The defining relations of geometric algebras of Type EC

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Contents

Notations

Throughout this talk,

- k: an algebraically closed field of characteristic 0.
- $A = k\langle x_1, \cdots, x_n\rangle/I$: a factor ring of a free k-algebra of n variables.
  - $\deg x_i = 1$.
  - A is a quadratic k-algebra, that is, I is generated by a subspace $I_2 \subset k\langle x_1, \cdots, x_n\rangle_2$.
- $\mathrm{GrMod}A$: the category of graded right A-modules and graded right A-module homomorphisms of degree 0.
- $\mathbb{P}^{n-1}(= \mathbb{P}^{n-1}_k)$: the $n-1$ dimensional projective space over k.

Geometric algebras

- $E \subset \mathbb{P}^{n-1}$: a closed subscheme, $\sigma \in \mathrm{Aut}_k E$.
- For a quadratic k-algebra $A = k\langle x_1, \cdots, x_n \rangle / I$,

$$\Gamma_A := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0, \ \forall f \in I_2\}.$$

Definition ([I. Mori, 2006])

$A = k\langle x_1, \ldots, x_n \rangle / I$: a quadratic k-algebra.

1. A satisfies (G1) $(\mathcal{P}(A) = (E, \sigma))$ $:\Longleftrightarrow \exists (E, \sigma)$ s.t. $\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}$.
2. A satisfies (G2) $(A = \mathcal{A}(E, \sigma))$ $:\Longleftrightarrow \exists (E, \sigma)$ s.t. $I_2 = \{f \in k\langle x_1, \ldots, x_n \rangle_2 \mid f(p, \sigma(p)) = 0, \ \forall p \in E\}$.
3. A: *geometric* $:\Longleftrightarrow$ A satisfies (G1), (G2) and $A = \mathcal{A}(\mathcal{P}(A))$.

- We call a geometric algebra Type EC if E is an elliptic curve in $\mathbb{P}^2$.

Example

Let $A = k\langle x, y\rangle/(f)$ where $f = xy - \alpha yx$, $\alpha \in k^{\times}$.
For $p = (a : b), q = (c : d) \in \mathbb{P}^1$,

$$\begin{aligned}(p, q) \in \Gamma_A(\subset \mathbb{P}^1 \times \mathbb{P}^1) &\iff f(p, q) = ad - \alpha bc = 0 \\ &\iff ad = \alpha bc \\ &\iff (c : d) = (a : \alpha b) \text{ in } \mathbb{P}^1 ,\end{aligned}$$

so

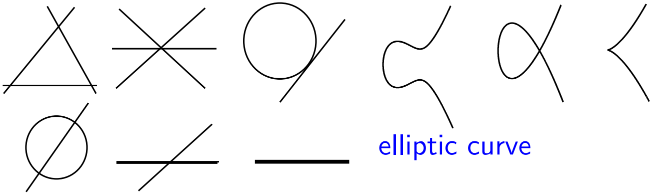
$$\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^1 \times \mathbb{P}^1 \,|\, p \in \mathbb{P}^1\},$$

where $\sigma \in \mathrm{Aut}_k \mathbb{P}^1$ is defined by $\sigma(p) := (a : \alpha b)$ for $p = (a : b) \in \mathbb{P}^1$. In fact, $A = \mathcal{A}(\mathbb{P}^1, \sigma)$ is a geometric algebra.

- The group $\mathrm{Aut}_k \mathbb{P}^{n-1}$ is isomorphic to $\mathrm{PGL}_n(k)$. The above automorphism $\sigma \in \mathrm{Aut}_k \mathbb{P}^1$ corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \in \mathrm{PGL}_2(k)$.

Motivations · Our goals

- An AS-regular algebra is one of the first classes of algebras studied in noncommutative algebraic geometry. It is known that a 3-dimensional quadratic AS-regular algebra generated in degree 1 is a geometric algebra and its geometric pair consists of $\mathbb{P}^2$ or the followings:



- Our goals are
  1. Find the defining relations of geometric algebras of Type EC (Main result 1).
  2. Give criterions that two geometric algebras of Type EC are graded k-algebra isomorphic and graded Morita equivalent (Main result 2).

Elliptic curve (Hesse form) · The j-invariant

Elliptic curve (Hesse form)

- We use a Hesse form

$$E = \mathcal{V}(f), f = x^3 + y^3 + z^3 - 3\lambda xyz \quad (\lambda \in k, \lambda^3 \neq 1).$$

  - An elliptic curve in $\mathbb{P}^2$ can be written by this form up to isomorphism.

- On an elliptic curve E in $\mathbb{P}^2$, we can define an addition with the zero element $0_E := (1 : -1 : 0)$; for $p = (a : b : c), q = (\alpha : \beta : \gamma) \in E$,

$$p + q := (ac\beta^2 - b^2\alpha\gamma : bc\alpha^2 - a^2\beta\gamma : ab\gamma^2 - c^2\alpha\beta).$$

- For $p \in E$, an automorphism $\sigma_p \in \mathrm{Aut}_k E$ is defined by $\sigma_p(q) := p + q$ for $q \in E$, called a translation.

