A categorical approach to algebras and coalgebras

Robert Wisbauer

University of Düsseldorf, Germany

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Overview

- Category theory

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- Adjoint functors

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- Monads and comonads

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- Adjoints and (co)monads

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- Distributive laws

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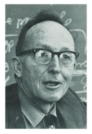
Category theory

Eilenberg - Mac Lane,
General Theory of natural equivalences
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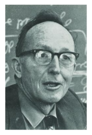
Category theory

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Category $\mathbb{A}$: objects and morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$

- functors $F : \mathbb{A} \to \mathbb{B}$;
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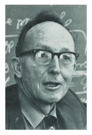

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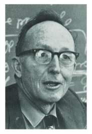

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Natural transformations $\psi : F \to G, \quad F, G : \mathbb{A} \to \mathbb{B}$

$$\begin{array}{ccc} A & & \\ \downarrow h & & \\ A', & & \end{array} \qquad \begin{array}{ccc} F(A) & \xrightarrow{\psi_A} & G(A) \\ F(h)\downarrow & & \downarrow G(h) \\ F(A') & \xrightarrow{\psi_{A'}} & G(A') \end{array}$$

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Adjoint pair of functors $F : \mathbb{A} \to \mathbb{B}$, $\mathbb{G} : \mathbb{B} \to \mathbb{A}$, bijection

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ε split epimorpism	G is separable

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Adjoint functors

Module categories

$$_R M \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(M, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

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- ε isomorphism: ${}_RM$ generator

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- ε monomorph and η epimorph (tilting theory)
  $\mathrm{Gen}(M) \simeq \mathrm{Cog}({}_SU)$, Brenner-Butler equivalence
  $U = \mathrm{Hom}(M, Q)$, Q cogenerator in $\mathrm{Gen}(M)$

Monads and comonads

Monads: $T : \mathbb{A} \to \mathbb{A}$ endofunctor

natural transformations: $m : TT \to T, \quad e : 1_{\mathbb{A}} \to T,$
with associativity and unitality conditions

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T-modules: $\varrho : T(A) \to A$

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T-morphisms $(A, \varrho) \xrightarrow{f} (A', \varrho'),$

$$\begin{array}{ccc} T(A) & \xrightarrow{T(f)} & T(A') \\ \varrho \downarrow & & \downarrow \varrho' \\ A & \xrightarrow{f} & A' \end{array}$$

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Free and forgetful functor - adjoint

$$\phi_T : \mathbb{A} \to \mathbb{A}_T, \quad A \mapsto (T(A), m_A : TT(A) \to T(A)),$$
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$$T = GF : \mathbb{A} \to \mathbb{A}, \qquad \text{product} \quad m : GFGF \xrightarrow{G\varepsilon F} GF,$$

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(K, δ, m') with Frobenius condition (without unit)

Adjoints and (co)monads

Adjoint pair: $F \dashv G : \mathbb{B} \to \mathbb{A}$, $\eta : 1_{\mathbb{A}} \to GF$, $\varepsilon : FG \to 1_{\mathbb{B}}$

ε split epi (G separable) : monad $T = GF : \mathbb{A} \to \mathbb{A}$

coproduct $\quad \delta' : GF \xrightarrow{G\varepsilon^{-1}F} GFGF, \qquad m \circ \delta' = 1_{GF}$

(T, m, δ') with Frobenius condition (without counit)

$$\begin{array}{ccc} TT & \xrightarrow{\delta T} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Tm} \\ T & \xrightarrow{\delta} & TT, \end{array} \qquad \begin{array}{ccc} TT & \xrightarrow{T\delta} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mT} \\ T & \xrightarrow{\delta} & TT, \end{array}$$

T separable monad

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(K, δ, m') with Frobenius condition (without unit)

K coseparable comonad

Adjoints and (co)monads

Module categories - ${}_R M_S$

$$M \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \qquad \mathrm{Hom}_R(M, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

