ATOM-MOLECULE CORRESPONDENCE IN GROTHENDIECK CATEGORIES

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Abstract. For a one-sided noetherian ring, Gabriel constructed two maps between the isomorphism classes of indecomposable injective modules and the two-sided prime ideals. In this paper, we give a categorical reformulation of these maps using the notion of Grothendieck category. Gabriel's maps become maps between the atom spectrum and the molecule spectrum in our setting, and these two spectra have structures of partially ordered sets. The main result in this paper shows that the two maps induce a bijection between the minimal elements of the atom spectrum and those of the molecule spectrum.

Key Words: Grothendieck category, Atom spectrum, Molecule spectrum, Prime ideal, Indecomposable injective object.

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1. Introduction

For a one-sided noetherian ring, Gabriel [1] described the relationship between indecomposable injective modules and two-sided prime ideals as follows.

Theorem 1 ([1]). *Let Λ be a right noetherian ring. Then we have two maps*

$$\frac{\{\,\text{indecomposable injective right } \Lambda\text{-modules}\,\}}{\cong} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\,\text{two-sided prime ideals of } \Lambda\,\}$$

characterized by the following properties.

1. *For each indecomposable injective right Λ-module I, the only associated (two-sided) prime of I is $\varphi(I)$.*
2. *For each two-sided prime ideal P of Λ, the injective envelope $E(\Lambda/P)$ of the right Λ-module Λ/P is the direct sum of a finite number of copies of the indecomposable injective Λ-module $\psi(P)$.*

Moreover, the composite $\varphi\psi$ is the identity map.

If the ring is commutative, then these maps are bijective ([7, Proposition 3.1]). In general, these maps are far from being bijective as the following example shows.

Example 2. The ring $\Lambda := \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ is a simple domain which is left and right noetherian. The only two-sided prime ideal of Λ is 0, while there exist infinitely many isomorphism classes of indecomposable injective right Λ-modules.

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In this paper, we generalize Theorem 1 to a certain class of Grothendieck categories as maps between the *atom spectrum* and the *molecule spectrum* of a Grothendieck category. Moreover, by using naturally defined partial orders on these spectra, we establish a bijection between the minimal elements of the atom spectrum and those of the molecule spectrum. This result would have been unknown even in the case of the category $\operatorname{Mod}\Lambda$ of right modules over a right noetherian ring Λ.

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2. Atom spectrum

Throughout this paper, let $\mathcal{A}$ be a Grothendieck category. In this section, we recall the definition of the atom spectrum of $\mathcal{A}$ and related notions.

The atom spectrum is defined by using monoform objects and an equivalence relation between them.

Definition 3.

(1) A nonzero object H in $\mathcal{A}$ is called *monoform* if for each nonzero subobject L of H, no nonzero subobject of H is isomorphic to a subobject of H/L.

(2) For monoform objects H_1 and H_2 in $\mathcal{A}$, we say that H_1 is *atom-equivalent* to H_2 if there exists a nonzero subobject of H_1 which is isomorphic to a subobject of H_2.

The following result is fundamental.

Proposition 4 ([3, Proposition 2.2])**.** *Every nonzero subobject of a monoform object is monoform.*

For a commutative ring R, all monoform objects in $\operatorname{Mod}R$ can be described in the following sense.

Proposition 5 ([9, p. 626])**.** *Let R be a commutative ring. Then a nonzero object H in* $\operatorname{Mod}R$ *is monoform if and only if there exist* $\mathfrak{p} \in \operatorname{Spec}R$ *and a monomorphism* $H \to k(\mathfrak{p})$ *in* $\operatorname{Mod}R$*, where* $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.

The atom equivalence is in fact an equivalence relation between the monoform objects in $\mathcal{A}$.

Definition 6. The *atom spectrum* $\operatorname{ASpec}\mathcal{A}$ of $\mathcal{A}$ is the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. Each element of $\operatorname{ASpec}\mathcal{A}$ is called an *atom* in $\mathcal{A}$. For each monoform object H in $\mathcal{A}$, the equivalence class of H is denoted by $\overline{H}$.

