Finite dimensional algebras arising from dimer models and their derived equivalences

Yusuke Nakajima

Kavli IPMU, University of Tokyo

September 19th, 2018

Motivations

2-dimensional case (Reiten-Van den Bergh, Bockland-Schedler-Wemyss, etc.)

$$\mathbb{C}[x,y] * G \overset{\text{Morita}}{\sim} \Pi \quad \underset{\text{preprojective algebra}}{\overset{\text{degree 0 part}}{\rightleftarrows}} \quad \mathbb{C}\widetilde{Q}$$

- Π: 2-CY algebra, gl.dim=2
- $\mathbb{C}\widetilde{Q}$: rep. infinite hereditary algebra, gl.dim=1

where $\widetilde{Q}$: extended Dynkin quiver of type ADE
$G \subset \mathrm{SL}(2,\mathbb{C})$: finite subgroup

Higher dim. case (cf. Keller, Minamoto-Mori, Herchend-Iyama-Oppermann)

$$\mathcal{A} \quad \underset{d\text{-preprojective algebra}}{\overset{\text{degree 0 part}}{\rightleftarrows}} \quad \mathcal{A}_0$$

- $\mathcal{A}$: d-CY alg. of GP 1, gl.dim=d
- $\mathcal{A}_0$: $(d-1)$-rep. infinite alg., gl.dim=d-1

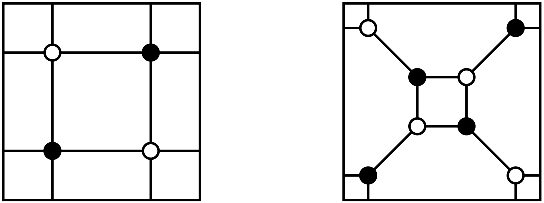
Some examples are obtained by **dimer models** (cf. Amiot-Iyama-Reiten).

What is a dimer model ?

Definition

A **dimer model** *(or* **brane tiling***) is a finite bipartite graph inducing a polygonal cell decomposition of the real two-torus* $\mathsf{T} := \mathbb{R}^2/\mathbb{Z}^2$.

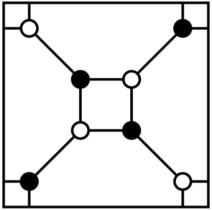
Therefore each node is colored either black or white so that each edge connects a black node to a white node.

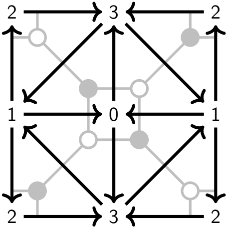


Quivers associated with dimer models

As the dual of a dimer model Γ, we define the quiver Q associated with Γ.

$$\begin{array}{rcl} \text{Dimer model } \Gamma & \longleftrightarrow & \text{Quiver } Q \\ \text{faces} & \longleftrightarrow & \text{vertices } Q_0 \\ \text{edges} & \longleftrightarrow & \text{arrows } Q_1 \end{array}$$





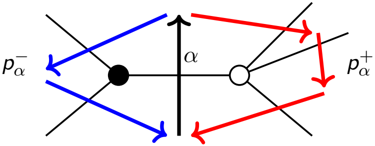
The orientation of arrows is determined so that the white node is on the right of the arrow.

(We also define a certain "potential W_Q" from a dimer model.)

Jacobian algebras arising from dimer models

- For the quiver Q associated with a dimer model, we consider the path algebra $\mathbb{C}Q$.

- For each arrow $\alpha \in Q_1$, $^{\exists}$two oppositely oriented cycles (αp_α^+ and αp_α^-) containing α as a boundary.



- Define the **Jacobian algebra** of Q (or Γ) as

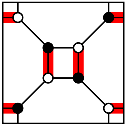
$$\mathcal{A} := \mathbb{C}Q/\langle p_\alpha^+ - p_\alpha^- \mid \alpha \in Q_1 \rangle.$$

Perfect matchings of dimer models

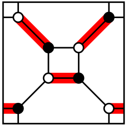
Definition

A **perfect matching** *of a dimer model* Γ *is a subset* D *of edges such that for any node* n *there is a unique edge in* D *containing* n *as the end point.*

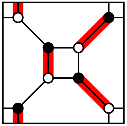
For example, perfect matchings of our dimer model are



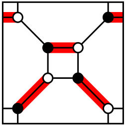
D_0



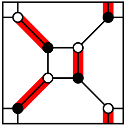
D_1



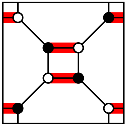
D_2



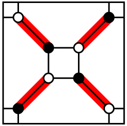
D_3



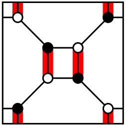
D_4



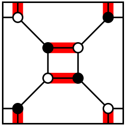
D_5



D_6



D_7



D_8

Calabi-Yau properties on Jacobian algebras

We consider

- Q : the quiver associated with a "consistent" dimer model Γ,
- $\mathcal{A}$: the Jacobian algebra of Q,
- D : a perfect matching of Γ.