Adjoints and (co)monads

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ε isomorphisms: ${}_R M$ generator

M_S fin. gen., projective, $R \simeq \mathrm{End}(M_S)$: comonad

$$M \otimes_S \mathrm{Hom}_R(M, -) \simeq M \otimes_S M^* \otimes_R - : {}_R\mathbb{M} \xrightarrow{\simeq} {}_R\mathbb{M}$$

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$$M^* \otimes_R \mathrm{Hom}_S(M^*, -) \simeq M^* \otimes_R M \otimes_S - : {}_S\mathbb{M} \to {}_S\mathbb{M}$$

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$M \otimes_S M^*$ R-coring, $M^* \otimes_R M$ S-coring

Adjoints and (co)monads

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$$M \otimes_S \mathrm{Hom}_R(M, -) \simeq \mathrm{Hom}_R(M \otimes_S \mathrm{Hom}_R(M, Q), -)$$

Adjoints and (co)monads

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R cogenerator in ${}_R\mathbb{M}$

$$\begin{aligned}\mathrm{Hom}_R(M \otimes_S M^*, -) &\simeq \mathrm{Hom}_R(M \otimes_S M^*, \mathrm{Hom}_R(M \otimes_S M^*, -)) \\ &\simeq \mathrm{Hom}_R(M \otimes_S M^* \otimes_R M \otimes_S M^*, -)\end{aligned}$$

$M \otimes_S M^*$ is R-ring

Adjoint monads and comonads

Adjoint endofunctors $F : \mathbb{A} \to \mathbb{A}, \quad G : \mathbb{A} \to \mathbb{A}$

$$\mathrm{Mor}_{\mathbb{A}}(F(X), Y) \xrightarrow{\varphi} \mathrm{Mor}_{\mathbb{A}}(X, G(Y)), \qquad \begin{array}{l} \eta : 1_{\mathbb{A}} \to GF, \\ \varepsilon : FG \to 1_{\mathbb{A}}. \end{array}$$

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F monad, $m : FF \to F$, $e : I_{\mathbb{A}} \to F$

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\mathrm{Mor}_{\mathbb{A}}(F(X), Y) & \xrightarrow{\varphi_{X,Y}} & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \\
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implies G **comonad,** $\underline{\delta} : G \to GG$, $\underline{\varepsilon} : G \to I_{\mathbb{A}}$

$$\underline{\delta} : G \xrightarrow{\eta G} GFG \xrightarrow{G\eta FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\varepsilon} GG,$$

$$\underline{\varepsilon} : G \xrightarrow{eG} FG \xrightarrow{\varepsilon} 1_{\mathbb{A}}\,.$$

Adjoint endofunctors

(F, m, e) monad, $F \dashv G$

$(G, \underline{\delta}, \underline{\varepsilon})$ is comonad, equivalence of categories $\mathbb{A}_F \simeq \mathbb{A}^G$

$$F(A) \xrightarrow{h} A \quad \mapsto \quad A \xrightarrow{\eta_A} GF(A) \xrightarrow{G(h)} G(A)$$

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$$\mathrm{Mor}_{\mathbb{A}^G}(\phi^G(A), \phi^G(A')) \simeq \mathrm{Mor}_{\mathbb{A}}(G(A), A')$$

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Module categories

Adjoint endofunctors $A \otimes_R -,\ \mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}$

$$\mathrm{Hom}_R(A \otimes_R X, Y) \xrightarrow{\simeq} \mathrm{Hom}_R(X, \mathrm{Hom}_R(A, Y))$$

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Frobenius monad $A \otimes_R - \simeq \mathrm{Hom}_R(A, -)$, $\mathbb{M}_A \simeq \mathbb{M}^A$

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Frobenius monad $\quad A \otimes_R - \simeq \mathrm{Hom}_R(A, -), \quad \mathbb{M}_A \simeq \mathbb{M}^A$

