A strongly quasi-hereditary structure on Auslander–Dlab–Ringel algebras

Mayu Tsukamoto

Osaka City University

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Notation

- A : artin algebra
- J : Jacobson radical of A
- mod A : the cat. of finitely generated right A-modules
- proj A : the full subcat. of mod A consisting of projective A-modules
- add M : the full subcat. of mod A whose objs are direct summands of fin. direct sums of $M \in \operatorname{mod} A$

- $\mathcal{C}$: Krull–Schmidt category
- $\mathcal{J}_{\mathcal{C}}$: Jacobson radical of $\mathcal{C}$
- $\mathcal{C}'$: full subcat. of $\mathcal{C}$ closed under isomorphism, direct sums and direct summands

Introduction

Question

Construct A-modules M with $\mathrm{gl.dim}\,\mathrm{End}_{\mathbf{A}}(\mathbf{M}) < \infty$

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Construct A-modules M with $\mathrm{gl.dim}\,\mathrm{End}_A(M) < \infty$

Answer

- **$M := \bigoplus_{i=1}^{m} A/J^i$ [Auslander (1971)]**
- **$\mathrm{End}_A(M)$: quasi-hereditary algebra [Dlab–Ringel (1989)]**

Remark

- **$\mathrm{End}_A(M)$: Auslander–Dlab–Ringel algebra**
- **A is a direct summand of $\bigoplus_{i=1}^{m} A/J^i$**
  $\therefore \forall A$: alg., $\exists B$: QH and $e \in B$: idemp. s.t. $A = eBe$
- **Realization of non-commutative schemes [Orlov (2018)]**

Aim

Rejective chains

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

are effective to study strongly quasi-hereditary(=:QH)algebras

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Theorem [T (2018)]

- $\mathcal{C}$:= proj A has a left rejective chain
  $\Leftrightarrow$ A : left-strongly QH algebra
- $\mathcal{C}$:= proj A has a rejective chain
  $\Leftrightarrow$ A : strongly QH algebra

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Aim

Give a (left-)strongly QH structure on Auslander–Dlab–Ringel algebras by using (left) rejective chains

Rejective subcategories

Definition [Iyama (2003)]

- $\mathcal{C}'$: **left rejective subcategory of $\mathcal{C}$ if $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a left adjoint with a unit η s.t. η_X is epic for all $X \in \mathcal{C}$**
- $\mathcal{C}'$: **rejective subcategory of $\mathcal{C}$ if $\mathcal{C}'$ is a left and right rejective subcategory of $\mathcal{C}$**

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B : factor algebra of A
Then mod B : rejective subcategory of mod A

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Example

B : factor algebra of A

Then $\operatorname{mod} B$: rejective subcategory of $\operatorname{mod} A$

$\because$ $\operatorname{mod} B \hookrightarrow \operatorname{mod} A$ has a left adjoint

$$- \otimes_A B : \operatorname{mod} A \to \operatorname{mod} B$$

with a unit η s.t. $\eta_X : X \to X \otimes_A B$ is epic $\forall X \in \operatorname{mod} A$

Rejective chains

Definition [Iyama (2003)]

A chain of subcategories of $\mathcal{C}$

$$\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$$

is called

- **left rejective chain if $\mathcal{C}_i$ is**
  - (a) **left rejective subcategory of $\mathcal{C}$**
  - (b) **cosemisimple subcategory of $\mathcal{C}_{i-1}$**
    (*i.e.*, the quotient cat. $\mathcal{C}_{i-1}/[\mathcal{C}_i]$ is semisimple)
- **rejective chain if it is a left and right rejective chain**

Quasi-hereditary algebras

- $\leq$: partial order on I (label set of simple A-modules)
- $\nabla(i)$: max. submod. of $E(i)$ s.t. $[\nabla(i) : S(j)] \neq 0 \Rightarrow j \leq i$

Definition [Cline-Parshall-Scott (1988)]

