ALGEBRAS SHARING THE SAME POSET OF SUPPORT τ-TILTING MODULES WITH TREE QUIVER ALGBERAS

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Abstract. This report is an announcement of our result in [3]. We give a generalization of Happel-Unger's result. More precisely, we characterize algebras whose support τ-tilting posets coincide with that of a tree quiver algebra.

1. Preliminary

In this section, we recall the definition and their fundamental results of support τ-tilting modules. Throughout this report, let $\Lambda = kQ/I$ be a finite dimensional algebra over an algebraically closed field k, where Q is a finite quiver and I an admissible ideal of kQ. We denote by $\mathsf{mod}\,\Lambda$ the category of finite dimensional right Λ-modules, and a *module* is always an object of this category. For a module M, $|M|$ denotes the number of pairwise non-isomorphic indecomposable direct summands of M.

1.1. Support τ-tilting modules.

The notion of support τ-tilting modules was introduced by Adachi-Iyama-Reiten as a generalization of that of tilting modules [2].

We denote by $\tau = \tau_\Lambda$ the Auslander-Reiten translation.

Definition 1. (1) A module M is said to be *τ-rigid* if $\mathrm{Hom}_\Lambda(M, \tau M) = 0$.

(2) A *τ-tilting module* M is defined to be τ-rigid with $|M| = |\Lambda|$.

(3) We say that a module M is *support τ-tilting* if there is an idempotent e of Λ such that M is a τ-tilting $\Lambda/(e)$-module.

Proposition 2. [2] *We have the following.*

1. *For any support τ-tilting module M, there exists a unique idempotent e of Λ such that M is a τ-tilting $\Lambda/(e)$-module.*
2. *Every support τ-tilting module is τ-rigid.*
3. *Any τ-rigid module is a direct summand of some support τ-tilting module.*
4. *(Support) tilting modules are (support) τ-tilting.*
5. *If Λ is hereditary, then M is a (support) τ-tilting module if and only if it is a (support) tilting one.*

We denote by $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ (resp. $\mathsf{s\text{-}tilt}\,\Lambda$) the set of (isomorphism classes of) basic support τ-tilting modules (resp. support tilting modules).

The detailed version of this paper has been submitted for publication elsewhere.

1.2. **Support τ-tilting posets.** $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ has a poset structure as follows (see [2]):

$$M \geq M' \overset{\text{def}}{\Leftrightarrow} \mathsf{fac}M \supset \mathsf{fac}M',$$

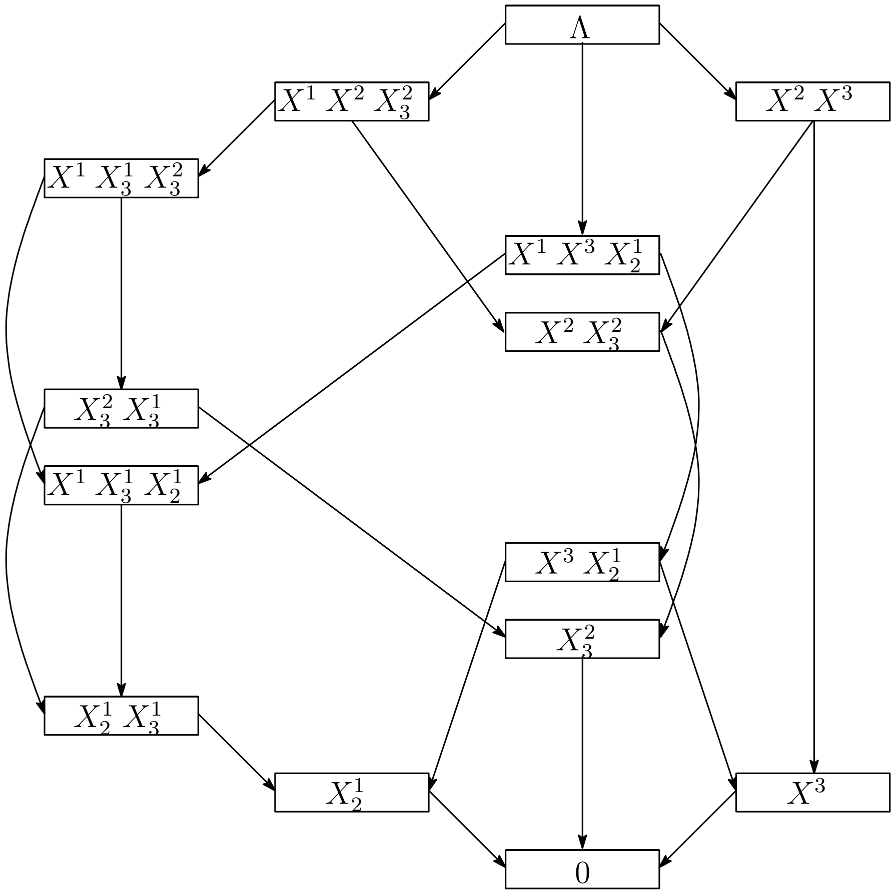
where $\mathsf{fac}M := \{X \in \mathsf{mod}\,\Lambda \mid M^{\oplus r} \overset{\exists}{\twoheadrightarrow} X \text{ for some } r > 0\}$.

Theorem 3. [2]

(1) *The Hasse quiver* $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda)$ *of* $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ *is* $|\Lambda|$*-regular.*
(2) *If* $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda)$ *has a finite connected component* $\mathcal{C}$*, then we have* $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda) = \mathcal{C}$*.*

We give an example of a support τ-tilting poset. Please refer to [1, 5], etc, for more examples.