The notion of atoms was originally introduced by Storrer [9] for module categories and generalized to arbitrary abelian categories in [3].

The next result shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 7 ([9, p. 631])**.** *Let R be a commutative ring. Then the map* $\operatorname{Spec}R \to \operatorname{ASpec}(\operatorname{Mod}R)$ *given by* $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ *is bijective.*

We can also generalize the notions of associated primes and support.

Definition 8. Let M be an object in $\mathcal{A}$.

(1) Define the subset $\operatorname{AAss} M$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{AAss} M = \{\, \alpha \in \operatorname{ASpec} \mathcal{A} \mid \alpha = \overline{H} \text{ for some monoform subobject } H \text{ of } M \,\}.$$

We call each element of $\operatorname{AAss} M$ an *associated atom* of M.

(2) Define the subset $\operatorname{ASupp} M$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{ASupp} M = \{\, \alpha \in \operatorname{ASpec} \mathcal{A} \mid \alpha = \overline{H} \text{ for some monoform subquotient } H \text{ of } M \,\}.$$

We call it the *atom support* of M.

Proposition 9. *Let R be a commutative ring, and let M be an R-module. Then the bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *in Proposition 7 induces bijections* $\operatorname{Ass} M \to \operatorname{AAss} M$ *and* $\operatorname{Supp} M \to \operatorname{ASupp} M$.

We introduce a partial order on the atom spectrum, which plays an crucial role in this paper.

Definition 10. For $\alpha, \beta \in \operatorname{ASpec} \mathcal{A}$, we write $\alpha \leq \beta$ if $\beta \in \operatorname{ASupp} H$ holds for each monoform object H in $\mathcal{A}$ with $\overline{H} = \alpha$.

The partial order on the atom spectrum is a generalization of the inclusion relation between prime ideals of a commutative ring.

Proposition 11. *Let R be a commutative ring. Then the bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *in Proposition 7 is an isomorphism between the partially ordered sets* $(\operatorname{Spec} R, \subset)$ *and* $(\operatorname{ASpec}(\operatorname{Mod} R), \leq)$.

We can generalize Matlis' correspondence in commutative ring theory.

Theorem 12 ([3, Theorem 5.9])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then we have a bijection*

$$\operatorname{ASpec} \mathcal{A} \xrightarrow{\sim} \frac{\{\, \textit{indecomposable injective objects in } \mathcal{A} \,\}}{\cong}$$

given by $\overline{H} \mapsto E(H)$.

For a locally noetherian Grothendieck category $\mathcal{A}$, the localizing subcategories of $\mathcal{A}$ can be classified by the localizing subsets of $\operatorname{ASpec} \mathcal{A}$.

Definition 13. A subset Φ of $\operatorname{ASpec} \mathcal{A}$ is called a *localizing subset* if there exists an object M in $\mathcal{A}$ such that $\Phi = \operatorname{ASupp} M$.

Theorem 14 ([2, Theorem 3.8], [6, Corollary 4.3], and [3, Theorem 5.5])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then we have a bijection*

$$\{\, \textit{localizing subcategories of } \mathcal{A} \,\} \xrightarrow{\sim} \{\, \textit{localizing subsets of } \operatorname{ASpec} \mathcal{A} \,\}$$

given by $\mathcal{X} \mapsto \operatorname{ASupp} \mathcal{X} := \bigcup_{M \in \mathcal{X}} \operatorname{ASupp} M$. *The inverse map is given by* $\Phi \mapsto \operatorname{ASupp}^{-1} \Phi$, *where*

$$\operatorname{ASupp}^{-1} \Phi = \{\, M \in \mathcal{A} \mid \operatorname{ASupp} \subset \Phi \,\}.$$

Moreover, the atom spectrum of the quotient category by a localizing subcategory can be described as follows.