$A \otimes_R -$ comonad (R-coring) $\Leftrightarrow$ $\mathrm{Hom}_R(A, -)$ monad

$$\widetilde{\mathbb{M}}^A \simeq \widetilde{\mathbb{M}}_{\mathrm{Hom}_R(A,-)},\quad A \otimes_R X \mapsto \mathrm{Hom}_R(A, X)$$

Module categories

Adjoint endofunctors $A\otimes_R -,\ \mathrm{Hom}_R(A,-) : {}_R\mathbb{M} \to {}_R\mathbb{M}$

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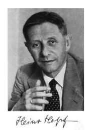
${}_RA$ fin. gen., projective $\mathrm{Hom}_R(A,-) \simeq A^*\otimes_R -$

$$A\otimes_R - \text{ monad} \quad \Leftrightarrow \quad A^*\otimes_R - \text{ comonad}$$

$$A\otimes_R - \text{ comonad} \quad \Leftrightarrow \quad A^*\otimes_R - \text{ monad}$$

$$\text{Frobenius} \quad A \simeq A^*, \quad \mathbb{M}_A \simeq \mathbb{M}^A$$

Hopf algebras



Heinz Hopf

Über die Topologie der Gruppen-Mannigfaltigkeiten und ihre Verallgemeinerungen, 1941

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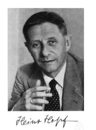

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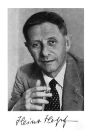

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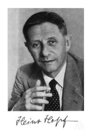

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equivalence $A \otimes_R - : \mathbb{M}_R \to {}_A^A\mathbb{M}$ (Hopf modules)

Composition of monads and comonads

Tensorproduct of R-algebras (A, m, e), (B, m', e')

$$A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \tau \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B$$

Distributive law: $\tau : B \otimes_R A \to A \otimes_R B$

$$\begin{array}{ccc} B \otimes B \otimes A & \xrightarrow{m' \otimes A} & B \otimes A \\ {\scriptstyle B \otimes \tau} \downarrow & & \downarrow {\scriptstyle \tau} \\ B \otimes A \otimes B \xrightarrow{\tau \otimes B} A \otimes B \otimes B & \xrightarrow{A \otimes m'} & A \otimes B, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e' \otimes A} & B \otimes A \\ & {\scriptstyle A \otimes e'} \searrow & \downarrow {\scriptstyle \tau} \\ & & A \otimes B. \end{array}$$

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Lifting of endofunctors

$$\begin{array}{ccc} {}_B\mathbb{M} & \xrightarrow{?} & {}_B\mathbb{M} \\ {\scriptstyle U_B}\downarrow & & \downarrow{\scriptstyle U_B} \\ \mathbb{M} & \xrightarrow{A \otimes_R -} & \mathbb{M}, \end{array} \qquad ? = A \otimes B \otimes -$$

Composition of monads and comonads

Liftings of endofunctors

$F, G : \mathbb{A} \to \mathbb{A}$, (F, m, e) monad, consider the diagram

$$\begin{array}{ccc} \mathbb{A}_F & \xrightarrow{\overline{G}} & \mathbb{A}_F \\ {\scriptstyle U_F}\downarrow & & \downarrow{\scriptstyle U_F} \\ \mathbb{A} & \xrightarrow{G} & \mathbb{A}, \end{array}$$

Composition of monads and comonads

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Questions

- does a lifting $\overline{G}$ exist ?

Composition of monads and comonads

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- F and G monads, when is $\overline{G}$ a monad ?

Composition of monads and comonads

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- F monad, G comonad, when is $\overline{G}$ a comonad ?

