implies fully faithfulness of $\mathcal{H}$ even if inj.dim $A \geq 1$. The following example tells us that it does not. Hence we need to modify strictly well-gradedness in order for $\mathcal{H}$ to be fully faithful.

Example 6. Let

$$A = KQ/(a^2 - bc, ab, cb, ca)$$

where Q is the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "a"] \arrow[r, bend left, "b"] & 2 \arrow[l, bend left, "c"] \end{tikzcd} .$$

The indecomposable projective A-modules P_1, P_2 and indecomposable injective A-modules I_1, I_2 are as follows.

$$P_1 = \begin{smallmatrix} & 1 & \\ 1 & & 2 \\ & 1 & \end{smallmatrix} \qquad P_2 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \qquad I_1 = P_1 \qquad P_2 = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$$

It is easy to see that A is an Iwanaga-Gorenstein algebra of inj.dim $A_A =$ inj.dim ${}_A A = 2$. Now we define a $\mathbb{Z}$-grading of A by $\deg a = 1$, $\deg b = 0$ and $\deg c = 2$. Then A becomes a strictly well-graded algebra.

In this example, the functor $\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A) \to \underline{\mathsf{CM}}^{\mathbb{Z}} A$ is not fully faithful. Indeed we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}} A = \mathsf{add} \left\{ \begin{smallmatrix} 1 & & 2 \\ & 1 & \end{smallmatrix} (i) \;\middle|\; i \in \mathbb{Z} \right\} \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, K).$$

On the other hands, ∇A is isomorphic to the algebra $KQ'/(\alpha\beta)$ where Q' is the quiver

$$\bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \xleftarrow{\gamma} \bullet .$$

It is obvious that there is no fully faithful functor from $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \nabla A)$ to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, K)$. Thus $\mathcal{H}$ is not fully faithful.

4. Our Results: the case inj.dim $A \geq 1$

Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded algebra with $A_\ell \neq 0$. In this section, we state results in [6] which gives an answer to the question for general case. First we introduce the derived version of strictly well-gradedness.

Definition 7. A graded algebra A is called *right homologically well-graded* if it satisfies

$$\mathbb{R}\mathrm{Hom}_A^{\mathbb{Z}}(A_0, A(i)) = 0$$

for all $i \neq \ell$.

Remark 8. We compare Definition 3 and Definition 7.

(1) If A is right homologically well-graded, then A is right strictly well-graded. Note that the converse does not hold.
(2) If A is self-injective, then it is right homologically well-graded if and only if it is right strictly well-graded.

Homologically well-gradedness is a necessary and sufficient condition for $\mathcal{H}$ to be fully faithful.

Theorem 9. *Assume that A is Iwanaga-Gorenstein. Then the following assertions hold.*

(1) *$\mathcal{H}$ is fully faithful if and only if A is right homologically well-graded.*
(2) *$\mathcal{H}$ gives an equivalence if and only if A is right homologically well-graded and A_0 is of finite global dimension.*

A class of right homologically well-graded Iwanaga-Gorenstien algebras is just a class of graded algebras which possess nice symmetry. This can be regarded as a higher analogue of Proposition 4.

Theorem 10. *The following conditions are equivalent.*

(a) *A is right homologically well-graded Iwanaga-Gorenstein.*
(b) *A is left homologically well-graded Iwanaga-Gorenstein.*
(c) *A satisfies (1) and (2).*
    (1) *A_ℓ is a cotilting bimodule over A_0.*
    (2) *$A(\ell) \simeq \mathbb{R}\mathrm{Hom}_{A_0}(A, A_\ell)$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}^{\mathbb{Z}} A)$.*

Here we recall the definition of cotilting bimodules.

Definition 11 ([7])**.** Let Λ be an algebra. A finite dimensional Λ-Λ-bimodule C is called a *cotilting bimodule* if it satisfies the following conditions.

(1) inj.dim $C_\Lambda < \infty$ and inj.dim ${}_\Lambda C < \infty$.
(2) $\mathrm{Ext}_\Lambda^{>0}(C, C) = 0$ and $\mathrm{Ext}_{\Lambda^{\mathrm{op}}}^{>0}(C, C) = 0$.
(3) The natural ring homomorphisms

$$\Lambda \to \mathrm{End}_\Lambda(C);\ \lambda \mapsto [c \to \lambda c]\,, \qquad \Lambda^{\mathrm{op}} \to \mathrm{End}_{\Lambda^{\mathrm{op}}}(C);\ \lambda \mapsto [c \to c\lambda]$$

are isomorphisms.

An injective cogenerator $D\Lambda$ is an example of cotitling bimodules.

Cotilting bimodules are important since it induces the following equivalences.

Theorem 12 ([7]). *Let Λ be an algebra and C a cotitling Λ-Λ-bimodule. Then there is an equivalence*

$$\mathbb{R}\mathrm{Hom}_{\Lambda}(-,C) : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \leftrightarrows \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda^{\mathrm{op}}) : \mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-,C).$$

In the rest of this section, we give two examples. First we consider trivial extensions.

Example 13. Let Λ be an algebra, C a finite dimensional Λ-Λ-bimodule, and $A = \Lambda \oplus C$ the trivial extension of Λ by C. We regard A as a $\mathbb{Z}$-graded algebra by the natural way. Then it follows from Theorem 10 that the following conditions are equivalent.

(a) A is a homologically well-graded Iwanaga-Gorenstein algebra.
(b) C is a cotilting bimodule.

If these conditions are satisfied, then

$$\mathcal{H} : \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \to \underline{\mathsf{CM}}^{\mathbb{Z}}\,A$$

is fully faithful. Moreover it is an equivalence if and only if Λ is of finite global dimension. Theorem 1 is the case $C = D\Lambda$ of this example.

Secondly, we recover the triangle equivalence given by Lu.

Example 14. Let Λ be an algebra of finite global dimension. We consider a finitely graded algebra $A = \Lambda \otimes_K K[x]/(x^{\ell+1})$ with $\deg x = 1$. One can easy to check that A satisfies the condition (2) in Theorem 10, and so it is a homologically well-graded Iwanaga-Gorenstein algebra. The Beilinson algebra ∇A of A is isomorphic to the algebra $U_\ell(\Lambda)$ of $\ell \times \ell$ upper triangular matrices over Λ. By Theorem 9 (2), we have a triangle equivalence

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,U_\ell(\Lambda)) \simeq \underline{\mathsf{CM}}^{\mathbb{Z}}\,A.$$

We remark that this triangle equivalence was shown by Lu [4]. His strategy is to use tilting theory for triangulated categories, and is different from us.

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