g-POLYTOPES OF BRAUER GRAPH ALGEBRAS

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Abstract. We introduce the notion of g-polytopes of finite dimensional algebras, which is firstly studied by Asashiba-Mizuno-Nakashima inspired by a work of L. Hille. We can regard it as a geometric realization of a simplicial complex of two-term presilting complexes in $\mathbb{R}^n$ where n is the number of simple modules. In [2], they show that g-polytopes of Brauer tree algebras are convex and symmetric with respect to origin. In this paper, we generalize their results to an arbitrary Brauer graph algebras.

1. Introduction

We introduce the notion of g-polytopes of finite dimensional algebras, which is firstly studied by [2] inspired by a work of L. Hille [3] in the study of a simplicial complex of tilting modules over path algebras of type $\mathbb{A}$. This is a lattice polytope in $\mathbb{R}^n$ defined by numerical data, called g-vectors, of two-term (pre)silting complexes, where n is the number of simple modules. We can regard it as a geometric realization of a simplicial complex of two-term presilting complexes whose j-dimensional faces consist of all basic two-term presilting complexes having $j+1$ indecomposable direct summands $(0 \leq j \leq n-1)$.

In [2], for an algebra having only finitely many two-term silting complexes, they give a condition for the convexity of g-polytopes in terms of silting mutation. Silting mutation plays an important role in τ-tilting theory. As an application, they show that the g-polytope $\triangle(A)$ of a Brauer tree algebra A is just the convex hull of g-vectors of all two-term indecomposable presilting complexes. In addition, $\triangle(A)$ is symmetric, that is, $\triangle(A) = -\triangle(A)$.

In this paper, we generalize their results to an arbitrary Brauer graph algebra A. In general, A has infinitely many two-term silting complexes, and $\triangle(A)$ is not convex nor symmetric (Section 4). However, the claims still hold after taking the closure $\overline{\triangle}(A)$ of $\triangle(A)$ in $\mathbb{R}^n$ (Theorem 7). Furthermore, we give an explicit description by calculating the fundamental domain of a natural group action on $\overline{\triangle}(A)$ (Theorem 8). A key observation is to determine all lattice points of $\overline{\triangle}(A)$ by using a geometric model of a classification of two-term tilting complexes established by Adachi-Aihara-Chan [1].

2. A simplicial complex of two-term silting complexes

Let A be a finite dimensional algebra over an algebraically closed field k. We denote by $\mathrm{proj}A$ the category of finitely generated right projective A-modules, by $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$ the homotopy category of bounded complexes of $\mathrm{proj}A$. Throughout this paper, we assume that every complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$ is basic, that is, it is isomorphic to a direct sum of indecomposable complexes which are mutually non-isomorphic, if otherwise specified.

The detailed version of this paper will be submitted for publication elsewhere.

We say that a complex $M = (M^i, d^i) \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$ is *two-term presilting* if the following conditions are satisfied:

- $M^i = 0$ for all integer $i \neq -1, 0$;
- $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)}(M, M[1]) = 0$.

A two-term presilting complex is said to be *two-term silting* if it has n indecomposable direct summands, where n is the number of simple A-modules. We denote by 2-silt$A$ the set of isomorphism classes of two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$.

Let $M \in$ 2-silt$A$ and X an indecomposable direct summand of M. Consider a triangle

$$X \xrightarrow{f} M' \to Y \to X[1] \tag{2.1}$$

such that f is a left minimal (addM/X)-approximation of X. We say that $\mu_X^-(M) := M/X \oplus Y$ is a left mutation of M when it is a two-term silting complex. Notice that M' is not basic in general.

