BRICKS OVER PREPROJECTIVE ALGEBRAS

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Abstract. The representation theory of preprojective algebras is strongly related to the corresponding Coxeter groups. For a Dynkin diagram Δ, there is a bijection S from the Coxeter group of type Δ to the set of semibricks in the module category of the preprojective algebra of type Δ. In this paper, we give a combinatorial way to construct the semibrick $S(w)$ in the case $\Delta = \mathbb{A}_n$.

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Notation

Throughout this paper, K is a field and A is a finite-dimensional K-algebra. The category of finite-dimensional left A-modules is denoted by $\mathsf{mod}\,A$. Unless otherwise stated, algebras and modules are finite-dimensional, and subcategories are full subcategories.

1. Lattices

In this paper, we deal with several lattices. We first recall the definition of lattices and join-irreducible elements.

Definition 1. Let $(L, \leq)$ be a poset, and $x, y \in L$.

(1) An element $z \in L$ is called the *meet* of x and y if z is the maximum element satisfying $z \leq x$ and $z \leq y$. In this case, we define $x \wedge y := z$.
(2) An element $z \in L$ is called the *join* of x and y if z is the minimum element satisfying $z \geq x$ and $z \geq y$. In this case, we define $x \vee y := z$.
(3) The poset $(L, \leq)$ is called a *lattice* if the meet $x \wedge y$ and the join $x \vee y$ are defined for any $x, y \in L$.
(4) The poset $(L, \leq)$ is called a *finite lattice* if $(L, \leq)$ is a finite poset and a lattice.

Definition 2. Let $(L, \leq)$ a lattice, and $x \in L$. Then, x is said to be *join-irreducible* if $x \neq \mathsf{min}\, L$ and there exist no $y, z \in L$ satisfying $x = y \vee z$, $y \neq x$, $z \neq x$. We write $\mathsf{j\text{-}irr}\, L$ for the set of join-irreducible elements in L.

The detailed version of this paper will be submitted for publication elsewhere.

2. Bricks and torsion-free classes

We recall some basic properties of bricks and semibricks in the point of view of τ-tilting theory from [2].

A full subcategory $\mathcal{F} \subset \mathsf{mod}\, A$ is called a *torsion-free class* if $\mathcal{F}$ is closed under submodules and extensions. We define $\mathsf{torf}\, A$ as the set of torsion-free classes in $\mathsf{mod}\, A$. Then, $\mathsf{torf}\, A$ is a poset with inclusion relations $\subset$. Moreover, the poset $(\mathsf{torf}\, A, \subset)$ is a lattice.

In the rest, we assume that $\mathsf{torf}\, A$ is a finite set. This is equivalent to that A is *τ-tilting finite* [6].

Now, we define bricks and semibricks as follows.

Definition 3. Let S be an A-module in $\mathsf{mod}\, A$.

(1) The module S is called a *brick* if the endomorphism algebra $\mathsf{End}_A(S)$ is a division K-algebra. We write $\mathsf{brick}\, A$ for the set of bricks in $\mathsf{mod}\, A$.
(2) The module S is called a *semibrick* if $S = \bigoplus_{i=1}^m S_i$ with each $S_i \in \mathsf{brick}\, A$ and $\mathsf{Hom}_A(S_i, S_j) = 0$ $(i \neq j)$. We write $\mathsf{sbrick}\, A$ for the set of semibricks in $\mathsf{mod}\, A$.

For a semibrick S, we define $\mathsf{F}(S)$ as the smallest torsion-free class containing S. Then, there are the following bijections.

Proposition 4. [1, 2] *The operation* F *induces bijections* $\mathsf{sbrick}\, A \to \mathsf{torf}\, A$ *and* $\mathsf{brick}\, A \to \mathsf{j\text{-}irr}(\mathsf{torf}\, A)$.

3. Canonical join representations

The result in the previous section leads to the following question:

> let $S \in \mathsf{sbrick}\, A$ and $S = \bigoplus_{i=1}^m S_i$ be a decomposition into bricks. Then, how are the torsion-free class $\mathsf{F}(S)$ and the join-irreducible torsion-free classes $\mathsf{F}(S_1), \dots, \mathsf{F}(S_m)$ related?

The answer is given by the notion of canonical join representations introduced by Reading [10].

Definition 5. Let L be a lattice, $x \in L$, and $U \subset \mathsf{j\text{-}irr}\, L$. Then, U is called a *canonical join representation* of x if the following conditions hold.

