AS-REGULARITY OF GEOMETRIC ALGEBRAS WHOSE POINT SCHEMES ARE PLANE ELLIPTIC CURVES

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Abstract. In noncommutative algebraic geometry, an Artin-Schelter regular algebra is one of the main interests to study. In this report, we give a simple condition that a geometric algebra introduced by Mori is a 3-dimensional quadratic AS-regular algebra whose point schemes are plane elliptic curves.

1. Geometric algebras

Throughout this report, let k be an algebraically closed field of characteristic 0. A graded k-algebra means an $\mathbb{N}$-graded algebra $A = \bigoplus_{i\in\mathbb{N}} A_i$. A *connected graded* k-algebra A is a graded k-algebra such that $A_0 = k$. We assume that every connected graded k-algebra A is finitely generated in degree 1, that is, it can be written as $A = k\langle x_1, \cdots, x_n\rangle/I$ where $\deg x_i = 1$ for any $i = 1, \cdots, n$ and I is a homogeneous two-sided ideal of $k\langle x_1, \cdots, x_n\rangle$. A connected graded k-algebra $A = k\langle x_1, \cdots, x_n\rangle/I$ is called a *quadratic algebra* if I is generated by homogeneous polynomials of degree two. We denote by $\mathbb{P}^{n-1}$ the $n-1$ dimensional projective space over k.

Let $(E, \mathcal{O}_E)$ be a scheme where $\mathcal{O}_E$ is the structure sheaf on E. An *invertible sheaf* on E is defined to be a locally free $\mathcal{O}_E$-module of rank 1. For a quadratic algebra $A = T(V)/(R)$ where V is a finite dimensional k-vector space and $R \subset V^{\otimes 2}$, we set

$$\mathcal{V}(R) := \{(p, q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p, q) = 0 \text{ for all } f \in R\}.$$

Let $E \subset \mathbb{P}(V^*)$ be a closed k-subscheme and σ an automorphism of E. For the rest of this report, we fix the following:

(a) $\pi : E \to \mathbb{P}(V^*)$ is the embedding, and
(b) $\mathcal{L} := \pi^*(\mathcal{O}_{\mathbb{P}(V^*)}(1))$.

In this case, $\mathcal{L}$ becomes an invertible sheaf on E. The map

$$\mu : \mathrm{H}^0(E, \mathcal{L}) \otimes \mathrm{H}^0(E, \mathcal{L}) \to \mathrm{H}^0(E, \mathcal{L}) \otimes \mathrm{H}^0(E, \mathcal{L}^\sigma) \to \mathrm{H}^0(E, \mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{L}^\sigma)$$

of k-vector spaces is defined by $v \otimes w \mapsto v \otimes w^\sigma$ where $\mathcal{L}^\sigma = \sigma^*\mathcal{L}$ and $w^\sigma = w \circ \sigma$.

For a quadratic algebra, a notion of *geometric algebra* was introduced by Mori [8].

Definition 1 ([8])**.** A quadratic algebra $A = T(V)/(R)$ is called *geometric* if there is a geometric pair (E, σ) where $E \subset \mathbb{P}(V^*)$ is a closed k-subscheme, and $\sigma \in \mathrm{Aut}_k E$ such that

- (G1): $\mathcal{V}(R) = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$, and

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- (G2): $R = \ker\mu$ with the identification

$$\mathrm{H}^0(E, \mathcal{L}) = \mathrm{H}^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(1)) = V \text{ as } k\text{-vector spaces.}$$

When A satisfies the condition (G2), we write $A = \mathcal{A}(E, \sigma)$.

Let $A = T(V)/(R)$ be a quadratic algebra. If $A = \mathcal{A}(E, \sigma)$ is a geometric algebra, then E is called the point scheme of A. If E is reduced, then the condition (G2) is equivalent to the condition (G2'): $R = \{f \in V^{\otimes 2} \mid f(p, \sigma(p)) = 0, \forall p \in E\}$ (see [8]).

Artin and Schelter [1] defined a class of regular algebras which is one of the main interests to study.