A pair $(\operatorname{mod} A, \leq)$: highest weight category (=: HWC) if there exists a short exact sequence

$$0 \to \nabla(i) \to E(i) \to E(i)/\nabla(i) \to 0$$

for any $i \in I$ with the following properties:

(a) $E(i)/\nabla(i) \in \mathcal{F}(\nabla)$;

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$(\operatorname{mod} A, \leq)$: HWC $\Leftrightarrow$ A : quasi-hereditary algebra

Strongly QH algebras

Definition [Ringel (2010)]

- **(A, $\leq$) : left-strongly quasi-hereditary**
  $\Leftrightarrow$ **(mod A, $\leq$) : HWC with** $\mathrm{inj.dim}\,\nabla \leq 1$
- **(A, $\leq$) : strongly quasi-hereditary**
  $\Leftrightarrow$ **(A, $\leq$) : left-strongly and right-strongly QH**

Strongly QH algebras

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Proposition 1 [T (2018)]

- **A : left-strongly QH**
  $\Leftrightarrow$ $\mathrm{proj}\, A$ has a left rejective chain
  $$\mathrm{proj}\, A = \mathrm{add}\, e_0 A \supset \mathrm{add}\, e_1 A \supset \cdots \supset \mathrm{add}\, e_n A = 0 \quad (*)$$
- **A : strongly QH $\Leftrightarrow$ $\mathrm{proj}\, A$ has a rejective chain $(*)$**

Global dimension

Theorem [Dlab–Ringel (1989), Ringel (2010)]

$$\mathrm{gl.dim}\mathbf{A} \leq \begin{cases} \mathbf{2(n-1)} & (\mathbf{A} : \text{QH}) \\ \mathbf{n} & (\mathbf{A} : \text{left-strongly QH}) \\ \mathbf{2} & (\mathbf{A} : \text{strongly QH}) \end{cases}$$

where n := the length of a left rejective chain

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where $n :=$ the length of a left rejective chain

Advantage of left-strongly QH

We can give a better upper bound for global dimension

Examples

Prototype of left-strongly QH algebra

QH algebra appeared in the proof of finiteness of representation dimension of artin algebras [Iyama (2003)]

Examples of left-strongly/right-strongly QH algebras

- **Auslander algebras [Ringel (2010)]**
- **Certain cluster tilted algebras for preprojective algebras [Iyama–Reiten (2011)]**
- **Nilpotent quiver algebras [Eiríksson–Sauter (2017)]**
- **Matrix algebras of semisimple d-systems [Coulembier (2017)]**
- **Algebras with global dimension at most two [T (2018)]**

Auslander–Dlab–Ringel algebras

Notation

- M : **semilocal module**
  i.e., **M is a direct sum of local modules**
  (local module $\Leftrightarrow$ its top is simple)
  e.g., **projective modules are semilocal**
- $\mathsf{m} := \ell\ell(\mathsf{M})$: **the Loewy length of $\mathsf{M} \in \operatorname{mod} \mathsf{A}$**
- $\widetilde{\mathsf{M}} := \bigoplus_{\mathsf{i}=1}^{\mathsf{m}} \mathsf{M}/\mathsf{M}\mathsf{J}^{\mathsf{i}}$
- $\mathsf{B} := \operatorname{End}_{\mathsf{A}}(\widetilde{\mathsf{M}})$: **Auslander–Dlab–Ringel (ADR) algebra of M**

Main Result

Theorem 2 [T (2018)]

$B := \mathrm{End}_A(\widetilde{M})$: ADR algebra of M
$\Rightarrow$ B : left-strongly QH algebra

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Corollary

B : QH algebra if

- M = A [Dlab–Ringel (1989)]
- M : semilocal module [Lin–Xi (1993)]

B : left-strongly QH if M = A [Conde (2016)]

Key Idea and Remark

Key Idea

- **B : left-strongly QH algebra**
  $\Leftrightarrow$ $\mathrm{proj}\,\mathrm{B}$ has a left rejective chain ($\because$ Proposition 1)
- **$\mathrm{proj}\,\mathrm{B} \simeq \mathrm{add}\,\widetilde{\mathrm{M}}$ as additive categories**
- **Construct a left rejective chain of $\mathrm{add}\,\widetilde{\mathrm{M}}$**