Beck, J., *Distributive laws*, 1969

$F, G : \mathbb{A} \to \mathbb{A}$ natural transformations $\lambda : FG \to GF$

Mixed distributive law (entwining): (F, m, e), (G, δ, ε)

lifting $\overline{G}$ comonad $\Leftrightarrow$ $\lambda : FG \to GF$ with comm. diagrams

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\begin{array}{ccccc}
FFG & & \xrightarrow{\quad m_G \quad} & & FG \\
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FGF & \xrightarrow{\lambda_F} & GFF & \xrightarrow{Gm} & GF,
\end{array}
\qquad
\begin{array}{ccccc}
FG & \xrightarrow{F\delta} & FGG & \xrightarrow{\lambda_G} & GFG \\
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GF & & \xrightarrow{\quad \delta_F \quad} & & GGF,
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$$\begin{array}{ccc} G & \xrightarrow{e_G} & FG \\ & {\scriptstyle Ge}\searrow & \downarrow{\scriptstyle \lambda} \\ & & GF, \end{array} \qquad \begin{array}{ccc} FG & \xrightarrow{F\varepsilon} & F \\ {\scriptstyle \lambda}\downarrow & \nearrow{\scriptstyle \varepsilon_F} & \\ GF. & & \end{array}$$

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Mixed modules: $\varrho_A : F(A) \to A$, $\varrho : A \to G(A)$, category $\mathbb{A}_F^G$

$$
\begin{array}{ccccc}
F(A) & \xrightarrow{\varrho_A} & A & \xrightarrow{\varrho^A} & G(A) \\
{\scriptstyle F(\varrho^A)}\downarrow & & & & \uparrow{\scriptstyle (\varrho_A)} \\
FG(A) & & \xrightarrow{\quad \lambda_A \quad} & & GF(A).
\end{array}
$$

Bimonad $B : \mathbb{A} \to \mathbb{A}$, (B, m, e), (B, δ, ε)

mixed distributive law $\lambda : BB \to BB$, compatibility

$$\begin{array}{ccccc} BB & \xrightarrow{m} & B & \xrightarrow{\delta} & BB \\ {\scriptstyle B\delta}\downarrow & & & & \uparrow{\scriptstyle Bm} \\ BBB & & \xrightarrow{\lambda_B} & & BBB. \end{array} \qquad B(A) \in \mathbb{A}_B^B$$

$\eta : 1_{\mathbb{A}} \to B$ monad morphism, $\quad \varepsilon : B \to 1_{\mathbb{A}}$ comonad morphism

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Category of (mixed) B-bimodules $\mathbb{A}^B_B$ - free functor

$$\phi^B_B : \mathbb{A} \to \mathbb{A}^B_B, \quad A \mapsto BB(A) \xrightarrow{m_A} B(A) \xrightarrow{\delta_A} BB(A).$$

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Hopf monads (antipode $S : B \to B$)

$$\phi^B_B \text{ equivalence} \quad \Leftrightarrow \quad BB \xrightarrow{B\delta} BBB \xrightarrow{mB} BB$$

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Finite dimensional algebras: $\mathrm{Hom}_K(A, -) \simeq A^* \otimes_K -$

adjoint pair	$(A \otimes_K -, A^* \otimes_K -)$,
counit	$\varepsilon : A \otimes_K A^* \to K,\ a \otimes f \mapsto f(a)$,
unit	$\eta : K \to A^* \otimes_K A,\ 1 \mapsto \sum a_i^* \otimes a_i$ (dual basis).

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A-coring $\quad A^* \otimes_K A \otimes_A -: \quad {}_A\mathbb{M} \to {}_A\mathbb{M}$,

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comonad	${}_{A^*}\widetilde{\mathbb{M}} \to {}_{A^*}\widetilde{\mathbb{M}}$, $A^* \otimes Y \mapsto A^* \otimes A \otimes Y$

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Nakayama functors, $D(-) := (-)^*$, $D(A) = A^*$

$$\begin{aligned} \nu(-) &:= D\,\mathrm{Hom}_A(-, {}_AA) : && A-\mathrm{mod} \to A-\mathrm{mod} \\ \nu^-(-) &:= \mathrm{Hom}_A(D(-), A_A): && A-\mathrm{mod} \to A-\mathrm{mod} \end{aligned}$$

$$\begin{aligned} \nu(X) &= D\,\mathrm{Hom}_A(X, {}_AA) &\simeq& \; D(A) \otimes_A X, \\ \nu^-(X) &= \mathrm{Hom}_A(D(X), A_A) &\simeq& \; \mathrm{Hom}_A(D(A), X), \end{aligned}$$