Definition 2 ([1])**.** A noetherian connected graded k-algebra A is called a *d-dimensional Artin-Schelter regular* (*AS-regular*) *algebra* if A satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,

(ii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

A geometric pair (E, σ) is called *regular* if $\mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra. Artin, Tate and Van den Bergh [2] showed that there is a one-to-one correspondence between 3-dimensional quadratic AS-regular algebras and regular geometric pairs.

Theorem 3 ([2])**.** *Let A be a quadratic algebra. Then A is a 3-dimensional AS-regular algebra if and only if A is isomorphic to a geometric algebra $\mathcal{A}(E, \sigma)$ which satisfies one of the following conditions*:

(1) *$E = \mathbb{P}^2$ and $\sigma \in \operatorname{Aut}_k \mathbb{P}^2$.*

(2) *E is a cubic curve in $\mathbb{P}^2$ and $\sigma \in \operatorname{Aut}_k E$ such that $\sigma^*\mathcal{L} \ncong \mathcal{L}$ and*

$$(\sigma^2)^*\mathcal{L} \otimes_{\mathcal{O}_E} \mathcal{L} \cong \sigma^*\mathcal{L} \otimes_{\mathcal{O}_E} \sigma^*\mathcal{L}.$$

Every 3-dimensional quadratic AS-regular algebra is a geometric algebra by Theorem 3, but the converse is not true. If $E \subset \mathbb{P}^2$ is singular, then a geometric algebra $\mathcal{A}(E, \sigma)$ is AS-regular for almost all $\sigma \in \operatorname{Aut}_k E$ (see [6]). In this report, we explain the case when $E \subset \mathbb{P}^2$ is non-singular, that is, an elliptic curve.

2. AS-regularity of geometric algebras of Type EC

We say that a geometric algebra $A = \mathcal{A}(E, \sigma)$ is *of Type EC* if E is an elliptic curve in $\mathbb{P}^2$. In this section, we give a simple criterion when a geometric algebra of Type EC is a 3-dimensional quadratic AS-regular algebra.

Let E be an elliptic curve in $\mathbb{P}^2$. In this report, we use a *Hesse form*

$$E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$$

where $\lambda \in k$ with $\lambda^3 \neq 1$. It is well-known that the *j-invariant* $j(E)$ classifies elliptic curves up to isomorphism, that is, two elliptic curves E and E' in $\mathbb{P}^2$ are isomorphic if and only if $j(E) = j(E')$ (see [4, Theorem IV 4.1(b)]). The *j-invariant* of a Hesse form is given by the following formula (see [3, Proposition 2.16]):

$$j(E) = \frac{27\lambda^3(\lambda^3 + 8)^3}{(\lambda^3 - 1)^3}.$$

For $p \in E$, we define $\operatorname{Aut}_k(E,p) := \{\sigma \in \operatorname{Aut}_k E \mid \sigma(p) = p\}$. It follows from [4, Corollary IV 4.7] that, for every point $p \in E$, $\operatorname{Aut}_k(E,p)$ becomes a cyclic group of order

$$|\operatorname{Aut}_k(E,p)| = \begin{cases} 2 & \text{if } j(E) \neq 0, 12^3, \\ 6 & \text{if } j(E) = 0, \\ 4 & \text{if } j(E) = 12^3. \end{cases}$$

For each point $o \in E$, we can define an addition $+$ on E so that $(E, o, +)$ is an abelian group with the zero element o and, for $p \in E$, the map σ_p defined by $\sigma_p(q) := p + q$ is a scheme automorphism of E, called the *translation* by a point p. In this report, we fix the group structure on E with the zero element $o_E := (1 : -1 : 0) \in E$. Every automorphism $\sigma \in \operatorname{Aut}_k E$ can be written as $\sigma = \sigma_p \tau^i$ where $p \in E$, τ is a generator of $\operatorname{Aut}_k(E, o_E)$ and $i \in \mathbb{Z}_{|\tau|}$ ([6, Proposition 4.5]). A generator of $\operatorname{Aut}_k(E, o_E)$ is given in the next lemma.

