A REMARK ON GRADED COUNTABLE COHEN-MACAULAY REPRESENTATION TYPE

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ABSTRACT. We show that there are only a finite number of isomorphism classes of graded maximal Cohen–Macaulay modules with fixed Hilbert series over Cohen–Macaulay rings of graded countable representation type.

Key Words: graded countable Cohen–Macaulay representation type, maximal Cohen–Macaulay modules, module variety.

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1. Introduction

In the representation theory of Cohen-Macaulay algebras, the various classes of algebras with respect to the complexity of the classification of maximal Cohen–Macaulay modules are given. For graded Cohen–Macaulay algebras, many authors including Eisenbud and Herzog [4], Stone [7], Drozd and Tovpyha [3] have studied. In this report, we investigate a graded Cohen–Macaulay algebra which is of countable Cohen–Macaulay representation type.

Let $R = \oplus_{i=0}^{\infty} R_i$ be a commutative positively graded affine k-algebra with $R_0 = k$ an algebraically closed uncoutable field. Let S be a graded Noetherian normalization. That is, S is a graded polynomial subring of R such that R is a finitely generated graded S-module. A finitely generated graded R-module M is said to be maximal Cohen-Macaulay (MCM) if M is graded free as a graded S-module. We say that a graded CM ring R is of graded countable CM representation type if there are infinitely but only countably many isomorphism classes of indecomposable graded MCM R-modules up to shift.

Theorem 1. *Let R be of graded countable CM representation type. For each graded free S-module F there are finitely many isomorphism classes of MCM R-modules which are isomorphic to F as graded S-modules. In other words, there are only a finite number of isomorphism classes of MCM R-modules with fixed Hilbert series.*

To prove the theorem we consider the analogy of a module variety for finitely generated modules over a finite dimensional algebra, which was introduced by Dao and Shipman [2].

2. Graded maximal Cohen–Macaulay modules

Throughout the talk, k is an algebraically closed uncountable field of characteristic 0 and $R = \oplus_{i=0}^{\infty} R_i$ is a commutative positively graded affine k-algebra with $R_0 = k$ and

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$R_+ = \oplus_{i>0}^{\infty} R_i$ For $i \in \mathbb{Z}$, $M(i)$ is defined by $M(i)_n = M_{n+i}$. We denote by $\mathrm{Hom}_R(M, N)_i$ consisting of homogenous morphisms of degree i.

In our setting we can take a graded Noetherian normalization S of R. That is, $S = k[y_1, ..., y_n] \subseteq R$ where $n = \dim R$ such that R is a finitely generated graded S-module (see [1, Theorem 1.5.17]).

Definition 2. A finitely generated graded R-module M is said to be maximal Cohen-Macaulay (MCM) if M is graded free as an S-module. In other words,

$$M \cong S \otimes_k V$$

for some finite dimensional graded k-module V.

We say that R is of graded countable CM representation type if there are infinitely but only countably many isomorphism classes of indecomposable graded MCM modules up to shift. The following definition is taken from [3].

Definition 3. [3, Definition 1.1.] We say that R is of graded *discrete* CM representation type if, for any fixed $r > 0$, there are only finitely many isomorphism classes of indecomposable graded MCM modules with rank r up to shift. Here the rank is taken over S.

One can show that if R is of discrete CM representation type then R is of countable CM representation type. Because there is only a countable set of graded MCM R-modules up to isomorphism and shift if R is of discrete. The converse does not hold in general.

Example 4. Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$. Then the graded Noetherian normalization of R is $S = k[y]$. It is known that R is of graded countable CM representation type whose indecomposable MCM R-modules are $I_n = (x, y^n)R$ for $n \geq 0$ up to shift. Note that $\mathrm{rank}_S\, I_n = 2$ since $I_n \cong S(-1) \oplus S(-n)$ where $S = k[y]$. But $I_n \not\cong I_m$ if $n \neq m$, so that R is not of graded discrete CM representation type.

The motivation of this report is to give the condition that the graded CM algebras of graded countable CM representation type is of graded discrete representation type.

3. A variety of graded MCM modules

In this section we recall the notion of a variety of graded MCM modules, which is the graded analogy of a module variety of modules over finite dimensional algebras. It is introduced by Dao and Shipman [2].

Given a graded MCM R-module M, since $M \cong S \otimes_k V$, there exists a degree 0 graded S-algebra homomorphism $\alpha : R \to \mathrm{End}_S(S \otimes_k V)$:

$$\alpha \in \mathrm{Hom}_{S-alg}(R, \mathrm{End}_S(S \otimes_k V))_0.$$

Then $\mathrm{Hom}_{S-alg}(R, \mathrm{End}_S(S \otimes_k V))_0$ is an algebraic variety over k. We denote it by $\mathrm{Rep}_S(R, V)(k)$.