Theorem 15 ([5, Theorem 5.17])**.** *Let $\mathcal{A}$ be a Grothendieck category, and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. Denote by $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{X}$ the canonical functor. Then we have a bijection*

$$\operatorname{ASpec} \mathcal{A} \setminus \operatorname{ASupp} \mathcal{X} \xrightarrow{\sim} \operatorname{ASpec} \frac{\mathcal{A}}{\mathcal{X}}$$

given by $\overline{H} \mapsto \overline{F(H)}$.

If the Grothendieck category $\mathcal{A}$ has a noetherian generator, then the set of minimal atoms in $\mathcal{A}$ has significant properties. Denote by $\operatorname{AMin} \mathcal{A}$ the set of minimal atoms in $\mathcal{A}$.

Theorem 16. *Let $\mathcal{A}$ be a Grothendieck category having a noetherian generator.*

(1) ([5, Proposition 4.7]) *For each $\alpha \in \operatorname{ASpec} \mathcal{A}$, there exists $\beta \in \operatorname{AMin} \mathcal{A}$ satisfying $\beta \leq \alpha$.*
(2) ([4, Theorem 4.4]) *$\operatorname{AMin} \mathcal{A}$ is a finite set.*
(3) *$\operatorname{ASpec} \mathcal{A} \setminus \operatorname{AMin} \mathcal{A}$ is a localizing subset of $\mathcal{A}$.*

Definition 17. Let $\mathcal{A}$ be a Grothendieck category having a noetherian generator. Define the *artinianization* $\mathcal{A}_{\mathrm{artin}}$ of $\mathcal{A}$ as the quotient category of $\mathcal{A}$ by the localizing subcategory $\operatorname{ASupp}^{-1}(\operatorname{ASpec} \mathcal{A} \setminus \operatorname{AMin} \mathcal{A})$.

It is easy to deduce that the artinianization has a generator of finite length. Moreover, the following result ensures that it is the module category of some right artinian ring.

Theorem 18 (Năstăsescu [8])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then the following assertions are equivalent.*

(1) *$\mathcal{A}$ has an artinian generator.*
(2) *$\mathcal{A}$ has a generator of finite length.*
(3) *There exists a right artinian ring Λ satisfying $\mathcal{A} \cong \operatorname{Mod} \Lambda$.*

3. Molecule spectrum

In this section, we introduce a new spectrum of a Grothendieck category, which we call the *molecule spectrum*. It is a generalization of the set of two-sided prime ideals of a ring. The definition uses the notion of closed subcategory.

Definition 19.

(1) A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *closed* if $\mathcal{C}$ is closed under subobjects, quotient objects, arbitrary direct sums, and arbitrary direct products.
(2) Let $\mathcal{C}$ and $\mathcal{D}$ be closed subcategories of $\mathcal{A}$. Denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of $\mathcal{A}$ consisting of all objects M in $\mathcal{A}$ such that there exists an exact sequence

$$0 \to L \to M \to N \to 0$$

with $L \in \mathcal{C}$ and $N \in \mathcal{D}$.

For each family of objects $\{M_i\}_{i\in I}$ in $\mathcal{A}$, we have the canonical monomorphism $\bigoplus_{i\in I} M_i \to \prod_{i\in I} M_i$ since $\mathcal{A}$ is a Grothendieck category. Therefore the closedness under arbitrary direct sums can be dropped from the definition of closed subcategory.

The following well-known result shows that closed subcategories of a Grothendieck category is a generalization of two-sided ideals of a ring.

Proposition 20. *Let Λ be a ring.*

(1) *We have a poset isomorphism*

$$(\{\, \textit{two-sided ideals of } \Lambda \,\}, \subset) \xrightarrow{\sim} (\{\, \textit{closed subcategories of } \operatorname{Mod}\Lambda \,\}, \supset)$$

given by $I \mapsto \operatorname{Mod}(\Lambda/I)$, where $\operatorname{Mod}(\Lambda/I)$ is identified with the full subcategory

$$\{\, M \in \operatorname{Mod}\Lambda \mid MI = 0 \,\}$$

of $\operatorname{Mod}\Lambda$.

