THICK SUBCATEGORIES OF DERIVED CATEGORIES OF ISOLATED SINGULARITIES

RYO TAKAHASHI

Abstract. Let R be a commutative noetherian local ring with residue field k. Denote by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of finitely generated R-modules. This article gives a classification of the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing k when R has an isolated singularity. If R is moreover Cohen–Macaulay and has minimal multiplicity, all the standard thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ are classified.

Key Words: Cohen–Macaulay ring, Derived category, Isolated singularity, Specialization-closed subset, Thick subcategory.

2010 *Mathematics Subject Classification*: 13D09.

1. Introduction

A *thick subcategory* of a triangulated category is by definition a full triangulated subcategory closed under direct summands. The notion of a thick subcategory has been introduced by Verdier [17] by the name of *épaisse* subcategory to develop the theory of Verdier localizations.

Classifying thick subcategories of triangulated categories is one of the most important subjects shared by homotopy theory, ring theory, algebraic geometry and representation theory. Classifying thick subcategories is one of the most important problems shared by homotopy theory, ring theory, algebraic geometry and representation theory. It was first done by Devinatz, Hopkins and Smith [4, 8] in the 1980s; they classified the thick subcategories of the triangulated category of compact objects in the p-local stable homotopy category. Later on, as an analogue of the Devinatz–Hopkins–Smith theorem for commutative rings, Hopkins and Neeman [7, 10] classified the thick subcategories of the derived category of perfect complexes over a commutative noetherian ring, and it was extended to a quasi-compact quasi-separated scheme by Thomason [16]. As an analogue of the Hopkins–Neeman theorem for finite groups, Benson, Carlson and Rickard [1] classified the thick subcategories of the stable category of finite dimensional representations of a finite p-group. It was extended to a finite group scheme by Friedlander and Pevtsova [6] and further generalized to the derived category of a finite group by Benson, Iyengar and Krause [2].

The celebrated Hopkins–Neeman theorem classifies the thick subcategories of perfect complexes over a commutative noetherian ring. To apply this for the whole derived category, let R be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then the theorem states that there is a bijection between the thick subcategories of the bounded

The detailed version [15] of this article will be submitted for publication elsewhere.

derived category $\mathsf{D^b}(R)$ of finitely generated R-modules and the specialization-closed subsets of $\operatorname{Spec} R$. This theorem especially says that any nonzero thick subcategory of $\mathsf{D^b}(R)$ contains k. Since this fact is itself clear, the essential part of the Hopkins–Neeman theorem asserts that for a regular local ring R taking the *supports* makes a bijection from the thick subcategories of $\mathsf{D^b}(R)$ containing k to the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$. The first main result of this article is the following theorem, which guatantees that this consequence of the Hopkins–Neeman theorem remains valid for a much wider class of rings, that is, the class of (catenary equidimensional) isolated singularities.

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a catenary equidimensional local ring with an isolated singularity. The assignments $f : \mathcal{X} \mapsto \operatorname{Supp}_R \mathcal{X}$ and $g : S \mapsto \operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S$ make mutually inverse bijections*

$$\left\{\begin{matrix}\textit{Thick subcategories of } \mathsf{D^b}(R)\\ \textit{containing } k\end{matrix}\right\} \underset{g}{\overset{f}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Specialization-closed subsets of } \operatorname{Spec} R\\ \textit{containing } \mathfrak{m}\end{matrix}\right\}.$$

The assumption that R is catenary and equidimensional is quite weak; local rings that appear in algebraic geometry usually satisfy this assumption. For example, one-dimensional local rings, Cohen–Macaulay local rings, complete local domains and their localizations at prime ideals are all catenary and equidimensional. Also, the assumption that R has an isolated singularity is a mild condition; it is a standard assumption in the representation theory of Cohen–Macaulay rings. For instance, this assumption is indispensable to get the Auslander–Reiten quiver of the maximal Cohen–Macaulay R-modules. Reduced local rings of dimension one and normal local rings of dimension two are Cohen–Macaulay rings with an isolated singularity. Moreover, it turns out that without the isolated singularity assumption, Theorem 1 is no longer true; see Remark 17.