(a) The join $\bigvee_{u \in U} u$ coincides with x.
(b) For any proper subset $U' \subsetneq U$, the join $\bigvee_{u' \in U'} u'$ does not coincide with x.
(c) Let $V \subset \mathsf{j\text{-}irr}\, L$ satisfy the conditions (a) and (b). Then, for any $u \in U$, there exists some $v \in V$ such that $u \leq v$.

For an element $x \in L$, if x admits a canonical join representation, then it is unique. The existence of a canonical join representation is not guaranteed, but the lattice $\mathsf{torf}\, A$ always admits a canonical join representation.

Theorem 6. *Let* $S \in \mathsf{sbrick}\, A$ *and* $S = \bigoplus_{i=1}^m S_i$ *be a decomposition into bricks. Then* $\mathsf{F}(S) = \bigvee_{i=1}^m \mathsf{F}(S_i)$ *is the canonical join representation.*

We remark that this theorem is generalized in [3, Proposition 3.2.5].

4. Coxeter groups and preprojective algebras

Now, we start to deal with Coxeter groups and preprojective algebras. We define the following symbols.

- Let Δ be a simply-laced Dynkin diagram with the vertices set Δ_0.
- Define W as the Coxeter group associated to Δ with its canonical generators s_i ($i \in \Delta_0$).
- We consider the right weak order $\leq$ on W. Then $(W, \leq)$ is a lattice [4].
- Define Π as the preprojective K-algebra associated to Δ.
- For $i \in \Delta_0$, we set e_i as the idempotent for the vertex i, and I_i as the ideal $\Pi(1-e_i)\Pi$.

Under this preparation, we can define the ideal $I(w)$ for each $w \in W$, which was firstly considered in [8, 5].

Definition 7. Let $w \in W$ and $w = s_{i_1}s_{i_2}\cdots s_{i_l}$ be a reduced expression. Then, we set $I(w) := I_{i_1}I_{i_2}\cdots I_{i_l} \subset \Pi$, and $J(w) := \Pi/I(w)$.

We remark that there may be several reduced expressions for an element $w \in W$, but that $I(w)$ does not depend on the choice of a reduced expression.

Mizuno gave the following remarkable bijection, which motivated our study.

Proposition 8. [9, Theorem 2.30] *There exists an isomorphism* $(W, \leq) \to (\mathsf{torf}\,\Pi, \subset)$ *of finite lattices given by* $w \mapsto \mathsf{Sub}\,J(w)$.

5. Coxeter groups and semibricks

By using Propositions 4 and 8 and the results in [1, 2], we obtain the following bijections.

Proposition 9. *There exists a bijection* $S\colon W \to \mathsf{sbrick}\,\Pi$ *given by* $w \mapsto \mathsf{soc}_{\mathsf{End}_\Pi(J(w))} J(w)$. *Moreover, it is restricted to a bijection* $S\colon \mathsf{j\text{-}irr}\,W \to \mathsf{brick}\,\Pi$.

Combining this and Theorem 6, we immediately get the following relationship between the Coxeter group and the semibricks.

Corollary 10. *Let* $S(w) \in \mathsf{sbrick}\,\Pi$ *and take* $w_1, \dots, w_m \in \mathsf{j\text{-}irr}\,W$ *such that* $S(w) = \bigoplus_{i=1}^m S(w_i)$. *Then,* $w = \bigvee_{i=1}^m w_i$ *is the canonical join representation.*

Therefore, we can determine the semibrick $S(w)$ by the following two steps.

(a) We explicitly give the canonical join representation $w = \bigvee_{i=1}^m w_i$ of each $w \in W$.
(b) We calculate the brick $S(w_i)$ for each i.

6. Canonical join representations in Coxeter groups

The aim of this section is to give the canonical join representation of $w \in W$ in the case $\mathbb{A}_n$.

In the rest, let $\Delta := \mathbb{A}_n$. We can identify W with the symmetric group $\mathfrak{S}_{n+1}$ by $s_i \mapsto (i \quad i+1)$. We express $w \in W$ in the form $(w(1), w(2), \dots, w(n+1))$.

We define some combinatorial notions.

Definition 11. Let $w \in W$, and $a, b \in \{1, 2, \dots, n+1\}$.

(1) A pair (a,b) is called an *inversion* of w if $w^{-1}(a) < w^{-1}(b)$ and $a > b$. We write $\mathsf{inv}(w)$ for the set of inversions of w.
(2) A pair (a,b) is called a *cover reflection* of w if $w^{-1}(a) = w^{-1}(b) - 1$ and $a > b$. We write $\mathsf{cov}(w)$ for the set of cover reflections of w.