Example 4. Let $Q = 1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$. For a pair (i, j) of $\{1, 2, 3\}$ with $i > j$, we denote by $X^i_j := e_i\Lambda/e_i\Lambda e_j\Lambda$. Also we set $X^i = e_i\Lambda$. Then $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ is given by the following.



2. Happel-Unger's result

Let Q be a finite connected acyclic quiver. We define a decorated quiver Q_{dec} of Q as follows:

(i) Vertices of Q_{dec} are those of Q.
(ii) We draw an arrow $i \to j$ on Q_{dec} if there is a unique arrow from i to j on Q.

(iii) We draw a decorated arrow $i \stackrel{*}{\to} j$ on Q_{dec} if there are at least two arrows from i to j on Q.

Note that we distinguish ordinary arrows $\to$ from decorated arrows $\stackrel{*}{\to}$.

Happel and Unger gave us the following result.

Theorem 5. [4, Theorem 6.4] *Let Q and Q' be finite connected acyclic quivers. If there is a poset isomorphism*

$$\mathsf{s\text{-}tilt}\, kQ \simeq \mathsf{s\text{-}tilt}\, kQ',$$

then Q_{dec} is isomorphic to Q'_{dec}.

The theorem above says that we can reconstruct tree quiver algebras from their posets of support (τ-)tilting modules.

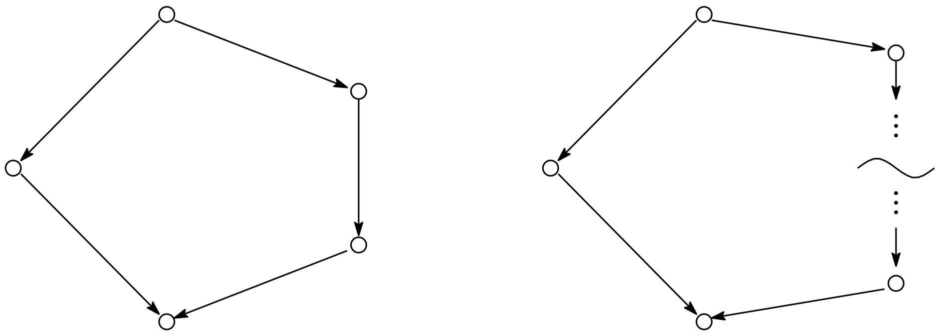
Example 6. Let $Q^{(m)}$ be a quiver with vertices $1, 2$ and m arrows from 1 to 2. Then we have

$$Q^{(m)}_{\text{dec}} = \begin{cases} 1 \to 2 & \text{if } m = 1 \\ 1 \stackrel{*}{\to} 2 & \text{if } m \geq 2 \end{cases}$$

Denote by Λ_m the path algebra of $Q^{(m)}$. Then Theorem 5 implies that

$$\mathsf{s\text{-}tilt}\, \Lambda_1 \not\simeq \mathsf{s\text{-}tilt}\, \Lambda_m \ (m \geq 2).$$

In fact, the Hasse quiver of the poset of support tilting modules of Λ_m is given by the following:



3. Main result

A motive of our work is to generalize Happel-Unger's result. We fix a tree quiver $\overrightarrow{\mathbb{T}}$ and its path algebra $\Gamma = k\overrightarrow{\mathbb{T}}$. As a main result of this report, we chracterize algebras whose support τ-tilting posets are isomorphic to $\mathsf{s\tau\text{-}tilt}\,\Gamma$.

Theorem 7. [3, Corollary 3.11] *Let Q be a finite quiver, I an admissible ideal of kQ and $\Lambda = kQ$. Then the following are equivalent.*

(i) $\mathsf{s\tau\text{-}tilt}\Lambda \simeq \mathsf{s\tau\text{-}tilt}\Gamma$.

(ii) *There is a quiver isomorphism $\sigma : Q \setminus \{\text{loops}\} \stackrel{\sim}{\to} \overrightarrow{\mathbb{T}}$ satisfying the following:*

  (a) $e_i \Lambda e_j = 0 \Leftrightarrow e_{\sigma(i)} \Gamma e_{\sigma(j)} = 0$.

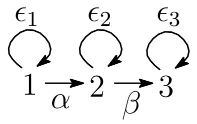
  (b) *If α is an arrow from i to j with $i \neq j$, then $e_i \Lambda \alpha = e_i \Lambda e_j = \alpha \Lambda e_j$.*

In particular, there are infinitely many algebras (up to Morita equivalence) satisfying $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Gamma$.

Remark 8. Under the condition (ii) of Theorem 7, there is an algebra epimorphism $\Lambda \twoheadrightarrow \Gamma$, and the tensor functor $- \otimes_\Lambda \Gamma$ induces a poset isomorphism

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,\Gamma.$$

Example 9. (1) Let Q be the following quiver:



Let $m \in \mathbb{Z}_{\geq 1}$ and I_m an ideal of kQ generated by

$$\epsilon_1^m,\ \epsilon_2^m,\ \epsilon_3^m,\ \epsilon_1\alpha,\ \alpha\epsilon_2,\ \epsilon_2\beta,\ \beta\epsilon_3.$$

Set $\Lambda_m := kQ/I_m$. Then we have a poset isomorphism

$$\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda_m \simeq \mathsf{s}\tau\text{-}\mathsf{tilt}\,k(1 \to 2 \to 3).$$

(2) Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, $I = (\alpha\beta)$ and $\Lambda = kQ/I$. Then $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$ is not isomorphic to $\mathsf{s}\tau\text{-}\mathsf{tilt}\,k(1 \to 2 \to 3)$. (In this case, there are 12 elements in $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$.)

References

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