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Adjoint pair

$$\begin{array}{rl} \mathrm{Hom}_A(D(A), -): & A-\mathrm{mod} \to A-\mathrm{mod}, \\ D(A) \otimes_A -: & A-\mathrm{mod} \to A-\mathrm{mod} \end{array}$$

Thank you !

Bimonads

Antipode - natural transformation $S : B \to B$

$$\begin{array}{ccccc} B & \xrightarrow{\varepsilon} & I & \xrightarrow{\eta} & B \\ \downarrow{\scriptstyle \delta} & & & & \uparrow{\scriptstyle \mu} \\ BB & & \xrightarrow[BS]{S_B} & & BB \end{array}$$

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$$\gamma : BB \xrightarrow{\delta_B} BBB \xrightarrow{Bm} BB$$

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Equivalent

(a) B has an antipode;

(b) γ is an isomorphism.

Braided bimonads

Consider $\mathbf{B} = (B, m, e, \delta, \varepsilon)$, where $B : \mathbb{A} \to \mathbb{A}$ is such that $\underline{B} = (B, m, e)$ is a monad and $\overline{B} = (B, \delta, \varepsilon)$ is a comonad.

Double entwinings

natural transformation $\tau : BB \to BB$ such that

(i) τ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;

(ii) τ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$.

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These conditions are obviously equivalent to

(iii) τ is a monad distributive law for the monad $\underline{B}$;

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Braided bimonads

Let $\tau : BB \to BB$ be a double entwining - commutative

$$\begin{CD}
BB @>{m}>> B @>{\delta}>> BB \\
@V{\delta\delta}VV @. @AA{mm}A \\
BBBB @>>{B\tau B}> @. BBBB,
\end{CD} \qquad (1)$$

$$\begin{CD}
BB @>{B\varepsilon}>> B \\
@V{m}VV @VV{\varepsilon}V \\
B @>{\varepsilon}>> 1,
\end{CD}
\qquad
\begin{CD}
1 @>{e}>> B \\
@V{e}VV @VV{\delta}V \\
B @>>{eB}> BB,
\end{CD}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{e} & B \\
 & \underset{=}{\searrow} & \downarrow{\scriptstyle \varepsilon} \\
 & & 1.
\end{array} \qquad (2)$$

Then $\quad \overline{\tau} : BB \xrightarrow{\delta B} BBB \xrightarrow{B\tau} BBB \xrightarrow{mB} BB$

is a **mixed distributive law** from the monad $\underline{B}$ to the comonad $\overline{B}$.

Braided bimonads

$\tau^2 = 1$ and τ satisfies the Yang-Baxter equation

$$\begin{array}{ccccc} BBB & \xrightarrow{\tau B} & BBB & \xrightarrow{B\tau} & BBB \\ {\scriptstyle B\tau}\downarrow & & & & \downarrow{\scriptstyle \tau B} \\ BBB & \xrightarrow[\tau B]{} & BBB & \xrightarrow[B\tau]{} & BBB\,, \end{array}$$

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BBB & \xrightarrow[\tau B]{} & BBB & \xrightarrow[B\tau]{} & BBB\,,
\end{array} \qquad \text{then}$$

BB is a bimonad

with multiplication, comultiplication and entwining structure

$$BBBB \xrightarrow{B\tau B} BBBB \xrightarrow{mm} BB\,,$$

$$BB \xrightarrow{\delta\delta} BBBB \xrightarrow{B\tau B} BBBB,$$

$$BBBB \xrightarrow{B\tau B} BBBB \xrightarrow{\tau\tau} BBBB \xrightarrow{B\tau B} BBBB\,.$$