We define the degree d_D on each arrow $a \in Q_1$ of Q as

$$d_D(a) = \begin{cases} 1 & \text{if } a \in D \\ 0 & \text{otherwise} \end{cases}$$

This makes the Jacobian algebra $\mathcal{A}$ a graded algebra.

Theorem (Broomhead, Amiot-Iyama-Reiten)

$\mathcal{A}$ *is a* **bimodule 3-Calabi-Yau algebra of Gorenstein parameter 1**, *that is,* $\mathcal{A} \in \mathrm{per}\mathcal{A}^e$ *and* $\exists P_\bullet$*: graded proj. resolution of* $\mathcal{A}$ *as* $\mathcal{A}^e$*-mod. s.t.*

$$P_\bullet \cong P_\bullet^\vee[3](-1)$$

where $(-)^\vee := \mathrm{Hom}_{\mathcal{A}^e}(-, \mathcal{A}^e)$.

2-representation infinite algebras arising from dimer models

We consider

- Q : the quiver associated with a consistent dimer model Γ,
- $\mathcal{A}$: the graded Jacobian algebra of Q whose grading induced by D

  (This is bimodule 3-CY algebra of GP 1),
- $\mathcal{A}_D$: the degree zero part of $\mathcal{A}$.

Theorem (Keller, Minamoto-Mori, Herchend-Iyama-Oppermann)

We assume that $\mathcal{A}_D$ is finite dimensional.
Then $\mathcal{A}_D$ is a **2-representation infinite algebra** *(or* **quasi 2-Fano algebra***), that is,*

$$\mathrm{gl.dim}\mathcal{A}_D \leq 2 \quad \text{and} \quad {\nu_2^-}^i(\mathcal{A}_D) \in \mathsf{mod}\mathcal{A}_D \text{ for all } i \geq 0$$

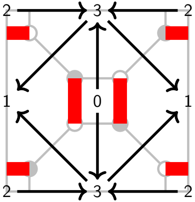
where $\nu_2^- := \nu^- \circ [2] : \mathrm{D^b}(\mathsf{mod}\mathcal{A}_D) \to \mathrm{D^b}(\mathsf{mod}\mathcal{A}_D)$.

*On the other hand, the "***3-preprojective algebra***" of $\mathcal{A}_D$ is $\mathcal{A}$.*

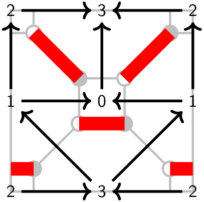
2-representation infinite algebras arising from dimer models

Question

When is the degree part $\mathcal{A}_D$ finite dimensional ?



$\mathcal{A}_{D_0}$: finite dimensional

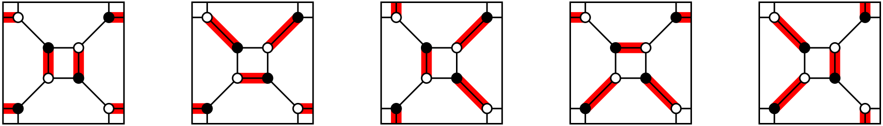


$\mathcal{A}_{D_1}$: not finite dimensional

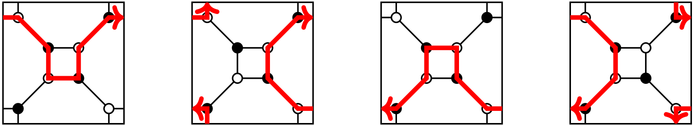
$\Rightarrow$ To understand this problem, consider the **perfect matching polygon**.

The perfect matching polygon

- Give the orientation to each edge of Γ: 
- Fix a perfect matching D' of Γ.
- For each perf. match. D, the difference $D - D'$ will be a 1-cycle on T.
- Consider $D - D'$ as the element in $[D - D'] \in \mathrm{H}_1(\mathrm{T}) \cong \mathbb{Z}^2$.



D_0 D_1 D_2 D_3 D_4



$[D_1 - D_0] = (1,0)$ $[D_2 - D_0] = (0,1)$ $[D_3 - D_0] = (-1,0)$ $[D_4 - D_0] = (0,-1)$

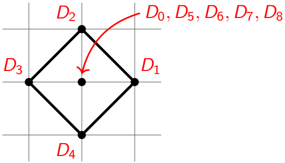
Remark that $[D_i - D_0] = (0,0)$ if $i = 0, 5, \cdots, 8$

The perfect matching polygon

- The **perfect matching polygon** of Γ:

$$\Delta_\Gamma := \mathrm{conv}\{[D - D'] \in \mathbb{Z}^2 \mid D \text{ is a perfect matching of } \Gamma\}$$

For our example, the PM polygon of our dimer model is



- This Δ_Γ is determined uniquely up to translations.
- D is called an **internal perfect matching** if it corresponds to an interior lattice point of Δ_Γ.