Example 5. Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$. Then $S = k[y]$ is a graded Noetherian normalization of R. Set $V = V_0 \oplus V_1$ where $V_0 = V_1 = k$. Then giving a graded

MCM R-module which is isomorphic to $S \otimes_k V = S \oplus S(-1)$ is equivalent to giving a $\mu \in \mathrm{End}_S(S \otimes_k V)_1$ with $\mu^2 = 0$. Note that

$$\mathrm{End}_S(S \otimes_k V)_1 = \left\{ \left(\begin{smallmatrix} ay & by^2 \\ c & dy \end{smallmatrix} \right) | a, b, c, d \in k \right\}.$$

Hence one can show that

$$\mathrm{Rep}_S(R, V)(k) = \mathrm{Hom}_{k\text{-alg}}(k[a, b, c, d]/(a^2 + bc, ab + bd, ac + bc, bc + d^2), k).$$

Remark 6. Dao and Shipman [2] introduced a functor $\mathrm{Rep}_S(R, V)$ from the category of commutative k-algebras to sets. For a commutative k-algebra T, they define the notion of T-flat family of V-framed R-modules ([2, Definition 2.1]), and $\mathrm{Rep}_S(R, V)(T)$ is a set of the modules. They show that $\mathrm{Rep}_S(R, V)$ is represented by an affine variety of finite type ([2, Proposition 2.2]).

Remark 7. (1) The algebraic group $G_V = \mathrm{Aut}_S(S \otimes_k V)_0$ acts on $\mathrm{Rep}_S(R, V)(k)$ by conjugation, and we have 1-1 correspondence;

$$\{G_V\text{-orbits in } \mathrm{Rep}_S(R, V)(k)\} \xleftrightarrow{1-1} \{M | M \cong S \otimes_k V \text{as graded } S\text{-modules } \} / \cong$$

(2) Note that $\mathrm{Rep}_S(R, V)(k)$ parameterizes graded MCM R-modules with fixed Hilbert series.

Now let us prove our main theorem. First we mention a lemma which is a key of our result.

Lemma 8. *Let $X \subseteq \mathbb{A}^n(k)$ be an algebraic set and let $X_i \subsetneq X$ be closed subsets with $\dim X_i \lneq \dim X$. Then X can be never represented by a countable union of X_i.*

Proof. We prove by induction on n. Suppose that $n = 1$. Then $\dim X_i = 0$, so that X_i is a finite set of points. Assume that $X = \cup_{i \geq 1} X_i$, and then X contains infinitely countably many points. This is a contradiction. Since k is an uncountable filed X must contain uncountably many points. Suppose that $n \geq 2$. Considering the irreducible decomposition of X, we may assume that X is irreducible. Then X is represented by $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ in $k[x_1.x_2, \cdots, x_n]$. We also put I_i ideals with $X_i = V(I_i)$. After renumbering, we may assume that X is not contained in $V(x_1 - c)$ for all $c \in k$. For each minimal prime ideal $\mathfrak{q}$ of I_i, there are a finite number of elements $c \in k$ such that $x_1 - c \in \mathfrak{q}$. Note that a number of minimal prime ideals are finite. Recall that k is an uncountable field. Thus we can take $\lambda \in k$ such that $x_1 - \lambda$ is neither contained in $\mathfrak{p}$ nor all minimal prime ideals of I_i for all i. Consider the quotient by $x_1 - \lambda$. Then we can reduce the case that $X = \cup_{i \geq 1} X_i$ is a closed subset of $\mathbb{A}_k^{n-1}$ preserving $\dim X_i \lneq \dim X$ for all i. Repeating this procedure we may assume that $n = 1$, and it is already investigated. □

Theorem 9. *Let $X \subseteq \mathbb{A}^n(k)$ be an algebraic set. Suppose that X is represented by a countable disjoint union of locally closed subsets, that is $X = \cup_{i \geq 1} Y_i$ where Y_i are locally closed and $Y_i \cap Y_j = \varnothing$ for $i \neq j$. Then it is a "finite" union.*

Proof. Let X_1, X_2, ... X_m be irreducible components of X. For each component X_k, we have

$$X_k = X_k \cap X = X_k \cap (\cup_{i \geq 1} Y_i) = \cup_{i \geq 1} (X_k \cap Y_i).$$