(2) *Let I and J be two-sided ideals of Λ. Then we have*

$$\operatorname{Mod}\frac{\Lambda}{IJ} = \operatorname{Mod}\frac{\Lambda}{J} * \operatorname{Mod}\frac{\Lambda}{I}$$

as a full subcategory of $\operatorname{Mod}\Lambda$, *that is, the isomorphism in (1) induces an isomorphism*

$$(\{\, \textit{two-sided ideals of } \Lambda \,\}, \cdot) \xrightarrow{\sim} (\{\, \textit{closed subcategories of } \operatorname{Mod}\Lambda \,\}, *)$$

of monoids.

(3) *Let M be a right Λ-module. Then the two-sided ideal $\operatorname{Ann}_\Lambda(M)$ corresponds to the smallest closed subcategory $\langle M \rangle_{\mathrm{closed}}$ of $\mathcal{A}$ containing M by the isomorphism in (1).*

We can generalize the notion of two-sided prime ideals of a ring to a Grothendieck category.

Definition 21. A nonzero closed subcategory $\mathcal{P}$ of $\mathcal{A}$ is called *prime* if for each closed subcategories $\mathcal{C}$ and $\mathcal{D}$ satisfying $\mathcal{P} \subset \mathcal{C} * \mathcal{D}$, we have $\mathcal{P} \subset \mathcal{C}$ or $\mathcal{P} \subset \mathcal{D}$.

Proposition 22. *Let Λ be a ring. Then the isomorphism in Proposition 20 (1) induces a poset isomorphism*

$$(\{\, \textit{two-sided prime ideals of } \Lambda \,\}, \subset) \xrightarrow{\sim} (\{\, \textit{prime closed subcategories of } \operatorname{Mod}\Lambda \,\}, \supset).$$

Although the set of prime closed subcategories can be used as the definition of the molecule spectrum, we use the notion of prime object instead, in order to clarify the similarity between the atom spectrum and the molecule spectrum.

Definition 23.

(1) A nonzero object H in $\mathcal{A}$ is called *prime* if for each nonzero subobject L of H, it holds that $\langle L \rangle_{\mathrm{closed}} = \langle H \rangle_{\mathrm{closed}}$.

(2) For prime objects H_1 and H_2 in $\mathcal{A}$, we say that H_1 is *molecule-equivalent* to H_2 if $\langle H_1 \rangle_{\mathrm{closed}} = \langle H_2 \rangle_{\mathrm{closed}}$.

Definition 24. The *molecule spectrum* $\operatorname{MSpec}\mathcal{A}$ of $\mathcal{A}$ is the quotient set of the set of prime objects in $\mathcal{A}$ by the molecule equivalence. Each element of $\operatorname{MSpec}\mathcal{A}$ is called a *molecule* in $\mathcal{A}$. For each prime object H in $\mathcal{A}$, the equivalence class of H is denoted by $\widetilde{H}$.

The following result shows that the molecule spectrum is also regarded as a generalization of the set of two-sided prime ideals of a ring. Although we assume the existence of a noetherian generator, this result can be also shown for the category $\operatorname{Mod}\Lambda$ of right modules over an arbitrary ring Λ by using classical ring-theoretic argument.

Proposition 25. *Let $\mathcal{A}$ be a Grothendieck category with a noetherian generator. Then we have a bijection*

$$\operatorname{MSpec}\mathcal{A} \xrightarrow{\sim} \{\, \textit{prime closed subcategories of } \mathcal{A} \,\}$$

given by $\widetilde{H} \mapsto \langle H \rangle_{\text{closed}}$. For each $\rho = \widetilde{H} \in \operatorname{MSpec}\mathcal{A}$, the prime closed subcategory $\langle H \rangle_{\text{closed}}$ corresponding to ρ is denoted by $\langle \rho \rangle_{\text{closed}}$.

$\operatorname{MSpec}\mathcal{A}$ has a partial order induced from the set of prime closed subcategories.