2-representation infinite algebras arising from dimer models

Theorem A (N., Bocklandt etc.)

Let Q be the quiver associated with a consistent dimer model Γ. Then, the following conditions are equivalent.

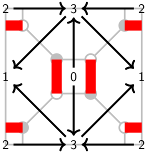
(1) *D is an internal perfect matching.*

(2) *$Q - D$ is an acyclic quiver.*

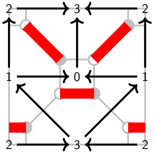
(3) *$\mathcal{A}_D$ is a finite dimensional algebra.*

When this is the case, $\mathcal{A}_D$ is a 2-representation infinite algebra.

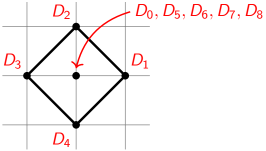
We recall that



$\mathcal{A}_{D_0}$: fin. dim.



$\mathcal{A}_{D_1}$: not fin. dim.

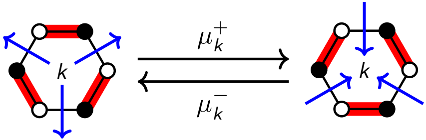


Δ_Γ: PM polygon

Is there a relationship between internal perfect matchings ?

Mutations of perfect matchings and derived equivalences

The **mutation of perfect matchings**:

e.g. 

In particular, if k is a source (resp. sink) of $Q - D$,

- $\mu_k^+(D)$ (resp. $\mu_k^-(D)$) is a perfect matching.
- k is a sink of $Q - \mu_k^+(D)$ (resp. a souce of $Q - \mu_k^-(D)$).
- $\mu_k^-(\mu_k^+(D)) = D$ (resp. $\mu_k^+(\mu_k^-(D)) = D$).

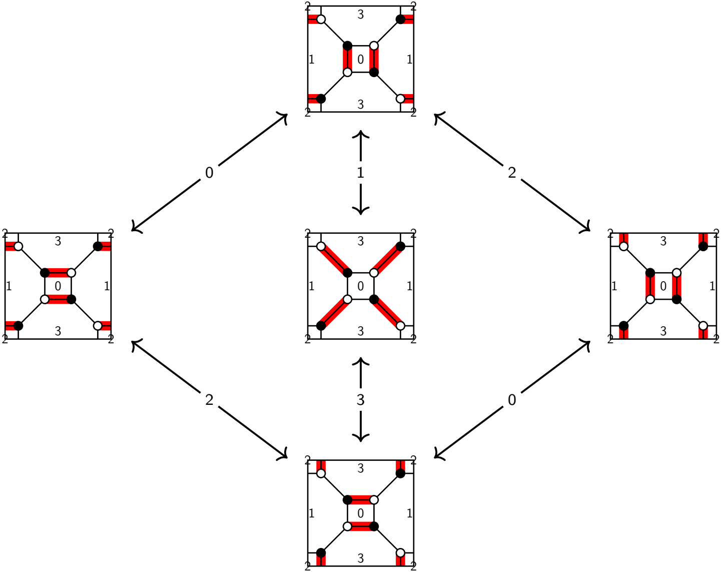
Theorem B (N.)

Let Γ be a consistent dimer model and Q be the associated quiver. Let D, D' be internal perfect matchings. Then, the followings are equivalent.

(1) *D and D' are "mutation equivalent".*

(2) *D and D' correspond to the same interior lattice point of Δ_Γ.*

Mutations of perfect matchings and derived equivalences



Mutations of perfect matchings and derived equivalences

Corollary (Theorem B + Iyama-Oppermann)

Let Γ be a consistent dimer model and Q be the associated quiver. If D, D' are internal perfect matchings corresponding to the same interior lattice point of Δ_Γ, then we have that

$$\mathrm{D^b}(\mathsf{mod}\mathcal{A}_D) \cong \mathrm{D^b}(\mathsf{mod}\mathcal{A}_{D'}).$$

Note that

- Iyama-Oppermann showed that if k is a source of $Q - D$ and $\mu_k^+(D) = D'$, then $\exists$tilting $\mathcal{A}_D$-module T_k s.t. $\mathsf{End}_{\mathcal{A}_D}(T_k) \cong \mathcal{A}_{D'}$.
- The converse of this corollary is not true, that is, even if $\mathrm{D^b}(\mathsf{mod}\mathcal{A}_D) \cong \mathrm{D^b}(\mathsf{mod}\mathcal{A}_{D'})$, D and D' might not correspond to the same interior lattice point.