The *g-vector* of M is the corresponding element $g^M \in K_0(\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)) \cong K_0(\mathrm{proj}A) \cong \mathbb{Z}^n$ in the Grothendieck group. For a given $M \in$ 2-silt$A$, let $C_{\leq 1}(M)$ be the convex hull of g-vectors of all indecomposable direct summands of M and 0 in $\mathbb{R}^n$. We clearly obtain the equality $C_{\leq 1}(M) = \{\sum a_X g^X \in \mathbb{R}^n \mid a_X \geq 0, \sum a_X \leq 1,\}$ where X runs over all indecomposable direct summands of M. The *g-polytope* of the algebra A is defined by

$$\triangle(A) := \bigcup_{M \in 2\text{-silt}A} C_{\leq 1}(M).$$

It is known that two distinct regions $C_{\leq 1}(M)$ and $C_{\leq 1}(N)$ intersect only at their boundary given by $C_{\leq 1}(U)$, where U is a maximal common direct summand of M and N. Therefore, $\triangle(A)$ can be regarded as a geometric realization of a simplicial complex of two-term presilting complexes in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$, whose j-dimensional faces consist of all two-term presilting complexes having $j+1$ indecomposable direct summands.

Now, we discuss the convexity of g-polytopes. The following observation is important.

Proposition 1. [2, Proposition 2.23] *Assume that $M, N \in$ 2-silt$A$ and N is a left mutation of M given by the triangle (2.1). Then the following conditions are equivalent*

(1) *$C_{\leq 1}(M) \cup C_{\leq 1}(N)$ is convex;*
(2) *the number of indecomposable direct summands of M' is at most* 2.

We say that the algebra A is *locally convex* if one of equivalent conditions in Proposition 1 is satisfied for all left mutation in 2-silt$A$.

The following is one of main results in [2].

Theorem 2. [2, Proposition 2.26] *Assume that* 2-silt$A$ *is finite. Then the following conditions are equivalent:*

(1) *A is locally convex;*
(2) *$\triangle(A)$ is convex.*

In this case, $\triangle(A)$ is the convex hull of g-vectors of all two-term indecomposable presilting complexes in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$.

3. g-POLYTOPES OF BRAUER GRAPH ALGEBRAS AND MAIN RESULTS

In this section, we study g-polytopes of Brauer graph algebras. In particular, we discuss the convexity and symmetry. Here, we say that a subset X of $\mathbb{R}^n$ is symmetric if $X = -X$. We will give several examples in the next section (Section 4).

Brauer graph algebras are also known as symmetric special biserial algebras. They are defined by combinatorial data called ribbon graphs, which are finite undirected graphs embedded in the oriented surface. For details, we refer to [4, Section 2]. We denote by $A_{\mathbb{G}}$ the Brauer graph algebra associated to a ribbon graph $\mathbb{G}$. Notice that, since $A_{\mathbb{G}}$ is symmetric, any silting complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj} A_{\mathbb{G}})$ is tilting.

A remarkable result in [1] is that every two-term tilting complex (and its g-vector) over $A_{\mathbb{G}}$ admits a combinatorial description as a certain collection of walks on $\mathbb{G}$. From this description, one can show that the endomorphism algebra of a two-term tilting complex is again a Brauer graph algebra and is derived equivalent to $A_{\mathbb{G}}$.

We begin with the following observation.

Proposition 3. *Brauer graph algebras are locally convex.*

Proof. Firstly, we show that every left mutation of $A_{\mathbb{G}}$ admits at most two indecomposable direct summands in the middle term of the associated mutation triangle. Remember that there is a one-to-one correspondence between indecomposable projective $A_{\mathbb{G}}$-modules P_e and edges e of $\mathbb{G}$. It is known that a left mutation $\mu^-_{P_e}(A_{\mathbb{G}})$ of A is given by a triangle

$$P_e \to P_i \oplus P_j \to P'_e \to P_e[1],$$

where i and j are edges of $\mathbb{G}$ appearing in a flip of $\mathbb{G}$ at e (See Figure 3), and the endomorphism algebra of $\mu^-_{P_e}(A_{\mathbb{G}})$ is isomorphic to a Brauer graph algebra $A_{\mu^-_e(\mathbb{G})}$. In particular, the middle term is a direct sum of at most two indecomposable modules.