It is well-known that $\mathsf{inv}(w) \subset \mathsf{inv}(w')$ is equivalent to $w \leq w'$.

The join-irreducible elements in W are characterized as follows.

Lemma 12. *Let $w \in W$. Then, w is join-irreducible if and only if there uniquely exists $l \in [1,n]$ such that $w(l) > w(l+1)$.*

In the case above, we say that w is an join-irreducible element of *type* l, and we set $R(w) := w([l+1,n+1])$. The correspondence $w \mapsto R(w)$ is an injection.

Reading obtained the following characterization of canonical join representations in the Coxeter group. This holds for any Coxeter groups of Dynkin type.

Proposition 13. [10, Theorem 10-3.9] *Let $w \in W$.*

(1) *For any $t \in \mathsf{cov}(w)$, there exists a minimum element w_t in the set $\{v \in W \mid v \leq w,\ t \in \mathsf{inv}(v)\}$.*
(2) *The canonical join representation of w is $w = \bigvee_{t \in \mathsf{cov}(w)} w_t$.*

Thus, the semibrick $S(w)$ has exactly $\# \mathsf{cov}(w)$ bricks as direct summands.

Let $t \in \mathsf{cov}(w)$. We can find w_t by the following observation: if $v \in W$ satisfies $v \leq w$ and $\mathsf{cov}(w) = \{t\}$, then $v = w_t$.

In the case that $\Delta = \mathbb{A}_n$, the join-irreducible element w_t is given as follows. This coincides with [10, Theorem 10-5.6].

Theorem 14. *Let $w \in W$, $t = (a,b) \in \mathsf{cov}(w)$. Then,*

$$R(w_t) = \{b\} \cup \{i \in [b+1, a-1] \mid w^{-1}(b) < w^{-1}(i)\} \cup [a+1, n+1].$$

Example 15. Let $n := 8$, and $w := (4,9,3,6,2,8,5,1,7)$. Then, the set $\mathsf{cov}(w)$ is $\{(9,3),(6,2),(8,5),(5,1)\}$, and

$$w_{(9,3)} = (1,2,4,9,3,5,6,7,8), \quad w_{(6,2)} = (1,3,4,6,2,5,7,8,9),$$
$$w_{(8,5)} = (1,2,3,4,6,8,5,7,9), \quad w_{(5,1)} = (2,3,4,5,1,6,7,8,9).$$

7. Description of bricks

In this section, we explicitly write down the structure of the brick $S(w)$ for $w \in \mathsf{j\text{-}irr}\, W$ in the case $\Delta = \mathbb{A}_n$. By using the result of [7] on $J(w)$, we have the following description of $S(w)$.

Theorem 16. *Let $w \in \mathsf{j\text{-}irr}\, W$. Then the brick $S(w)$ is given as follows.*

- *Take the unique $(a,b) \in \mathsf{cov}(w)$, and set $V := [a, b-1]$.*
- *The brick $S(w)$ has a K-basis $(\langle i \rangle)_{i \in V}$ with $\langle i \rangle \in e_i S(w)$.*
- *Place a symbol i for each $i \in V$, which denotes the one-dimensional vector subspace $K\langle i \rangle \subset S(w)$.*
- *For $i \in [a, b-2]$, write an arrow $i \to i+1$ if $i+1 \in R(w)$ and $i+1 \to i$ if $i+1 \notin R(w)$*

Example 17. Let $n := 8$, and $w := (4, 9, 3, 6, 2, 8, 5, 1, 7)$ as in Example 15. Then, the semibrick $S(w)$ is the direct sum of the following bricks:

$$\begin{aligned}
S(w_{(9,3)}) &= S(1,2,4,9,3,5,6,7,8) = && 3 \leftarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8, \\
S(w_{(6,2)}) &= S(1,3,4,6,2,5,7,8,9) = && 2 \leftarrow 3 \leftarrow 4 \rightarrow 5, \\
S(w_{(8,5)}) &= S(1,2,3,4,6,8,5,7,9) = && 5 \leftarrow 6 \rightarrow 7, \\
S(w_{(5,1)}) &= S(2,3,4,5,1,6,7,8,9) = && 1 \leftarrow 2 \leftarrow 3 \leftarrow 4.
\end{aligned}$$

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