Several related results to Theorem 1 have been obtained so far. The author [14] classifies the thick subcategories of $\mathsf{D^b}(R)$ containing R and k when R is a Gorenstein local ring that is locally a hypersurface on the punctured spectrum. Stevenson [12] obtains a complete classification of the thick subcategories of $\mathsf{D^b}(R)$ in the case where R is a complete intersection. Thus, our next goal is to classify the thick subcategories of $\mathsf{D^b}(R)$ for a non-complete-intersection local rings. However, this problem itself turns out to be quite hard; indeed, there seems even to be no example of a non-complete-intersection ring R such that all the thick subcategories of $\mathsf{D^b}(R)$ are classified. So it would be a reasonable approach to consider classifying the thick subcategories satisfying a certain condition which all the thick subcategories satisfy over complete intersections. The standard condition is such a one: We say that a thick subcategory of $\mathsf{D^b}(R)$ is *standard* if it contains a nonzero object of finite projective dimension. Dwyer, Greenlees and Iyengar [5] prove that if R is a complete intersection, then every nonzero thick subcategory of $\mathsf{D^b}(R)$ is standard. As an application of Theorem 1, we obtain the following classification theorem of standard thick subcategories.

Theorem 2. *Let R be a nonregular local ring with an isolated singularity. Suppose that R is either*

(1) *a hypersurface, or*
(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then there is a one-to-one correspondence

$$\left\{\begin{matrix}\textit{Standard thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \underset{\Gamma}{\overset{\Lambda}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\} \sqcup \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\}.$$

Here, the maps Λ *and* Γ *are defined by:*

$$\Lambda(\mathcal{X}) = \begin{cases}(\operatorname{Supp}\mathcal{X}, 1) & \textit{if } \mathcal{X} \subseteq \mathsf{D}_{\mathsf{perf}}(R),\\ (\operatorname{Supp}\mathcal{X}, 2) & \textit{if } \mathcal{X} \nsubseteq \mathsf{D}_{\mathsf{perf}}(R),\end{cases}$$

$$\Gamma((S,i)) = \begin{cases}(\operatorname{Supp}^{-1} S) \cap \mathsf{D}_{\mathsf{perf}}(R) & \textit{if } i = 1,\\ \operatorname{Supp}^{-1} S & \textit{if } i = 2.\end{cases}$$

In the next Section 2 we make several necessary definitions and fundamental properties. In Sections 3 and 4 we give some comments on the above two theorems.

2. Basic definitions

Let us begin with fixing our conventions.

Convention 3. Throughout (the rest of) this article, let R be a commutative noetherian ring. We assume that all modules are finitely generated, and that all subcategories are nonempty, full and closed under isomorphism.

We denote by $\operatorname{\mathsf{mod}} R$ the category of (finitely generated) R-modules, by $\mathsf{C}^{\mathsf{b}}(R)$ the category of bounded complexes of (finitely generated) R-modules and by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of $\operatorname{\mathsf{mod}} R$. Note that $\operatorname{\mathsf{mod}} R$ and $\mathsf{C}^{\mathsf{b}}(R)$ are abelian categories and $\mathsf{D}^{\mathsf{b}}(R)$ is a triangulated category.

We make the definitions of thick subcategories of $\operatorname{\mathsf{mod}} R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$.

Definition 4. (1) A subcategory $\mathcal{X}$ of $\operatorname{\mathsf{mod}} R$ is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences.
(2) A subcategory $\mathcal{X}$ of $\mathsf{C}^{\mathsf{b}}(R)$ is called *thick* if it is closed under direct summands and shifts and satisfies the 2-out-of-3 property for short exact sequences.
(3) A subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property for exact triangles.

Let $\mathcal{C}$ be one of the categories $\operatorname{\mathsf{mod}} R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$. For each subcategory $\mathcal{M}$ of $\mathcal{C}$ we denote by $\mathsf{thick}_{\mathcal{C}}\,\mathcal{M}$ the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{M}$, and call it the *thick closure* of $\mathcal{M}$ in $\mathcal{C}$.

Remark 5. (1) Every Serre subcategory of $\operatorname{\mathsf{mod}} R$ is thick.
(2) The category $\mathsf{thick}_{\operatorname{\mathsf{mod}} R} R$ consists of the R-modules of finite projective dimension.
(3) A thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ is closed under shifts. In fact, each object M of $\mathsf{D}^{\mathsf{b}}(R)$ admits exact triangles $M \to 0 \to M[1] \rightsquigarrow$ and $M[-1] \to 0 \to M \rightsquigarrow$ in $\mathsf{D}^{\mathsf{b}}(R)$. Since $\mathcal{X}$ contains 0, this shows that if M is in $\mathcal{X}$, then so is $M[\pm 1]$, and by induction so is $M[n]$ for all $n \in \mathbb{Z}$.
(4) Let R be a local ring with residue field k. Then $\mathsf{thick}_{\operatorname{\mathsf{mod}} R} k$ consists of the R-modules of finite length. In particular, a thick subcategory $\mathcal{X}$ of $\operatorname{\mathsf{mod}} R$ contains k if and only if $\mathcal{X}$ contains all the R-modules of finite length.

