Auslander correspondence for triangulated categories

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Background

Theorem (Auslander correspondence)

There exists a bijection between

1. *Finite abelian categories.*
2. *Finite dimensional algebras with* $\operatorname{gl.dim} \leq 2 \leq \operatorname{dom.dim}$.

A relationship between **categorical** and **homological** properties.

Aim

Give a triangulated analogue.

=Give a homological characterization of ‘finite’ triangulated categories.

Notations and setup

- k: field.
- 'category'=k-linear, Hom-finite, Krull-Schmidt category.
- a category $\mathcal{C}$ is **finite** if $\sharp \operatorname{ind} \mathcal{C} < \infty$
- for a finite category $\mathcal{C} = \operatorname{add} M$,
  $\operatorname{End}_{\mathcal{C}}(M)$: the **Auslander algebra** of $\mathcal{C}$.

Finiteness for triangulated categories

1. Finite.
2. '[1]-finite'.

Finite case

A homological characterization of Auslander algerbras of finite triangulated categories:

Theorem 1

k: perfect field. The following are equivalent for a basic finite dimensional k-algebra A:

1. *A is the basic Auslander algebra of a finite triangulated category.*
2. *A is self-injective and $\Omega^3 \simeq (-)_\alpha$ on $\underline{\mathrm{mod}}\, A$ for some automorphism α of A.*

Remark

Saying nothing about the triangle structures.

Proof of (1) ⇒ (2)

- A is self-injective since $\mathcal{T}$ is.
- Each triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow$$

in $\mathcal{T}$ yields an exact sequence

$$\begin{array}{ccccccccc} (-,Z[-1]) & \to & (-,X) & \longrightarrow & (-,Y) & \longrightarrow & (-,Z) & \to & (-,X[1]) \\ & \searrow & \nearrow & & & & \searrow & \nearrow & \\ & M[-1] & & & & & & M & \end{array}$$

in $\operatorname{mod}\mathcal{T} \simeq \operatorname{mod} A$, so $\Omega^3 M \simeq M[-1]$ in $\underline{\operatorname{mod}}\, A$.
- [1] can induce an automorphism of A since it is basic.

Sketch of (2) $\Rightarrow$ (1)

We want to show that $\operatorname{proj} A$ admits a structure of a triangulated category.

Proposition (Amiot)

Let $\mathcal{A}$ be a k-linear category such that $\operatorname{mod}\mathcal{A}$ is naturally Frobenius. Let S be an automorphism of $\mathcal{A}$ and extend this to $\operatorname{mod}\mathcal{A} \to \operatorname{mod}\mathcal{A}$ (by $M \mapsto M \circ S^{-1}$). Assume there exists an exact sequence

$$0 \longrightarrow 1 \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow S \longrightarrow 0$$

of exact functors from $\operatorname{mod}\mathcal{A}$ to $\operatorname{mod}\mathcal{A}$ such that the X^i's values in $\mathcal{A}$. Then, $\mathcal{A}$ has a structure of a triangulated category with suspension S.

The triangles are given by $X^0M \to X^1M \to X^2M \to SX^0M$ with $M \in \operatorname{mod}\mathcal{A}$.

Sketch of (2) $\Rightarrow$ (1)

Such an exact sequence of functors can be obtained by considering **bimodules** as functors.

Proposition (a variant of Green-Snashall-Solberg)

Let A be a ring-indecomposable non-semisimple finite dimensional k-alebra such that A/J_A is separable over k and $n > 0$. Then, the following are equivalent.

1. $\Omega^n(A/J_A) \simeq A/J_A$.
2. *A is self-injective and $\Omega^n \simeq (-)_\alpha$ on $\underline{\mathrm{mod}}\, A$ for some automorphism α of A.*
3. *There exists an automorphism of α of A such that $\Omega^n_{A^e}(A) \simeq {}_1A_\alpha$.*

Sketch of (2) $\Rightarrow$ (1)

Assume A is self-injective and $\Omega^3 \simeq (-)_\alpha$. Then, $\Omega^3_{A^e}(A) \simeq {}_1A_\sigma$ for some automorphism σ, so there exists an exact sequence

$$0 \longrightarrow A \longrightarrow P^0 \longrightarrow P^1 \longrightarrow P^2 \longrightarrow {}_1A_{\sigma^{-1}} \longrightarrow 0$$

in $\operatorname{mod} A^e$ with $P^i \in \operatorname{proj} A^e$, which gives a desired exact sequence of functors.