- The j-invariant of an elliptic curve is given by $j(E) = \frac{27\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3}$.
- $E \cong E'$ if and only if $j(E) = j(E')$.

Automorphism group

- $T := \{\sigma_p \in \mathrm{Aut}_k E \mid p \in E\}$: the set of translations.
- $\mathrm{Aut}_k(E, 0_E) := \{\sigma \in \mathrm{Aut}_k E \mid \sigma(0_E) = 0_E\}$.

Lemma

$\mathrm{Aut}_k(E, 0_E) = \langle \tau \rangle$ and τ is given by

$$\begin{cases} \text{(i) } \tau(a:b:c) := (b:a:c), \ (\text{the case of } j(E) \neq 0, 12^3, |\tau| = 2), \\ \text{(ii) } \tau(a:b:c) := (b:a:\varepsilon c), \ (\text{the case of } j(E) = 0, |\tau| = 6), \\ \text{(iii) } \tau(a:b:c) := (\varepsilon^2 a + \varepsilon b + c : \varepsilon a + \varepsilon^2 b + c : a + b + c), \\ \quad (\text{the case of } j(E) = 12^3, |\tau| = 4), \end{cases}$$

for $(a:b:c) \in E$, where ε is a primitive 3rd root of unity.

Proposition

$\mathrm{Aut}_k E = \{\sigma_p \tau^i \mid \sigma_p \in T, \ i \in \mathbb{Z}_d\} \cong T \rtimes \mathrm{Aut}_k(E, 0_E)$, where $d := |\tau|$.

3-torsion points · Geometric algebras of Type EC

We call $p \in E$ a 3-torsion point if $3p = 0_E$.

- $E[3] := \{p \in E \mid 3p = 0_E\}$: the set of 3-torsion points.
  - $E[3]$ is a finite set.
- $T[3] := \{\sigma \in T \mid \sigma^3 = \mathrm{id}\} = \{\sigma_p \in T \mid p \in E[3]\}$.
- $\mathrm{Aut}_k(\mathbb{P}^2, E) := \{\sigma \in \mathrm{Aut}_k\mathbb{P}^2 \mid \sigma|_E \in \mathrm{Aut}_k E\}$.
- $\mathrm{Aut}_k(E, 0_E) \leq \mathrm{Aut}_k(\mathbb{P}^2, E)$.

Proposition

1. $T \cap \mathrm{Aut}_k(\mathbb{P}^2, E) = T[3]$.
2. $\mathrm{Aut}_k(\mathbb{P}^2, E) = \{\sigma_p \circ \tau^i \mid p \in E[3], i \in \mathbb{Z}_d\} \cong T[3] \rtimes \mathrm{Aut}_k(E, 0_E)$, where $d := |\tau|$.

Lemma

Let E be an elliptic curve in $\mathbb{P}^2$, $p \in E$ and $i \in \mathbb{Z}_d$ where $d = |\tau|$. Then $\mathcal{A}(E, \sigma_p\tau^i)$: geometric algebra of Type EC $\iff p \in E \setminus E[3]$.

Sklyanin algebras

Sklyanin algebras

Let $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$ be an elliptic curve in $\mathbb{P}^2$ and $p = (a : b : c) \in E \setminus E[3]$. Then

$$A = \mathcal{A}(E, \sigma_p) = k\langle x, y, z\rangle / \begin{pmatrix} ayz + bzy + cx^2 \\ azx + bxz + cy^2 \\ axy + byx + cz^2 \end{pmatrix}.$$

This algebra $A = \mathcal{A}(E, \sigma_p)$ is a 3-dimensional Sklyanin algebra.

Main result 1

Theorem [IM]

Every geometric algebra $\mathcal{A}(E, \sigma_p\tau^i)$ of Type EC is isomorphic to one of the following algebras $k\langle x, y, z\rangle/(f_1, f_2, f_3)$ where $p = (a : b : c) \in E \setminus E[3]$ and ε is a primitive 3rd root of unity.