Let us recall the relationships among the categories $\mathsf{mod}\, R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$. Among these three categories there are natural functors

$$\mathsf{mod}\, R \xrightarrow{\alpha} \mathsf{C}^{\mathsf{b}}(R) \xrightarrow{\beta} \mathsf{D}^{\mathsf{b}}(R).$$

Each object C in $\mathsf{C}^{\mathsf{b}}(R)$ is sent by β to the same complex C, while each morphism $g : X \to Y$ in $\mathsf{C}^{\mathsf{b}}(R)$ is sent by β to the roof $X \xleftarrow{1} X \xrightarrow{g} Y$ in $\mathsf{D}^{\mathsf{b}}(R)$. For an object $M \in \mathsf{mod}\, R$ (resp. $C \in \mathsf{C}^{\mathsf{b}}(R)$) we often use the same letter M (resp. C) to denote $\alpha(M)$ (resp. $\beta(C)$).

Next we recall the defintion of a specialization-closed subset.

Definition 6. A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if S contains $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in S$. Here, for an ideal I of R we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I.

Remark 7. (1) A specialization-closed subset of $\operatorname{Spec} R$ is nothing but a (possibly infinite) union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.
(2) Let R be a local ring with maximal ideal $\mathfrak{m}$. Then a specialization-closed subset of $\operatorname{Spec} R$ is nonempty if and only if it contains $\mathfrak{m}$.

Now we introduce the notion of supports for the module category $\mathsf{mod}\, R$.

Definition 8. (1) For each module $M \in \mathsf{mod}\, R$ we denote by $\operatorname{Supp}_R M$ the set of prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \not\cong 0$ in $\mathsf{mod}\, R_{\mathfrak{p}}$, and call this the *support* of M in $\mathsf{mod}\, R$.
(2) For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ we set $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp}_R X$, and call this the *support* of $\mathcal{X}$.
(3) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}^{-1}_{\mathsf{mod}\, R} S$ the subcategory of $\mathsf{mod}\, R$ consisting of all modules whose supports are contained in S.

Remark 9. (1) For an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$ it holds that $\operatorname{Supp}_R M = \operatorname{Supp}_R L \cup \operatorname{Supp}_R N$.
(2) Let M be an R-module. Then $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology.
(3) Let S be a set of prime ideals of R. Then $\operatorname{Supp}^{-1}_{\mathsf{mod}\, R} S$ is a Serre subcategory of $\mathsf{mod}\, R$, and in particular, a thick subcategory of $\mathsf{mod}\, R$.

Next we introduce the notion of supports for the derived category $\mathsf{D}^{\mathsf{b}}(R)$.

Definition 10. (1) Let X be an object of $\mathsf{D}^{\mathsf{b}}(R)$. Then the following sets of prime ideals of R are the same.
- $\operatorname{Supp}_R \mathrm{H}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X)_{\mathfrak{p}} \not\cong 0 \text{ in } \mathsf{mod}\, R_{\mathfrak{p}}\}$.
- $\bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R \mathrm{H}^i(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}^i(X)_{\mathfrak{p}} \not\cong 0 \text{ in } \mathsf{mod}\, R_{\mathfrak{p}} \text{ for some } i \in \mathbb{Z}\}$.
- $\{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \not\cong 0 \text{ in } \mathsf{D}^{\mathsf{b}}(R_{\mathfrak{p}})\}$.
- $\{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \not\cong 0 \text{ in } \mathsf{D}^{-}(R_{\mathfrak{p}})\}$.

Here $\mathsf{D}^{-}(R_{\mathfrak{p}})$ stands for the derived category of bounded-above $R_{\mathfrak{p}}$-complexes. We denote these four sets by $\operatorname{Supp}_R X$ and call it the *support* of X in $\mathsf{D}^{\mathsf{b}}(R)$.
(2) For a subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ we set $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp}_R X$ and call it the *support* of $\mathcal{X}$.

(3) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S$ the subcategory of $\mathsf{D^b}(R)$ consisting of objects whose supports are contained in S.