[1]-finite triangulated categories

Another finiteness for triangulated categories:

Definition

A triangulated category $\mathcal{T}$ is [1]-*finite* if

1. There exists $M \in \mathcal{T}$ such that $\mathcal{T} = \mathrm{add}\{M[i] \mid i \in \mathbb{Z}\}$.
2. For any $X, Y \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[i]) = 0$ for almost all $i \in \mathbb{Z}$.

In this case, we say M is a [1]-*additive generator* for $\mathcal{T}$.

Example

Λ: representation-finite hereditary algebra with $\mathrm{mod}\,\Lambda = \mathrm{add}\, M$.
$\Rightarrow D^b(\mathrm{mod}\,\Lambda)$ is [1]-finite, M is a [1]-additive generator for $D^b(\mathrm{mod}\,\Lambda)$.

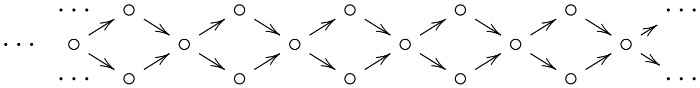
Known results

By the results of Xiao-Zhu and Riedtmann's 'knitting' argument, we know

Proposition

$\mathcal{T}$: [1]*-finite triangulated category over an algebraically closed field.*

1. *The AR-quiver of* $\mathcal{T}$ *is* $\mathbb{Z}Q$ *for some Dynkin quiver* Q.
2. $\mathcal{T}$ *is standard, hence* $\mathcal{T} \simeq k(\mathbb{Z}Q)$ *as* k*-linear categories.*



$\mathbb{Z}Q$ for type A_3

Graded projectivization

How to build the 'Auslander algebras' for [1]-finite triangulated categories?

Recall

$\mathcal{C}$: finite category with add $M = \mathcal{C}$. Setting $\Gamma = \mathrm{End}_{\mathcal{C}}(M)$, we have an equivalence

$$\mathrm{Hom}_{\mathcal{C}}(M, -) \colon \mathcal{C} \xrightarrow{\simeq} \operatorname{proj} \Gamma.$$

Γ does not depend on M up to Morita equivalence.

Graded projectivization

Proposition

$\mathcal{C}$: category with an automorphism F. Assume $\mathcal{C} = \operatorname{add}\{F^i M \mid i \in \mathbb{Z}\}$ for some $M \in \mathcal{C}$. Set $\Gamma = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(M, F^i M)$. Then, there exists an equivalence

$$\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(M, F^i(-)) \colon \underset{\circlearrowleft F}{\mathcal{C}} \xrightarrow{\simeq} \underset{\circlearrowleft (1)}{\operatorname{proj}^{\mathbb{Z}} \Gamma}.$$

Γ does not depend on M up to graded Morita equivalence.
Here, Graded rings A, B are *graded Morita equivalent* if there exists an equivalence

$$\underset{\circlearrowleft (1)}{\operatorname{Mod}^{\mathbb{Z}} A} \xrightarrow{\simeq} \underset{\circlearrowleft (1)}{\operatorname{Mod}^{\mathbb{Z}} B}.$$

[1]-Auslander algebra

Apply the above proposition to triangulated categories.

Definition

$\mathcal{T}$: [1]-finite triangulated category with [1]-additive generator M. We call

$$C = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(M, M[i])$$

the [1]-*Auslander algebra* of $\mathcal{T}$.