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Remark

- $M = A$
  $\Rightarrow$ Construct a left rejective chain by "length order"
- M : semilocal
  $\Rightarrow$ "length order" does not necessarily induce a left rejective chain

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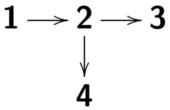
Example : M=A

M = A ⇒ Construct a left rejective chain by "length order"

$A :=$



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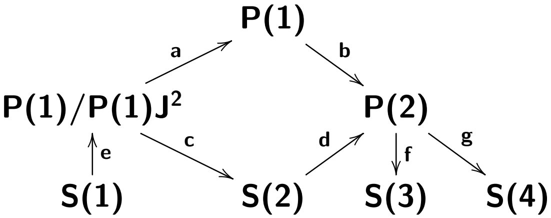

with relations $ab - cd, ec, df$ and dg

$\mathcal{C}_0$	$\supset$	$\mathcal{C}_1$	$\supset$	$\mathcal{C}_2$	$\supset$	$\mathcal{C}_3 = 0$
add $\widetilde{A}$	$\supset$	$\mathcal{C}_0 \setminus P(1)$	$\supset$	$\mathcal{C}_1 \setminus \{P(1)/P(1)J^2, P(2)\}$	$\supset$	$\mathcal{C}_2 \setminus \{S(i)\}$

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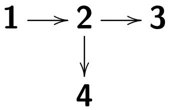

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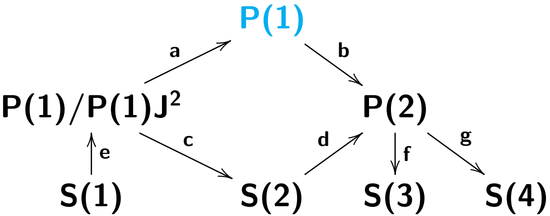

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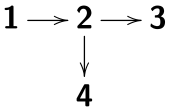

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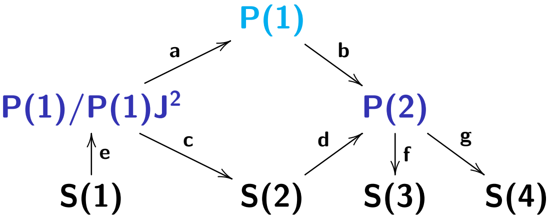

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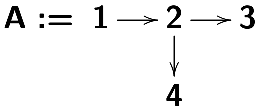

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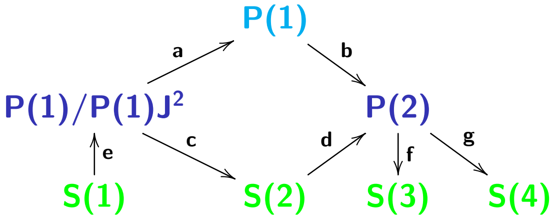

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Example : M : semilocal

$\mathbf{M := P(1) \oplus P(1)/S(3) \oplus P(1)/S(4) \oplus P(2)/S(3)}$: **semilocal**
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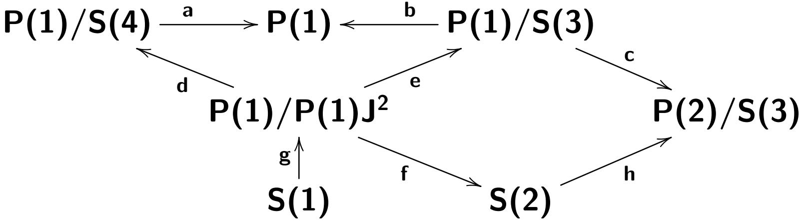