The supports for $\mathsf{D^b}(R)$ have the same notation as those for $\operatorname{\mathsf{mod}} R$, but there would be no danger of confusion since the support of an object M of $\operatorname{\mathsf{mod}} R$ is equal to the support of the object $\beta\alpha(M)$ of $\mathsf{D^b}(R)$.

Remark 11. (1) (a) One has $\operatorname{Supp}_R(X \oplus Y) = \operatorname{Supp}_R X \cup \operatorname{Supp}_R Y$ for $X, Y \in \mathsf{D^b}(R)$.
(b) One has $\operatorname{Supp}_R(X[n]) = \operatorname{Supp}_R X$ for $X \in \mathsf{D^b}(R)$ and $n \in \mathbb{Z}$.
(c) Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D^b}(R)$. Then for any permutation A, B, C of X, Y, Z one has $\operatorname{Supp}_R A \subseteq \operatorname{Supp}_R B \cup \operatorname{Supp}_R C$.
(2) For $X \in \mathsf{D^b}(R)$ the subset $\operatorname{Supp}_R X$ of $\operatorname{Spec} R$ is closed in the Zariski topology.
(3) For a subset S of $\operatorname{Spec} R$ the subcategory $\operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S$ of $\mathsf{D^b}(R)$ is thick.
(4) Let $\mathcal{X}$ be a subcategory of $\mathsf{D^b}(R)$. Then $\operatorname{Supp}_R \mathcal{X}$ is a specialization-closed subset of $\operatorname{Spec} R$. Furthermore, it holds that $\operatorname{Supp}_R \mathcal{X} = \operatorname{Supp}_R(\operatorname{\mathsf{thick}}_{\mathsf{D^b}(R)} \mathcal{X})$, because $\operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{X})$ is a thick subcategory containing $\mathcal{X}$, whence contains $\operatorname{\mathsf{thick}} \mathcal{X}$.

Let us recall the defintion of (the derived category of) perfect complexes.

Definition 12. A *perfect* complex is by defintion a bounded complex of finitely generated projective modules. We denote by $\mathsf{D_{perf}}(R)$ the subcategory of $\mathsf{D^b}(R)$ consisting of perfect complexes. This is a thick subcategory of $\mathsf{D^b}(R)$, and hence a triangulated category. For each subset S of $\operatorname{Spec} R$ we set $\operatorname{Supp}^{-1}_{\mathsf{D_{perf}}(R)} S = (\operatorname{Supp}^{-1}_{\mathsf{D^b}(R)} S) \cap \mathsf{D_{perf}}(R)$.

Remark 13. (1) Every thick subcategory of $\mathsf{D_{perf}}(R)$ is a thick subcategory of $\mathsf{D^b}(R)$.
(2) One has $\operatorname{\mathsf{thick}}_{\mathsf{D^b}(R)} R = \mathsf{D_{perf}}(R)$.
(3) For a set S of prime ideals of R, the subcategory $\operatorname{Supp}^{-1}_{\mathsf{D_{perf}}(R)} S$ of $\mathsf{D_{perf}}(R)$ is thick.

Finally, we recall the definitions of a hypersurface, a Cohen–Macaulay ring with minimal multiplicity and a disjoint union of sets.

Definition 14. (1) A local ring R is called a *hypersurface* if the completion of R is isomorphic to a quotient of a regular local ring by a nonzero element.
(2) Let R be a Cohen–Macaulay local ring. Then R satisfies the inequality

$$\mathrm{e}(R) \geq \operatorname{edim} R - \dim R + 1, \tag{2.1}$$

where $\mathrm{e}(R)$ and $\operatorname{edim} R$ denote the multiplicity of R and the embedding dimension of R, respectively. We say that R has *minimal multiplicity* (or *maximal embedding dimension*) if the equality of (2.1) holds.
(3) Let A_1, A_2 be sets whose intersection is possibly nonempty. The *disjoint union* of A_1 and A_2 is defined as

$$A_1 \sqcup A_2 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) = \{(x, 1), (y, 2) \mid x \in A_1, y \in A_2\}.$$

In the case where $A_1 \cap A_2$ is empty, the set $A_1 \sqcup A_2$ is identified with the union $A_1 \cup A_2$, namely, it is the usual disjoint union.

3. Comments on Theorem 1

The essentail part of the proof of Theorem 1 is played by the following result. This is shown by using contradiction, considering complexes of objects in $\mathsf{C}^{\mathsf{b}}(R)$ and applying the Hopkins–Neeman theorem [10, Theorem 1.5].