Lemma 4 ([6, Theorem 4.6])**.** *A generator τ of $\operatorname{Aut}_k(E, o_E)$ is given by*

(i) $\tau_1(a : b : c) := (b : a : c)$ *if* $j(E) \neq 0, 12^3$,
(ii) $\tau_2(a : b : c) := (b : a : c\varepsilon)$ *if* $\lambda = 0$ *(so that* $j(E) = 0$*)*,
(iii) $\tau_3(a : b : c) := (a\varepsilon^2 + b\varepsilon + c : a\varepsilon + b\varepsilon^2 + c : a + b + c)$ *if* $\lambda = 1 + \sqrt{3}$ *(so that* $j(E) = 12^3$*)*

where ε is a primitive 3rd root of unity.

A point $p \in E$ is called *n-torsion* if $np = o_E$. We set $E[n] := \{p \in E \mid np = o_E\}$. It is known that $|E[n]| = n^2$ (see [4, IV Example 4.8.1]).

Lemma 5 ([6, Lemma 4.14])**.** *For $p \in E$ and $i \in \mathbb{Z}_{|\tau|}$, $A = \mathcal{A}(E, \sigma_p \tau^i)$ is a geometric algebra of Type EC if and only if $p \in E \setminus E[3]$.*

For a geometric algebra of Type EC, we give a simple condition when it is AS-regular. For each $i \in \mathbb{Z}_{|\tau|}$, we define

$$U_{\tau^i} := \{p \in E \mid p - \tau^i(p) \in E[3]\}.$$

Theorem 6 ([7, Theorem 4.3])**.** *For $p \in E$ and $i \in \mathbb{Z}_{|\tau|}$, $\mathcal{A}(E, \sigma_p \tau^i)$ is a 3-dimensional quadratic AS-regular algebra if and only if $p \in U_{\tau^i} \setminus E[3]$.*

If $i = 0$, then it follows from Theorem 6 that $\mathcal{A}(E, \sigma_p)$ is AS-regular if and only if $p \in E \setminus E[3]$. For $p \in E \setminus E[3]$, $\mathcal{A}(E, \sigma_p)$ is called a 3-dimensional Sklyanin algebra. In [8], we show that if $i \neq 0$, then U_{τ^i} is a finite set. From this result, we have that, for each fixed elliptic curve E, there are only finitely many 3-dimensional quadratic AS-regular algebras $\mathcal{A}(E, \sigma)$ other than 3-dimensional Sklyanin algebras up to isomorphism.

Example 7. (1) Assume that $j(E) \neq 0, 12^3$. By Lemma 4, a generator of $\operatorname{Aut}_k(E, o_E)$ is given by

$$\tau_1(a : b : c) = (b : a : c).$$

Since $p - \tau_1(p) = 2p$ for $p \in E$, we have that

$$U_{\tau_1} = E[6],$$

so $|U_{\tau_1}| = 36$. It follows from Theorem 6 that

$$\mathcal{A}(E, \sigma_p \tau_1) \text{ is AS-regular } \iff p \in E[6] \setminus E[3].$$

(2) Assume that $\lambda = 0$ so that $j(E) = 0$. By Lemma 4, a generator of $\mathrm{Aut}_k(E, o_E)$ is given by

$$\tau_2(a:b:c) = (b:a:c\varepsilon),$$

where ε is a primitive 3rd root of unity. By calculations, we have that

$$U_{\tau_2} = E[3],$$

so $|U_{\tau_2}| = 9$. It follows from Theorem 6 that $\mathcal{A}(E, \sigma_p\tau_2)$ is never AS-regular.

(3) Assume that $\lambda = 1+\sqrt{3}$ so that $j(E) = 12^3$. By Lemma 4, a generator of $\mathrm{Aut}_k(E, o_E)$ is given by

$$\tau_3(a:b:c) = (a\varepsilon^2 + b\varepsilon + c : a\varepsilon + b\varepsilon^2 + c : a+b+c),$$

where ε is a primitive 3rd root of unity. By calculations, we have that

$$U_{\tau_3} = E[3] \sqcup \{(1:1:\lambda) + r \mid r \in E[3]\},$$

so $|U_{\tau_3}| = 18$. It follows from Theorem 6 that

$$\mathcal{A}(E, \sigma_p\tau_3) \text{ is AS-regular } \iff p \in \{(1:1:\lambda) + r \mid r \in E[3]\}.$$

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