Definition 26. Let $\mathcal{A}$ be a Grothendieck category with a noetherian generator. For $\rho, \sigma \in \operatorname{MSpec}\mathcal{A}$, we write $\rho \leq \sigma$ if $\langle \rho \rangle_{\text{closed}} \supset \langle \sigma \rangle_{\text{closed}}$ holds.

The partial order on $\operatorname{MSpec}\mathcal{A}$ can be also defined for $\operatorname{Mod}\Lambda$, where Λ is an arbitrary ring, and we can show the following proposition.

Proposition 27. *Let Λ be a ring. Then we have a poset isomorphism*

$$(\{\, \textit{two-sided prime ideals of } \Lambda \,\}, \subset) \xrightarrow{\sim} (\operatorname{MSpec}\Lambda, \leq)$$

given by $P \mapsto \widetilde{\Lambda/P}$.

Denote by $\operatorname{MMin}\mathcal{A}$ the set of minimal elements of $\operatorname{MSpec}\mathcal{A}$.

4. Atom-molecule correspondence

From now on, let $\mathcal{A}$ be a Grothendieck category having a noetherian generator and satisfying the Ab4* condition, that is, direct product preserves exactness. For a right noetherian ring Λ, the category $\operatorname{Mod}\Lambda$ satisfies this assumption. The following theorem is our main result.

Theorem 28.

(1) *We have a surjective poset homomorphism*

$$\varphi\colon \operatorname{ASpec}\mathcal{A} \to \operatorname{MSpec}\mathcal{A}$$

given by $\overline{H} \mapsto \widetilde{H}$, where H is taken to be a prime monoform object in $\mathcal{A}$.

(2) *The map φ induces a poset isomorphism*

$$\operatorname{AMin}\mathcal{A} \xrightarrow{\sim} \operatorname{MMin}\mathcal{A}.$$

(3) *There exists an injective poset homomorphism*

$$\psi\colon \operatorname{MSpec}\mathcal{A} \to \operatorname{ASpec}\mathcal{A}$$

satisfying the following properties.

(a) *For each* $\rho \in \operatorname{MSpec} \mathcal{A}$*, the image* $\psi(\rho)$ *is the smallest element in the set*

$$\{\, \alpha \in \operatorname{ASpec} \mathcal{A} \mid \rho \leq \varphi(\alpha) \,\}.$$

(b) *The composite* $\varphi\psi$ *is the identity map on* $\operatorname{MSpec} \mathcal{A}$*.*
(c) *The map* ψ *induces a poset isomorphism* $\operatorname{MSpec} \mathcal{A} \xrightarrow{\sim} \Im\psi$*.*
(d) *For each* $\alpha \in \operatorname{ASpec} \mathcal{A}$ *and* $\rho \in \operatorname{MSpec} \mathcal{A}$*, we have*

$$\psi(\rho) \leq \alpha \iff \rho \leq \varphi(\alpha).$$

This theorem is proved by using the next two lemmas. Recall that an object H in $\mathcal{A}$ is called *compressible* if every nonzero subobject of H has a subobject which is isomorphic to H. In our setting, every compressible object in $\mathcal{A}$ is monoform.

Lemma 29. *For every* $\alpha \in \operatorname{AMin} \mathcal{A}$*, there exists a compressible object* H *satisfying* $\alpha = \overline{H}$*.*

A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *weakly closed* if $\mathcal{X}$ is closed under subobjects, quotient objects, and arbitrary direct sums. The following lemma explains why minimal elements of $\operatorname{ASpec} \mathcal{A}$ behave nicely.

Lemma 30. *Let* M *be an object in* $\mathcal{A}$ *satisfying* $\operatorname{AAss} M \subset \operatorname{AMin} \mathcal{A}$*. Let* $\mathcal{X}$ *be the smallest weakly closed subcategory containing* M*. Then* $\mathcal{X}$ *is closed under arbitrary direct products, that is,* $\mathcal{X} = \langle M \rangle_{\text{closed}}$*.*

The Ab4* condition is used in the proof of Lemma 30. The proof also depends on the fact that $\operatorname{AMin} \mathcal{A}$ is a finite set.

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