Proposition 15. *Let $(R, \mathfrak{m}, k)$ be a catenary equidimensional local ring with an isolated singularity. Let X be a non-acyclic bounded complex of R-modules. Then one has*

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, X\} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}.$$

Remark 16. The equality in Proposition 15 is no longer true if we remove k from the left-hand side; the equality

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)} X = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}$$

holds for $X = R$ if and only if $\mathsf{D}_{\mathsf{perf}}(R) = \mathsf{D}^{\mathsf{b}}(R)$, if and only if R is regular. This is one of the reasons why we consider thick subcategories containing k.

We should remark that unless R has only an isolated singularity, Theorem 1 does not necessarily hold. To be more precise, if R does not have an isolated singularity, then there may exist a thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ containing k such that $\mathcal{X} \neq \operatorname{Supp}^{-1} S$ for all nonempty specialization-closed subsets S of $\operatorname{Spec} R$.

Remark 17. Let $(R, \mathfrak{m}, k)$ be a local ring, and suppose that R does not have an isolated singularity. Set $\mathcal{X} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, R\}$. Then $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, but $\mathcal{X} \neq \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ for all subsets S of $\operatorname{Spec} R$.

Note that in the above remark $\mathcal{X} \neq \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ for all subsets S of $\operatorname{Spec} R$, not necessarily nonempty specialization-closed ones.

As a consequence of Theorem 1, we obtain the following one-to-one correspondence without prime ideals.

Corollary 18. *Let R be a catenary equidimensional local ring with an isolated singularity. Then one has a one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing } k\end{array}\right\} \underset{\psi}{\overset{\phi}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{array}{c}\textit{Nonzero thick subcategories}\\ \textit{of } \mathsf{D}_{\mathsf{perf}}(R)\end{array}\right\},$$

where ϕ, ψ are defined by $\phi(\mathcal{X}) = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$ and $\psi(\mathcal{Y}) = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$ for subcategories $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ and $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$.

Proof. Let S be a specialization-closed subset of $\operatorname{Spec} R$ containing $\mathfrak{m}$. Take a sequence $\boldsymbol{x}$ of elements of R which generates $\mathfrak{m}$. Then $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} S$ contains the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$, and hence it is a nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Conversely, for any nonzero thick subcategory $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$, the support $\operatorname{Supp}_R \mathcal{Y}$ contains $\mathfrak{m}$. Thus, the Hopkins–Neeman theorem [10, Theorem 1.5] implies that Supp_R and $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)}$ make mutually inverse bijections between the nonzero thick subcategories of $\mathsf{D}_{\mathsf{perf}}(R)$ and the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$.

Let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, and let $\mathcal{Y}$ be a nonzero thick subcategory of $\mathsf{D}_{\mathsf{perf}}(R)$. Combining our Theorem 1 with the above one-to-one correspondence, one has only to verify the equalities

(1) $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} \operatorname{Supp} \mathcal{X} = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$,

(2) $\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} \operatorname{Supp} \mathcal{Y} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$.

We have $\mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R) \subseteq \operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)}(\operatorname{Supp} \mathcal{X}) \subseteq \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)}(\operatorname{Supp} \mathcal{X}) = \mathcal{X}$, where the last equality follows from Theorem 1. This shows the equality (1). On the other hand, it holds that $\operatorname{Supp} \mathcal{Y} = \operatorname{Supp}(\mathcal{Y} \cup \{k\}) = \operatorname{Supp}(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\}))$, where the second equality follows from the fact that $\mathcal{Y}$ is nonzero. Applying $\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)}$ and using Theorem 1, we obtain the equality (2). $\square$

4. Comments on Theorem 2

The *stable derived category* $\mathsf{D}_{\mathsf{sg}}(R)$ of R, which is also called the *singularity category* of R, is defined as the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(R)$ by $\mathsf{D}_{\mathsf{perf}}(R)$. This has been introduced by Buchweitz [3] in relation to maximal Cohen–Macaulay modules over Gorenstein rings, and explored by Orlov [11] in relation to the Homological Mirror Symmetry Conjecture.

The essential part of the proof of Theorem 2 is played by the following result on the stable derived category. The assertion immediately follows from [13, Main Theorem] in the case (1). As for the case (2), it is shown by taking a minimal reduction of the maximal ideal.