1. If $j(E) \neq 0, 12^3$, then for $p = (a : b : c) \in E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz)$, $(\lambda^3 \neq 1)$,

$$\sigma_p\tau^0 \begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases}$$

$$\sigma_p\tau \begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases}$$

(ii) If $j(E) = 0$, then for $p = (a : b : c) \in E = \mathcal{V}(x^3 + y^3 + z^3)$,

$$\sigma_p \tau^0 \begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases} \qquad \sigma_p \tau \begin{cases} f_1 = axz + b\varepsilon zy + cyx, \\ f_2 = a\varepsilon zx + byz + cxy, \\ f_3 = ay^2 + bx^2 + c\varepsilon z^2. \end{cases}$$

$$\sigma_p \tau^2 \begin{cases} f_1 = ayz + b\varepsilon^2 zy + cx^2, \\ f_2 = a\varepsilon^2 zx + bxz + cy^2, \\ f_3 = axy + byx + c\varepsilon^2 z^2. \end{cases} \qquad \sigma_p \tau^3 \begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases}$$

$$\sigma_p \tau^4 \begin{cases} f_1 = ayz + b\varepsilon zy + cx^2, \\ f_2 = a\varepsilon zx + bxz + cy^2, \\ f_3 = axy + byx + c\varepsilon z^2. \end{cases} \qquad \sigma_p \tau^5 \begin{cases} f_1 = axz + b\varepsilon^2 zy + cyx, \\ f_2 = a\varepsilon^2 zx + byz + cxy, \\ f_3 = ay^2 + bx^2 + c\varepsilon^2 z^2. \end{cases}$$

- (iii) If $j(E) = 12^3$, then for $p = (a:b:c) \in E = \mathcal{V}(x^3 + y^3 + z^3 - 3(1+\sqrt{3})xyz)$,

$$\sigma_p\tau^0 \begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2. \end{cases} \quad \sigma_p\tau^2 \begin{cases} f_1 = axz + bzy + cyx, \\ f_2 = azx + byz + cxy, \\ f_3 = ay^2 + bx^2 + cz^2. \end{cases}$$

$$\sigma_p\tau \begin{cases} f_1 = a(\varepsilon x + \varepsilon^2 y + z)z + b(x+y+z)y + c(\varepsilon^2 x + \varepsilon y + z)x, \\ f_2 = a(x+y+z)x + b(\varepsilon^2 x + \varepsilon y + z)z + c(\varepsilon x + \varepsilon^2 y + z)y, \\ f_3 = a(\varepsilon^2 x + \varepsilon y + z)y + b(\varepsilon x + \varepsilon^2 y + z)x + c(x+y+z)z. \end{cases}$$

$$\sigma_p\tau^3 \begin{cases} f_1 = a(\varepsilon^2 x + \varepsilon y + z)z + b(x+y+z)y + c(\varepsilon x + \varepsilon^2 y + z)x, \\ f_2 = a(x+y+z)x + b(\varepsilon x + \varepsilon^2 y + z)z + c(\varepsilon^2 x + \varepsilon y + z)y, \\ f_3 = a(\varepsilon x + \varepsilon^2 y + z)y + b(\varepsilon^2 x + \varepsilon y + z)x + c(x+y+z)z. \end{cases}$$

Main result 2

Theorem [IM]

Let $A = \mathcal{A}(E, \sigma_p\tau^i)$ and $A' = \mathcal{A}(E, \sigma_q\tau^j)$ be geometric algebras of Type EC where $p, q \in E \setminus E[3]$ and $i, j \in \mathbb{Z}_d$, $d := |\tau|$.

1. $A \cong A'$ if and only if
   1. $i = j$, and
   2. there exist $r \in E[3]$ and $l \in \mathbb{Z}_d$ such that $q = \tau^l(p) + r - \tau^i(r)$.
2. $\mathrm{GrMod} A \cong \mathrm{GrMod} A'$ if and only if
   1. $p - \tau^{j-i}(p) \in E[3]$, and
   2. there exist $r \in E[3]$ and $l \in \mathbb{Z}_d$ such that $q = \tau^l(p) + r$.

Examples

Let $A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ and $A' = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ with

$$\begin{cases} f_1 = ayz + bzy + cx^2, \\ f_2 = azx + bxz + cy^2, \\ f_3 = axy + byx + cz^2, \end{cases} \quad \begin{cases} g_1 = axz + bzy + cyx, \\ g_2 = azx + byz + cxy, \\ g_3 = ay^2 + bx^2 + cz^2, \end{cases}$$

where $p = (a : b : c) \in \mathbb{P}^2$ satisfies $abc \neq 0$ and $(a^3 + b^3 + c^3)^3 \neq (3abc)^3$. Then A and A' are geometric algebras of Type EC with $A = \mathcal{A}(E, \sigma_p)$ and $A' = \mathcal{A}(E, \sigma_p \tau^{\frac{d}{2}})$ where $E = \mathcal{V}(x^3 + y^3 + z^3 - 3\lambda xyz) \subset \mathbb{P}^2$, $\lambda = \frac{a^3+b^3+c^3}{3abc}$ and $d = |\tau|$. By main result 2,

1. $A \ncong A'$.
2. $\mathrm{GrMod} A \cong \mathrm{GrMod} A'$ if and only if $p - \tau(p) = 2p \in E[3]$ if and only if $p \in E[6]$ where $E[6]$ is the set of 6-torsion points of E.