Proposition

1. *C is a finite dimensional algebra.*
2. *We have an equivalence $\mathcal{T} \simeq \mathrm{proj}^{\mathbb{Z}}\, C$ such that $[1] \leftrightarrow (1)$.*
3. *C is self-injective and $\Omega^3 \simeq (-1)$ on $\underline{\mathrm{mod}}^{\mathbb{Z}}\, C$.*

Proof of (3)

- C is self-injective since $\mathcal{T}$ is.
- Each triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow$$

in $\mathcal{T}$ yields an exact sequence

$$\begin{array}{ccccccccc} (-,Z[-1]) & \to & (-,X) & \longrightarrow & (-,Y) & \longrightarrow & (-,Z) & \to & (-,X[1]) \\ & \searrow & \nearrow & & & & & \searrow \quad \nearrow & \\ & M[-1] & & & & & & M & \end{array}$$

in $\operatorname{mod}\mathcal{T} \simeq \operatorname{mod}^{\mathbb{Z}} C$, so $\Omega^3 M \simeq M(-1)$ in $\underline{\operatorname{mod}}^{\mathbb{Z}} C$.

[1]-Auslander correspondence

Theorem 2

k: algebraically closed field. There exists a bijection between

1. *[1]-finite algebraic triangulated categories / triangle equivalence*
2. *Finite dimensional graded self-injective algebras such that $\Omega^3 \simeq (-1)$ / graded Morita equivalence*
3. *Disjoint union of Dynkin diagrams of type A, D, and E.*

The correspondences are given by

- From (1) to (2): taking the [1]-Auslander algebra.
- From (2) to (1): $C \mapsto \operatorname{proj}^{\mathbb{Z}} C$.
- From (1) to (3): taking the tree type of the AR-quiver of $\mathcal{T}$.
- From (3) to (1): $Q \mapsto k(\mathbb{Z}Q)$.

Uniqueness of triangle structures

(3) to (1) (or (2) to (1)) says:

Proposition

Q: Dynkin quiver, $k(\mathbb{Z}Q)$: its mesh category. Then, $k(\mathbb{Z}Q)$ has the unique structure of an algebraic triangulated category up to equivalence.

On the other hand, $k(\mathbb{Z}Q)$ has a structure of an algebraic triangulated category $D^b(\operatorname{mod} kQ)$.

Corollary

Any [1]-finite algebraic triangulated category over an algebraically closed field k is triangle equivalent to $D^b(\operatorname{mod} kQ)$ for some Dynkin quiver Q.

Remark

The uniqueness of algebraic triangle structures (up to equivalence) holds for $K^b(\operatorname{proj} \Lambda)$ for certain ring Λ.

Graded Iwanaga-Gorenstein algebras

- A **graded** Noetherian algebra Λ is *Iwanaga-Gorenstein* if $\operatorname{inj.dim}\Lambda < \infty$ on each side.
- We have the category

$$\mathsf{CM}^{\mathbb{Z}}\Lambda = \{X \in \operatorname{mod}^{\mathbb{Z}}\Lambda \mid \operatorname{Ext}_{\Lambda}^{>0}(X,\Lambda) = 0\}$$

  of **graded** *Cohen-Macaulay* Λ-modules.
- $\mathsf{CM}^{\mathbb{Z}}\Lambda$ is naturally Frobenius, hence the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}\Lambda$ is algebraically triangulated.
- Λ is *CM-finite* if $\mathsf{CM}^{\mathbb{Z}}\Lambda$ is finite up to degree shift.

The triangle equivalence

$\Lambda = \bigoplus_{i \geq 0} \Lambda_i$: positively graded Iwanaga-Gorenstein algebra such that

- each Λ_i is finite dimensional over k.
- $\operatorname{gl.dim} \Lambda_0 < \infty$.

Assume Λ is CM-finite.

Theorem 3

$\underline{\mathrm{CM}}^{\mathbb{Z}} \Lambda$ is $[1]$-finite, and therefore, if k is algebraically closed,

1. *The AR-quiver of $\underline{\mathrm{CM}}^{\mathbb{Z}} \Lambda$ is $\mathbb{Z}Q$ for some Dynkin quiver Q.*
2. *There exists a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}} \Lambda \simeq D^b(\operatorname{mod} kQ)$.*

e.g.

- (commutative) simple singularities
- finite dimensional representation-finite self-injective algebras
- representation-finite Gorenstein orders