Since endomorphism algebras of two-term tilting complexes over $A_{\mathbb{G}}$ is a Brauer graph algebra, we get the assertion by Proposition 1 and the previous discussion. $\square$

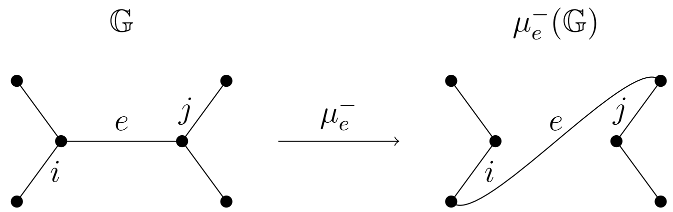

FIGURE 1. A flip of a ribbon graph $\mathbb{G}$ at e

We are interested in Brauer graph algebras having only finitely many two-term tilting complexes. Such algebras are completely determined by their ribbon graphs.

Proposition 4. [1, Theorem 1.1(2)] *For a ribbon graph $\mathbb{G}$, the following conditions are equivalent:*

(1) 2-tilt$A_{\mathbb{G}}$ *is finite;*
(2) $\mathbb{G}$ *contains at most one cycle of odd length, and no cycle of even length.*

An aim of [2] is to study g-polytopes of Brauer graph algebras $A_{\mathbb{G}}$ of the tree (=a graph without cycles). In this case, $\triangle(A_{\mathbb{G}})$ is convex by Theorem 2, Proposition 3 and 4. More strongly, they give an explicit description by using convex polytopes associated to root systems. Here, *root polytopes* of type $\mathbb{A}$ and $\mathbb{C}$ of dimension n are the convex hull of

$$\{e_i - e_j; 1 \le i, j \le n+1\} \quad \text{and} \quad \{\pm 2e_i; 1 \le i \le n, \pm e_i \pm e_j; 1 \le i \neq j \le n\}$$

respectively. Let $\mathcal{P}_{\mathbb{A}_n}$ (resp., $\mathcal{P}_{\mathbb{C}_n}$) be the image in $\mathbb{R}^n$ with canonical basis $\alpha_i := e_i - e_{i+1}$ for $1 \le i \le n$ (resp., $\beta_i := e_i - e_{i+1}$ for $1 \le i \le n-1$ and $\beta_n := 2e_n$).

Theorem 5. [2, Theorem 1.1] *Let $\mathbb{G}$ be a tree. Then $\triangle(A_{\mathbb{G}})$ is convex and symmetric. In addition, there exists a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\triangle(A_{\mathbb{G}})) = \mathcal{P}_{\mathbb{A}_n}$ and $|\det f| = 1$.*

On the other hand, as an analog of Theorem 5, we prove that the remained ribbon graphs of finite type correspond to root polytopes of type $\mathbb{C}$.

Theorem 6. *Let $\mathbb{G}$ be a ribbon graph containing precisely one cycle of odd length, and no cycle of even length. Then $\triangle(A_{\mathbb{G}})$ is convex and symmetric. In addition, there exists a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\triangle(A_{\mathbb{G}})) = \mathcal{P}_{\mathbb{C}_n}$ and $|\det f| = 1$.*

Now, we study a Brauer graph algebra $A_{\mathbb{G}}$ for an arbitrary ribbon graph $\mathbb{G}$. We will see in Section 4 that $\triangle(A_{\mathbb{G}})$ is not convex nor symmetric in general. However, we find that the claims still hold after taking the closure. Let $\overline{\triangle}(A_{\mathbb{G}})$ be the closure of $\triangle(A_{\mathbb{G}})$ with respect to the natural topology on $\mathbb{R}^n$.