Braided bimonads

Opposite bimonad

Given τ as above, an opposite bimonad B^{op} can be defined for B with multiplication

$$m \cdot \tau : BB \xrightarrow{\tau} BB \xrightarrow{m} B$$

and comultiplication

$$\tau \cdot \delta : B \xrightarrow{\delta} BB \xrightarrow{\tau} BB.$$

If B has an antipode S, then $S : B^{\mathrm{op}} \to B$ is a bimonad morphism provided that

$$\tau \cdot BS = SB \text{ and } \tau \cdot BS = SB.$$

Module categories

Coalgebras - comultiplication and counit, $C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

with coassociativity and counitality conditions.

Module categories

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C-comodules - category ${}^C\mathbb{M}$

$$\varrho^M : M \to C \otimes_R M,$$

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with compatibility conditions.

Adjoint functors - $M \in {}_R\mathbb{M}$, $X \in {}^C\mathbb{M}$

$$U^C : {}^C\mathbb{M} \to {}_R\mathbb{M}, \quad C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M},$$

Module categories

Coalgebras - comultiplication and counit, $C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

with coassociativity and counitality conditions.

C-comodules - category ${}^C\mathbb{M}$

$$\varrho^M : M \to C \otimes_R M,$$

with compatibility conditions.

Adjoint functors - $M \in {}_R\mathbb{M}$, $X \in {}^C\mathbb{M}$

$$U^C : {}^C\mathbb{M} \to {}_R\mathbb{M}, \quad C \otimes_R - : {}_R\mathbb{M} \to {}^C\mathbb{M},$$

$$\mathrm{Hom}^C(M, C \otimes_R X) \to \mathrm{Hom}_R(M, X), \quad f \mapsto \varepsilon_X \circ f.$$

Modules and comodules on ${}_R\mathbb{M}$

Equivalent for $C \in {}_R\mathbb{M}$:

(a) $C \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$ is a comonad (C is an R-coalgebra);

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Monads and Comonads in ${}_R\mathbb{M}$

Correspondence of categories

$$\mathrm{Hom}^C(C,-) : {}^C\mathbb{M} \to \mathbb{M}_{[C,-]}, \quad M \mapsto \mathrm{Hom}^C(C,M).$$

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Equivalence of Kleisli categories

For any $X \in {}_R\mathbb{M}$,

$$C \otimes_R X \mapsto \mathrm{Hom}^C(C, C \otimes_R X) \simeq \mathrm{Hom}_R(C,X),$$

$$\mathrm{Hom}_R(C,X) \mapsto C \otimes_{[C,-]} \mathrm{Hom}_R(C,X) \simeq C \otimes_R X.$$

Modules and comodules in ${}_R\mathbb{M}$

The monads ${}_R C^* \otimes_R -$ and $[C, -]$

$$\beta_M : C^* \otimes_R M \to \mathrm{Hom}_R(C, M), \quad f \otimes m \mapsto [c \mapsto f(c)m].$$

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This yields a functor

$$\begin{array}{rccc} F: & \mathbb{M}_{[C,-]} & \longrightarrow & {}_{C^*}\mathbb{M} \\ \mathrm{Hom}_R(C, M) \to M & & \mapsto & C^* \otimes_R M \xrightarrow{\beta_M} \mathrm{Hom}_R(C, M) \to M. \end{array}$$

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$$\begin{array}{cccc} G: & \mathbb{M}^C & \longrightarrow & \mathbb{M}^{(C^*,-)} \simeq {}_{C^*}\mathbb{M} \\ M \to C \otimes_R M & & \mapsto & M \to C \otimes_R M \xrightarrow{\alpha_M} \mathrm{Hom}_R(C^*, M). \end{array}$$

