The chamber structures of the Grothendieck groups coming from bricks

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Setting

- K: a field.
- A: a finite-dimensional K-algebra.
- $\operatorname{proj} A$: the category of fin. dim. proj. A-modules.
- $P_1, \ldots, P_n$: all non-iso. indec. proj. A-modules.
- $\operatorname{mod} A$: the category of fin. dim. A-modules.
- $S_1, \ldots, S_n$: all non-iso. simple A-modules (S_i is the top of P_i).
- $K_0(\mathcal{C})$: the Grothendieck group of $\mathcal{C}$.

Euler form

Proposition

(1) $(P_i)_{i=1}^n$ is a $\mathbb{Z}$-basis of $K_0(\mathrm{proj}\,A) \cong K_0(\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,A))$.

(2) $(S_i)_{i=1}^n$ is a $\mathbb{Z}$-basis of $K_0(\mathrm{mod}\,A) \cong K_0(\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A))$.

Definition

Euler form is the bilinear form
$\langle ?, ? \rangle \colon K_0(\mathrm{proj}\,A) \times K_0(\mathrm{mod}\,A) \longrightarrow \mathbb{Z}$ defined by

$$\langle T, X \rangle := \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)}(T, X[k])$$

for $T \in \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\,A)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\,A)$.

Standard dual bases

Proposition

$(P_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ give dual bases w.r.t. Euler form.

For any $i, j \in \{1, \ldots, n\}$,

$$\langle P_i, S_j \rangle = \begin{cases} \dim_K \operatorname{End}_A(S_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Koenig–Yang correspondence

We will find other dual bases in terms of

- (2-term) basic silting objects in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$,
- (2-term) simple-minded collections in $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

Proposition [Koenig–Yang, Brüstle–Yang]

There exist bijections

$$\operatorname{silt} A \to \operatorname{smc} A, \quad 2\text{-}\operatorname{silt} A \to 2\text{-}\operatorname{smc} A$$

sending T to the set of simple objects of the heart of the t-structure associated to T.

Dual bases property

Proposition [Koenig–Yang]

Let $T \in \operatorname{silt} A$ be sent to $\mathcal{X} \in \operatorname{smc} A$.
Then, there exist $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying

- $T = \bigoplus_{i=1}^n T_i$, $\mathcal{X} = \{X_i\}_{i=1}^n$, and
- $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ give dual bases w.r.t. Euler form.

For any $i, j \in \{1, \dots, n\}$,

$$\langle T_i, X_j \rangle = \begin{cases} \dim_K \operatorname{End}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(X_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Additional setting

Setting in the rest

Let $T \in$ 2-silt A be sent to $\mathcal{X} \in$ 2-smc A.
We take $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying

- $T = \bigoplus_{i=1}^n T_i$, $\mathcal{X} = \{X_i\}_{i=1}^n$, and
- $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ give dual bases w.r.t. Euler form.

For any $i, j \in \{1, \dots, n\}$,

$$\langle T_i, X_j \rangle = \begin{cases} \dim_K \mathrm{End}_{\mathrm{D^b}(\mathrm{mod}\, A)}(X_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Cones for objects in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$

Now we consider the real-valued Grothendieck group

$$K_0(\operatorname{proj} A)_{\mathbb{R}} := K_0(\operatorname{proj} A) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n.$$

For $U = \bigoplus_{i=1}^m U_i \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ with U_i: indec,
we define the cone $C(U)$ associated to U by

$$C(U) := \{a_1[U_1] + \cdots + a_m[U_m] \mid a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}\} \subset K_0(\operatorname{proj} A)_{\mathbb{R}}.$$

Later, we often assume $U = T \in 2\text{-}\operatorname{silt} A$.

Cones and mutations

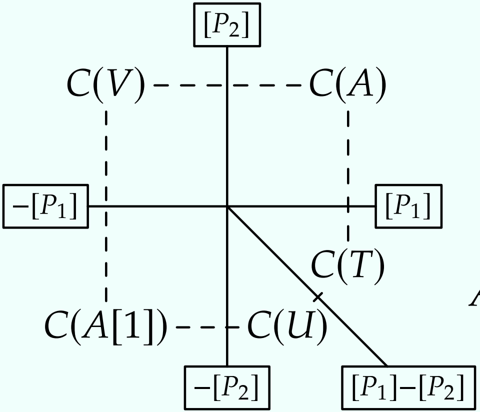
Remark

If $T, T' \in$ 2-silt A and add $T \cap$ add $T' =$ add U, then $C(T) \cap C(T') = C(U)$.