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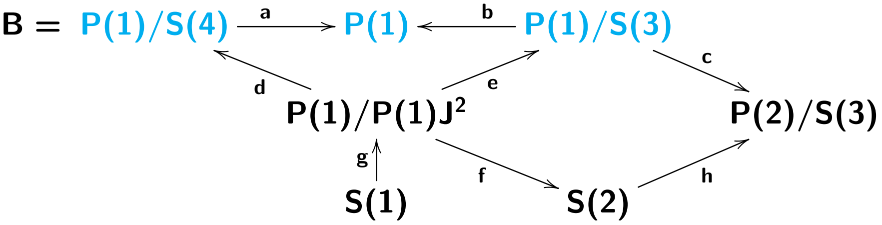

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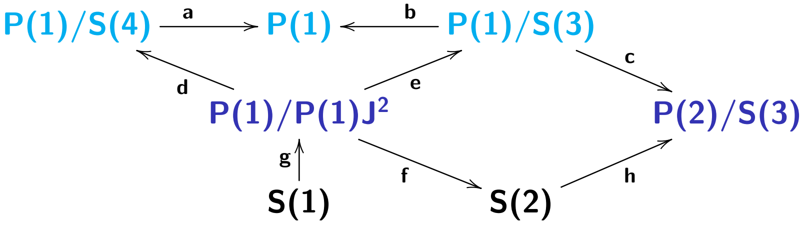

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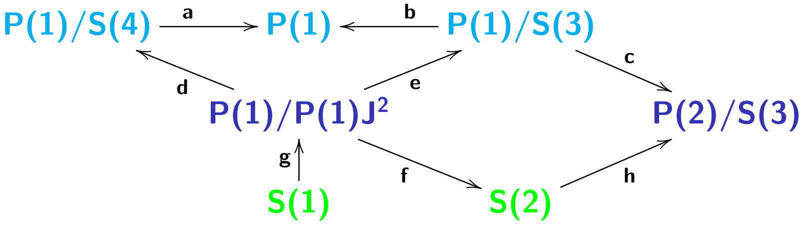

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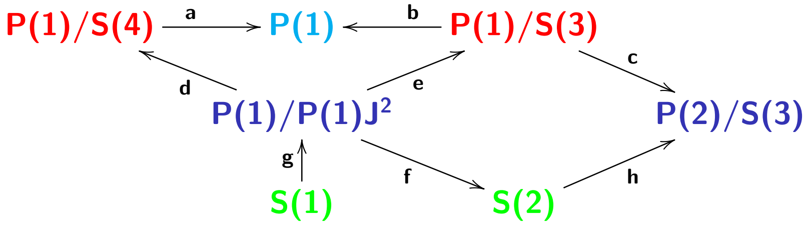

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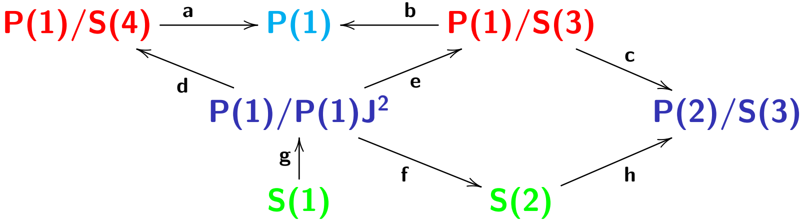

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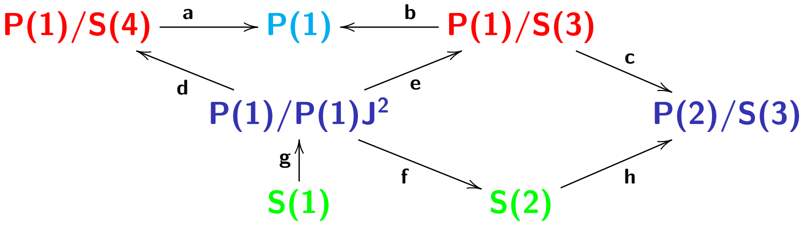

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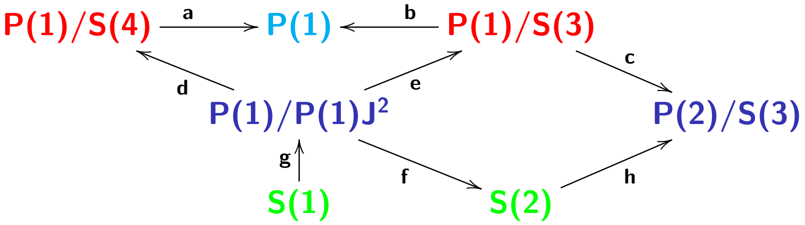