Note that $X_k \cap Y_i$ are locally closed for all i. By Lemma 8, there exists j such that $\dim X_k = \dim X_k \cap Y_j$, so that $X_k = \overline{X_k \cap Y_j}$. Since $X_k \cap Y_j$ is open in $\overline{X_k \cap Y_j} = X_k$,

$$X_{k,2} := X_k \backslash (X_k \cap Y_j)$$

is closed and $\dim X_{k,2} \subsetneq \dim X_k$. We decompose $X_{k,2}$ into its irreducible components $X_{k,2,1}$, $X_{k,2,2}$, ... $X_{k,2,m'}$ and apply this argument for each components $X_{k,2,j}$. Then we also obtain the closed subsets $X_{k,3,j'}$ with $\dim X_{k,3,j'} \lneq \dim X_{k,2,j'}$ for all j'. Repeating this at most $\dim X$ times, we achieve the case that each components are of dimension 0. Since a closed subset of dimension 0 is a finite set of points, these subset can be represented by a finite union of (locally) closed subsets which derive from Y_i. Note that a number of Y_i which are appeared in this arguments is finite. Hence we obtain the assertion. □

Remark 10. Let X be a G-variety. Namely X is a variety equipped with an action of the group G. For $x \in X$, we denote by $\mathcal{O}(x)$ the G-orbit of x. Then one can show the following statements hold.

(1) $\mathcal{O}(x)$ is locally closed. See [8, Proposition 21.4.3(i)].
(2) $\mathcal{O}(x) = \mathcal{O}(y)$ if and only if $\overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$ for $x, y \in X$. See [6, Proposition 3.5] for instance.

Corollary 11. *Let R be a graded countable CM representation type. For each finite dimensional graded k-module V, there are finitely many isomorphism classes of graded MCM R-modules which are isomorphic to $S \otimes_k V$.*

Proof. According to Remark 7, it is enough to show that $\mathrm{Rep}_S(R, V)(k)$ consists of finitely many orbits. It follows from Theorem 9 and Remark 10. □

Corollary 11 says that, if R is of graded countable CM representation type, there are only finitely many graded MCM R-modules with a fixed Hilbert series. It is natural to ask what happens if we fix the Hilbert polynomial instead of the Hilbert series. We have the following example.

Example 12. Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$ and $I_n = (x, y^n)R$. Then Hilbert polynomials of I_n are 2 for all n and $I_n \not\cong I_m$ if $n \neq m$.

At the end of this report, we investigate the relation between graded countable CM representation type and graded discrete CM representation type. The following theorem is due to Dao and Shipman. We say that R is with an isolated singularity if each graded localization $R_{(\mathfrak{p})}$ is regular for each graded prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq R_+$.

Theorem 13. [2, Theorem 3.1] *Assume that R is with an isolated singularity. For each $r > 0$ there exists $\alpha_r > 0$ such that if M is an indecomposable graded MCM R module with rank r then*

$$g_{max}(M) - g_{min}(M) < \alpha_r,$$

where $g_{max}(M) = max\{m | (M/S_+M)m \neq 0\}$ and $g_{min}(M) = min\{m | (M/S_+M)m \neq 0\}$.

One can deduce from the theorem that for a MCM graded R-module of rank r up to shift, only finitely many fnite dimensional graded k-modules V come into question. Indeed, for an indecomposable graded MCM modules M with rank r, we shift M to normalize it, and then $g_{min}(M) = 0$. Thus $0 \leq g_{max}(M) < \alpha_r$. Note that α_r depends on only r. Since $M/S_+M \cong V$ as k-modules, we obtain the claim.

For a graded MCM R-module M with rank r, we have $M \cong S(k_1) \oplus S(k_2) \oplus \cdots \oplus S(k_r)$ with $k_1 \leq k_2 \leq \cdots \leq k_r$. We call the tuple $(k_1, k_2, \cdots, k_r)$ the type of M. By the definition, $g_{min}(M) = k_1$ and $g_{max}(M) = k_r$.

Corollary 14. *Let R be of graded countable CM representation type. Suppose that R is with an isolated singularity. Then R is of graded discrete CM representation type.*

Proof. According to Theorem 13, for each $r > 0$,

$$\{M : \text{indecomposable MCM } R\text{-modules with rank } r\}/ <\cong, \text{shift}>$$
$$\subseteq \bigcup_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_r \leq \alpha_r} \mathrm{Rep}_S(R, V)(k).$$

Since $\mathrm{Rep}_S(R, V)(k)$ consists of finitely many orbits (Corollary 11), the left hand side is a finite set. (Compare with [2, Corollary A]. See also [5, Theorem 3.16].) $\square$

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