- $C(T)$ has n walls $C(T/T_i)$ with $i \in \{1, \dots, n\}$.
- $C(T/T_i)$ is $(n-1)$-dimensional.
- $C(T/T_i)$ corresponds to the mutation of T at T_i.
- $C(T/T_i)$ is orthogonal to $[X_i] \in K_0(\text{mod}\, A)$ w.r.t. Euler form.
- If $T \not\cong T' \in$ 2-silt A, then $C(T)^\circ \cap C(T') = \emptyset$ ($C(T)^\circ$ is the interior of $C(T)$).

Example

Let $A = K(1 \to 2)$.



$$\begin{aligned}
A &= P_1 \oplus P_2 \\
T &= P_1 \oplus (P_2 \to P_1) \\
U &= P_2[1] \oplus (P_2 \to P_1) \\
V &= P_1[1] \oplus P_2 \\
A[1] &= P_1[1] \oplus P_2[1]
\end{aligned}$$

Numerical torsion(-free) classes

Each $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ gives
$\theta := \langle \theta, ? \rangle \colon K_0(\operatorname{mod} A) \to \mathbb{R}$: a $\mathbb{Z}$-linear form.

Definition [Baumann–Kamnitzer–Tingley]

For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, we define

$$\overline{\mathcal{T}}_\theta := \{M \in \operatorname{mod} A \mid \forall N\text{: quotient of } M,\ \theta(N) \geq 0\},$$
$$\overline{\mathcal{F}}_\theta := \{M \in \operatorname{mod} A \mid \forall L\text{: submod. of } M,\ \theta(L) \leq 0\}$$

called the numerical torsion(-free) class.

Func. fin. $\Longrightarrow$ numerical

Remark [Adachi–Iyama–Reiten]

There exist bijections

$$\begin{aligned} \text{2-silt}\, A &\to \{\text{func. fin. torsion classes}\}, \\ T &\mapsto \mathcal{T}_T := \operatorname{Fac} H^0(T), \\ \text{2-silt}\, A &\to \{\text{func. fin. torsion-free classes}\}, \\ T &\mapsto \mathcal{F}_T := \operatorname{Sub} H^{-1}(\nu T). \end{aligned}$$

Moreover, $(\mathcal{T}_T, \mathcal{F}_T)$ is a torsion pair.

Proposition [Yurikusa]

$T \in \text{2-silt}\, A,\ \theta \in C(T)^\circ \Longrightarrow \overline{\mathcal{T}}_\theta = \mathcal{T}_T,\ \overline{\mathcal{F}}_\theta = \mathcal{F}_T.$

NTF equivalence

Definition

We define NTF equivalence $\sim$ in $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by

$$\theta \sim \theta' :\Longleftrightarrow \overline{\mathcal{T}}_{\theta} = \overline{\mathcal{T}}_{\theta'}, \ \overline{\mathcal{F}}_{\theta} = \overline{\mathcal{F}}_{\theta'},$$

and $[\theta] \subset K_0(\operatorname{proj} A)_{\mathbb{R}}$ as the NTF equiv. class of θ.

Proposition

For $T \in \text{2-silt}\, A$, $C(T)^{\circ}$ is an NTF equiv. class.

Question

How can we extend the observation outside the cones?

Semistable subcategories

Definition [King]

For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, we define $\mathcal{W}_\theta \subset \operatorname{mod} A$ by

$$\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta \subset \operatorname{Ker}\langle \theta, ? \rangle$$

called the semistable subcategory.

Remark

(1) If $\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$: NTF equivalent, then $\mathcal{W}_{(1-r)\theta + r\theta'}$ is constant for $r \in [0,1]$.

(2) If $T \in$ 2-silt A and $\theta \in C(T)^\circ$, then $\mathcal{W}_\theta = \{0\}$.