Module categories

Adjoint functors between module categories ${}_R\mathbb{M}$, ${}_S\mathbb{M}$ by ${}_RP_S$

$$P \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}.$$

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Adjunction - $N \in {}_R\mathbb{M}, \quad X \in {}_S\mathbb{M}$

$$\begin{array}{rcl} \mathrm{Hom}_R(P \otimes_S X, N) & \to & \mathrm{Hom}_S(X, \mathrm{Hom}_R(P, N)), \\ f & \mapsto & [x \mapsto f(- \otimes x)]. \end{array}$$

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Counit and unit

$$\begin{aligned} \varepsilon_M : &\quad P \otimes_S \mathrm{Hom}_R(P, M) \to M, \quad p \otimes f \mapsto f(p), \\ \eta_X : &\quad X \to \mathrm{Hom}_R(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x]. \end{aligned}$$

Module categories

equivalent

(a) ε_M is an epi(iso)morphism for all $M \in {}_R\mathbb{M}$;

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equivalent

(a) η and ε are isomorphisms;

(b) $P \otimes_S -$ is an equivalence (with inverse $\mathrm{Hom}_R(P, -)$);

(c) ${}_RP$ is a finitely generated, projective generator and $S = \mathrm{End}_R(P)$.

Module categories

equivalent if $S = \mathrm{End}_R(P)$

(a) ε_M is a monomorphism for all $M \in {}_R\mathbb{M}$,
η_X is an epimorphism for all $X \in {}_S\mathbb{M}$;

(b) $(P \otimes_S -, \mathrm{Hom}_R(P,-))$ induces an equivalence

$$P \otimes_S - : {}_S\mathbb{M}_{Hom_R(P,P\otimes_S -)} \longrightarrow {}_R\mathbb{M}^{P\otimes_S \mathrm{Hom}_R(P,-)}.$$

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In this case P is a $*$-*module* (in the sense of Menini-Orsatti).

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Tilting modules P

P $*$-module and for any $M \in {}_R\mathbb{M}$, there is monomorphism $M \to P \otimes_S X$, for some $X \in {}_S\mathbb{M}$.

(implies Brenner-Butler equivalence)

Module categories

${}_RP$ finitely generated and projective $\quad \mathrm{Hom}_R(P,-) \simeq P^* \otimes_R -$

adjoint pair	$(P \otimes_R -, P^* \otimes_R -)$,
counit	$\varepsilon : P \otimes_R P^* \to R,\ p \otimes f \mapsto f(p)$,
unit	$\eta : R \to P^* \otimes_R P,\ \ 1 \mapsto \sum p_i^* \otimes p_i$ (dual basis).

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$P^* \otimes_R P \otimes_S -$ is a monad on ${}_S\mathbb{M}$ (S-ring)

$$P^* \otimes_R P \xrightarrow{\simeq} End_R(P),\ f \otimes p \mapsto [x \mapsto f(x)p].$$

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$P \otimes_S P^* \otimes_R -$ is a comonad on ${}_R\mathbb{M}$ (R-coring) - coproduct

$$P \otimes_S P^* \to P \otimes_S P^* \otimes_R P \otimes_S P^*,\; p \otimes f \;\mapsto \sum p \otimes p_i^* \otimes p_i \otimes f.$$

Module categories

$P = R^n$ $\quad \mathrm{End}_R(R^n) \simeq M_n(R)$, R commutative

adjoint pair $\quad (R^n \otimes_R -, (R^n)^t \otimes_R -)$,

counit $\quad \varepsilon : R^n \otimes_R (R^n)^t \to R$, evaluation,

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Sweedler coring: $h : R \to A$, $P = {}_A A_R$, 1975

$$\begin{aligned} \delta : A \otimes_R A &\to A \otimes_R A \otimes_A A \otimes_R A, \\ a \otimes b &\mapsto a \otimes 1 \otimes_A 1 \otimes b\,; \\ \varepsilon = m : A \otimes_R A &\to A, \\ a \otimes b &\mapsto ab\,. \end{aligned}$$

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