Theorem 7. *Let $\mathbb{G}$ be a ribbon graph.*

(1) *$\overline{\triangle}(A_{\mathbb{G}})$ is convex and symmetric.*
(2) *If a ribbon graph $\mathbb{G}'$ is obtained by iterated flip from $\mathbb{G}$, then the isomorphism between Grothendieck groups $K_0(\mathrm{proj}A_{\mathbb{G}})$ and $K_0(\mathrm{proj}A_{\mathbb{G}'})$ induces a linear transformation $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $f(\overline{\triangle}(A_{\mathbb{G}})) = \overline{\triangle}(A_{\mathbb{G}'})$ and $|\det f| = 1$.*

The following result declare the shape of the closure of g-polytopes.

Theorem 8. *Let $\mathbb{G}$ be a ribbon graph. Then we have a decomposition*

$$\overline{\triangle}(A_{\mathbb{G}}) = \triangle(A_{\mathbb{G}_0}) \times H$$

where $\mathbb{G}_0$ is a subgraph of $\mathbb{G}$ containing at most one cycle of odd length and no cycle of even length, and H is a subspace of $\mathbb{R}^n$. Furthermore, $\triangle(A_{\mathbb{G}_0})$ is the fundamental domain of a group action of the additive group H on $\overline{\triangle}(A_{\mathbb{G}})$, which is isomorphic to root polytopes of type $\mathbb{A}$ or $\mathbb{C}$.

In fact, the above H is determined by "cycles" of $\mathbb{G}$. A key observation is to determine all lattice points of $\overline{\triangle}(A_{\mathbb{G}})$ by using a geometric model of a classification of two-term tilting complexes established in [1].

4. Examples

We describe g-polytopes of Brauer graph algebras $A_{\mathbb{G}}$ for several ribbon graphs $\mathbb{G}$.

(a) For ribbon graphs having 1 edge, they are either a tree or a loop. In both cases, there are precisely two two-term tilting complexes $A_{\mathbb{G}}$ and $A_{\mathbb{G}}[1]$. Therefore, the associated g-polytope is just an interval $[-1, 1]$ in $\mathbb{R}$, which is of type $\mathbb{A}_1$.

(b) In Figure 2, we describe three ribbon graphs having 2 edges and the closure of the associated g-polytopes by dotted areas in $\mathbb{R}^2$. Notice that the Brauer graph algebra of the right is the trivial extension of a path algebra of the Kronecker quiver. In this case, there are infinitely many two-term tilting complexes for $A_{\mathbb{G}}$. In Theorem 8, a decomposition of $\overline{\triangle}(A_{\mathbb{G}})$ is provided by $\triangle(A_{\mathbb{G}_0}) = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x \leq 1, y = 0\}$ of a subgraph $\mathbb{G}_0$ of $\mathbb{G}$ with edge set $\{1\}$, and a 1-dimensional subspace $H = \{(x, y) \mid y = -x\} \subset \mathbb{R}^2$.

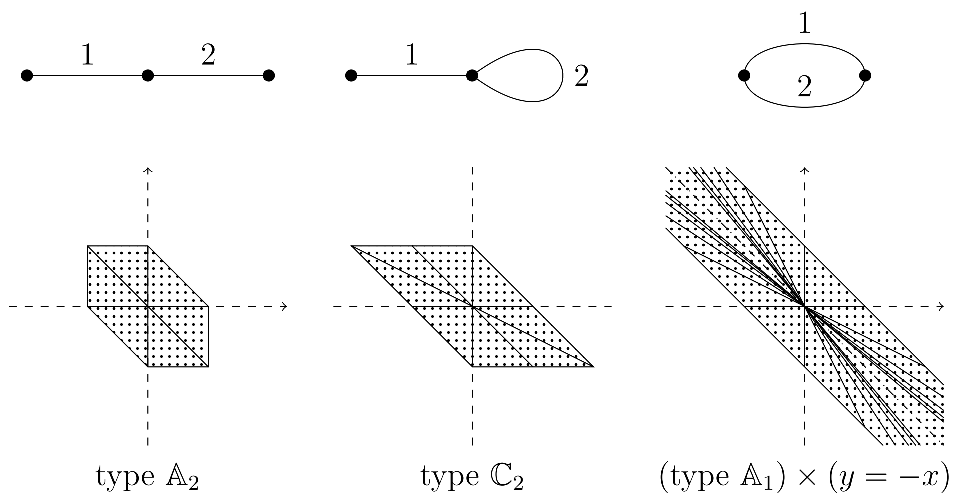

Figure 2. $\overline{\triangle}(A_{\mathbb{G}})$ for ribbon graphs $\mathbb{G}$ having 2 edges.