Walls defined for bricks

Definition

$S \in \operatorname{mod} A$: brick $:\Longleftrightarrow$ $\operatorname{End}_A(S)$: a division K-alg.
$\operatorname{brick} A := \{$all bricks in $\operatorname{mod} A\}$.

Definition [Brüstle–Smith–Treffinger]

For $S \in \operatorname{brick} A$, we define $\Theta_S \subset K_0(\operatorname{proj} A)_{\mathbb{R}}$ by

$$\Theta_S := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid S \in \mathcal{W}_\theta\} \subset \operatorname{Ker}\langle ?, S\rangle$$

called the wall for S.

Remark

For $T \in \text{2-silt}\, A$ and $S \in \operatorname{brick} A$, $C(T)^\circ \cap \Theta_S = \emptyset$.

Walls of cones vs. walls for bricks

Remark [Brüstle–Yang]

Any $\mathcal{X} \in$ 2-smc A satisfies $\mathcal{X} \subset (\text{brick}\, A) \cup (\text{brick}\, A)[1]$.

Proposition

Let $T \in$ 2-silt A be sent to $\mathcal{X} \in$ 2-smc A.
We take $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying

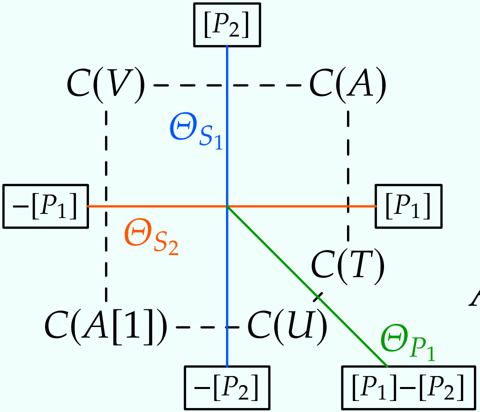
- $T = \bigoplus_{i=1}^n T_i$, $\mathcal{X} = \{X_i\}_{i=1}^n$, and
- $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ give dual bases w.r.t. Euler form.

Set $S \in$ brick A so that $X_i \in \{S, S[1]\}$, then

$$C(T/T_i) \subset \Theta_S.$$

Example

Let $A = K(1 \longrightarrow 2)$, then brick $A = \{S_2, P_1, S_1\}$.



$$
\begin{aligned}
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T &= P_1 \oplus (P_2 \longrightarrow P_1) \\
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A[1] &= P_1[1] \oplus P_2[1]
\end{aligned}
$$

NTF equiv. and cones

Proposition

Let $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, then TFAE.

- **(a)** The NTF equiv. class $[\theta]$ is an open set.
- **(b)** The interior $[\theta]^\circ$ is nonempty.
- **(c)** There exists $T \in \text{2-silt}\, A$ such that $\theta \in C(T)^\circ$.

Corollary

There exists a bijection

$$\text{2-silt}\, A \longrightarrow \left\{ \begin{array}{c} \text{NTF equiv. classes} \\ \text{with nonempty interiors} \end{array} \right\}, \quad T \mapsto C(T)^\circ.$$

NTF equiv. characterized by walls

Proposition

Let $\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, then TFAE.

(a) θ and θ' are not NTF equivalent.

(b) $\exists S \in \operatorname{brick} A$, $\exists_1 r \in [0, 1]$, $(1-r)\theta + r\theta' \in \Theta_S$.

Corollary

Let $\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, then TFAE.

(a) θ and θ' are NTF equivalent.

(b) $\mathcal{W}_{(1-r)\theta+r\theta'}$ is constant for $r \in [0, 1]$.

Cones = chambers

Theorem [A]

As subsets of $K_0(\text{proj}\, A)_{\mathbb{R}}$,

$$\coprod_{T \in \text{2-silt}\, A} C(T)^\circ = K_0(\text{proj}\, A)_{\mathbb{R}} \setminus \overline{\left(\bigcup_{S \in \text{brick}\, A} \Theta_S \right)},$$

where LHS is the decomp. to connected components.

There exist no cones $C(T)$ for $T \in$ 2-silt A
where the walls Θ_S for $S \in$ brick A lie densely.

Thank you for your attention.