Proposition 19. *Let R be a local ring with an isolated singularity. Suppose that R is either*

(1) *a hypersurface, or*

(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then $\mathsf{D}_{\mathsf{sg}}(R)$ has no nontrivial thick subcategory.

Let us give some examples of a ring satisfying Theorem 2(2).

Example 20. Let k be an infinite field, and let x, y be indeterminates over k. Then it is easy to observe that $k[[x, y]]/(x^2, xy, y^2)$, $k[[x, y, z]]/(x^2 - yz, y^2 - zx, z^2 - xy)$ and $k[[x^3, x^2y, xy^2, y^3]]$ are non-Gorenstein rings satisfying the condition (2) in Theorem 2. In general, normal local domains of dimension two with rational singularities satisfy Theorem 2(2); see [9, Theorem 3.1].

Remark 21. (1) Theorem 2(1) can also be deduced from [12, Theorem 4.9].
(2) Theorem 2(2) especially says the following.

> *Let R be a Cohen–Macaulay local ring with an isolated singularity and infinite residue field, and assume that R has minimal multiplicity. Let $\mathcal{X}$ be a standard thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ which is not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Then $\mathcal{X}$ contains the residue field of R.*

This statement is no longer true without the assumption that R has minimal multiplicity.

Indeed, let $R = k[x, y]/(x^2, y^2)$ with k a field, and let $\mathcal{X}$ be the thick closure of R and $R/(x)$ in $\mathsf{D}^{\mathsf{b}}(R)$. Then R is an artinian complete intersection local ring, and $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. As $\mathcal{X}$ contains R, it is standard. As $R/(x)$ has infinite projective dimension as an R-module, $\mathcal{X}$ is not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Note that both R and $R/(x)$ have complexity at most one. Since the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of objects having complexity at most one is thick, every object in $\mathcal{X}$ have complexity at most one. Since k has complexity two, $\mathcal{X}$ does not contain k.

Consequently, the assumption in Theorem 2(2) that R has minimal multiplicity is indispensable.

References

[1] D. J. Benson; J. F. Carlson; J. Rickard, Thick subcategories of the stable module category, *Fund. Math.* **153** (1997), no. 1, 59–80.

[2] D. J. Benson; S. B. Iyengar; H. Krause, Stratifying modular representations of finite groups, *Ann. of Math. (2)* **174** (2011), no. 3, 1643–1684.

[3] R.-O. Buchweitz, Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings, `http://hdl.handle.net/1807/16682`.

[4] E. S. Devinatz; M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, I, *Ann. of Math. (2)* **128** (1988), no. 2, 207–241.

[5] W. Dwyer; J. P. C. Greenlees; S. Iyengar, Finiteness in derived categories of local rings, *Comment. Math. Helv.* **81** (2006), no. 2, 383–432.

[6] E. M. Friedlander; J. Pevtsova, Π-supports for modules for finite group schemes, *Duke Math. J.* **139** (2007), no. 2, 317–368.

[7] M. J. Hopkins, Global methods in homotopy theory, *Homotopy theory (Durham, 1985)*, 73–96, London Math. Soc. Lecture Note Ser., 117, *Cambridge Univ. Press, Cambridge*, 1987.

[8] M. J. Hopkins; J. H. Smith, Nilpotence and stable homotopy theory, II, *Ann. of Math. (2)* **148** (1998), no. 1, 1–49.

[9] C. Huneke; K.-i. Watanabe, Upper bound of multiplicity of F-pure rings, *Proc. Amer. Math. Soc.* (to appear), `arXiv:1310.0584`.

[10] A. Neeman, The chromatic tower for $D(R)$, With an appendix by Marcel Bökstedt, *Topology* **31** (1992), no. 3, 519–532.

[11] D. O. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, *Proc. Steklov Inst. Math.* **246** (2004), no. 3, 227–248.

[12] G. Stevenson, Duality for bounded derived categories of complete intersections, *Bull. Lond. Math. Soc.* **46** (2014), no. 2, 245–257.

[13] R. Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

[14] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.

[15] R. Takahashi, Thick subcategories over isolated singularities, in preparation.

[16] R. W. Thomason, The classification of triangulated subcategories, *Compos. Math.* **105** (1997), no. 1, 1–27.

[17] J.-L. Verdier, Catégories dérivées, état 0, SGA $4\frac{1}{2}$, 262–308, Lecture Notes in Mathematics, 569, *Springer-Verlag, Berlin-New York*, 1977.

Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602, Japan
E-mail address: `takahashi@math.nagoya-u.ac.jp`