(c) We observe ribbon graphs having 3 edges. We give three examples in Figure 3. The ribbon graph of the tree in center is obtained from the left one by a flip at the edge 1. In Theorem 7(2), the induced linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $e'_1 := -e_1 + e_2$, $e'_2 := e_2$ and $e'_3 := e_3$. By Proposition 5, both of them are isomorphic to a root polytope of type $\mathbb{A}_3$.

For the right one, we find that $\triangle(A_{\mathbb{G}})$ itself is not convex nor symmetric, but its closure is. A decomposition of $\overline{\triangle}(A_{\mathbb{G}})$ is provided by a 1-dimensional subspace $H := \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, z = -y\} \subset \mathbb{R}^3$ and a subgraph $\mathbb{G}_0$ of $\mathbb{G}$ with edge set $\{1, 2\}$. Namely, the fundamental domain is given by the intersection of $\overline{\triangle}(A_{\mathbb{G}})$ and xy-plane and is a hexagon of type $\mathbb{A}_2$ in (b).

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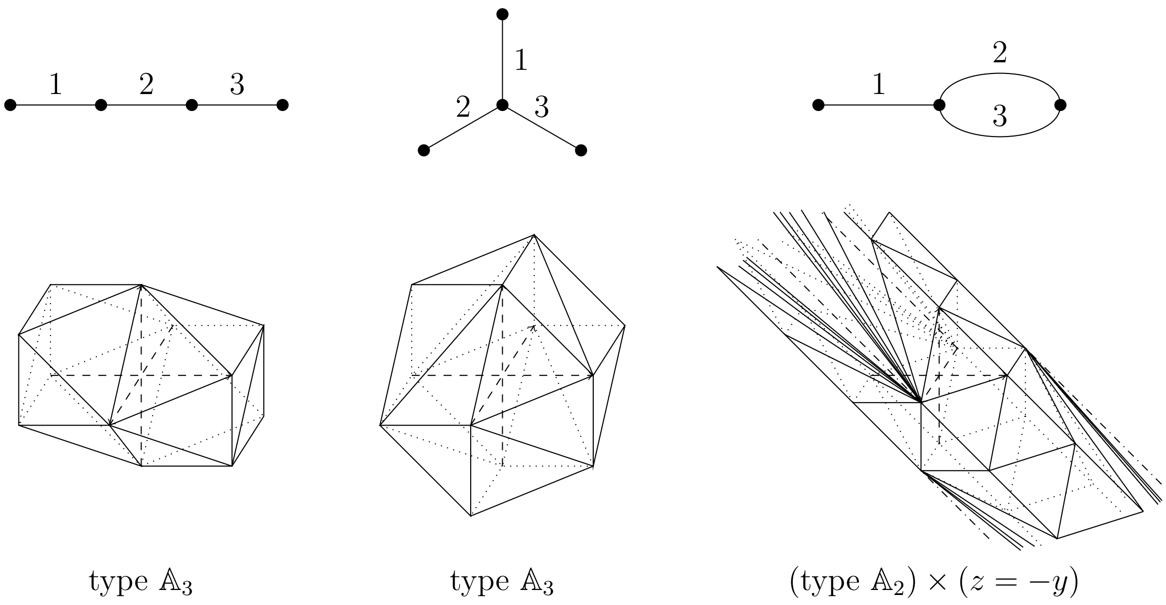

FIGURE 3. Three examples of $\triangle(A_{\mathbb{G}})$ for ribbon graphs $\mathbb{G}$ having 3 edges.

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