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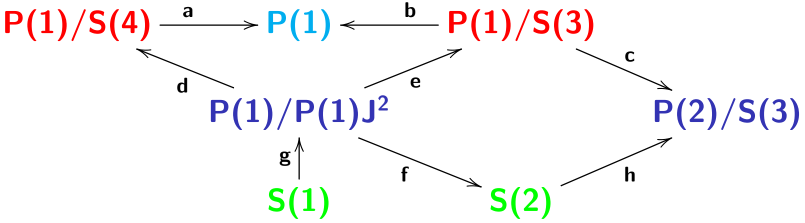

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Lemma

Lemma 3 [Iyama (2003)]

$\mathsf{B} = \mathsf{Be} \oplus \mathsf{Bf}$

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Lemma

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Lemma 4

$\exists \mathsf{X}$ **: indec. direct summand of** $\widetilde{\mathsf{M}}$ **s.t.**

- $\ell\ell(\mathsf{X}) = \mathsf{m} = \ell\ell(\mathsf{M})$
- **natural surj.** $\varphi : \mathsf{X} \twoheadrightarrow \mathsf{X}/\mathsf{XJ}^{\mathsf{m}-1}$ **induces an isom.**

$$\mathsf{Hom}_{\mathsf{A}}(\mathsf{X}/\mathsf{XJ}^{\mathsf{m}-1}, \widetilde{\mathsf{M}}) \xrightarrow{-\circ\varphi} \mathcal{J}_{\mathsf{mod\,A}}(\mathsf{X}, \widetilde{\mathsf{M}})$$

Sketch of Proof of Theorem 2

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1. **$\exists X$: indec. direct summand of $\widetilde{M}$ s.t. nat. surj. $\varphi : X \twoheadrightarrow X/XJ^{m-1}$ induces an isom. $\mathrm{Hom}_A(X/XJ^{m-1}, \widetilde{M}) \xrightarrow{-\circ\varphi} \mathcal{J}_{\mathrm{mod}\,A}(X, \widetilde{M})$ ($\because$ Lemma 4)**

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5. $\exists N$: semilocal module s.t. $\widetilde{M}/X = \widetilde{N}$

Strongly QH algebras and global dimension ≤ 2

Fact

- $\mathrm{gl.dim} A \leq 2 \Rightarrow A$: left-strongly QH [T (2018)]
- A : strongly QH $\Rightarrow \mathrm{gl.dim} A \leq 2$ [Ringel (2010)]

The converses do not hold in general

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The converses do not hold in general

Aim

Study the difference between $\mathrm{gl.dim} \leq 2$ **and strongly QH about ADR algebras**

Remark

B : Auslander algebra of A
B : strongly QH $\Leftrightarrow$ **A : Nakayama algebra**

Strongly QH ADR algebras

Theorem 5 [T (2018)]

Assume $\mathbf{M} = \mathbf{A}$ and $\mathbf{A}$: non-semisimple
Then the following are equivalent:

(1) **B : strongly QH algebra**

(2) $\mathrm{gl.dim}\mathbf{B} = 2$

(3) $\mathbf{J} \in \mathbf{add}\,\widetilde{\mathbf{A}}$

Sketch of Proof

(1)⇒(2): **B : strongly QH** $\Rightarrow \mathrm{gl.dim}\mathbf{B} \leq 2$ **($\because$ Ringel)**

(2)⇔(3): **[Smalø(1978)]**

(3)⇒(1): **Lemma 6**

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- **$\operatorname{add} \widetilde{\mathbf{A}} \supset \operatorname{add} \widetilde{\mathbf{A}}/\mathbf{X} \supset \cdots \supset \mathbf{0}$: left rejective ch. in Thm 2**

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