2-DIMENSIONAL QUANTUM BEILINSON ALGEBRAS

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Abstract. A Fano algebra introduced by Minamoto is roughly speaking a finite dimensional algebra of finite global dimension which is derived equivalent to a (noncommutative) Fano variety [3]. Over such an algebra, a notion of regular module was introduced by Herschend, Iyama and Oppermann from the view point of representation theory of finite dimensional algebras [2]. In this article, we will recall the definitions of a Fano algebra and a regular module, and then explicitly calculate algebraic spaces parameterizing isomorphism classes of simple regular modules over typical examples of Fano algebras, namely, 2-dimensional quantum Beilinson algebras, using techniques of noncommutative algebraic geometry [5].

1. Motivation

Throughout, let k be an algebraically closed field of characteristic 0. All algebras in this article are algebras over k. For a finite dimensional algebra R, we denote by $\operatorname{mod} R$ the category of finite dimensional right R-modules, and $D^b(\operatorname{mod} R)$ the bounded derived category of $\operatorname{mod} R$. If $\operatorname{gldim} R = d < \infty$, then we define an autoequivalence ν_d of $D^b(\operatorname{mod} R)$ by $\nu_d(X) := X \otimes^L_R DR[-d]$ where $DR = \operatorname{Hom}_k(R, k)$.

Definition 1. [3] A finite dimensional algebra R is called d-dimensional Fano if

(1) $\operatorname{gldim} R = d < \infty$, and
(2) $\nu_d^{-i}(R) \in \operatorname{mod} R$ for all $i \geq 0$.

If R is d-dimensional Fano as above, then we define the preprojective algebra of R by

$$\Pi R := T_R(\nu_d^{-1}(R)) = T_R(\operatorname{Ext}^d_R(DR, R))$$

as a graded algebra.

Theorem 2. [3] *A finite dimensional algebra is 1-dimensional Fano if and only if it is a hereditary algebra of infinite representation type.*

Remark 3. By the above theorem, Herschend, Iyama and Oppermann [2] call a d-dimensional Fano algebra R a d-representation infinite algebra. Moreover, they call R d-representation tame if ΠR is noetherian as an algebra.

For a hereditary algebra R of infinite representation type (that is, a 1-dimensional Fano algebra by the above theorem), classifying regular modules is essential in understanding $\operatorname{mod} R$. The notion of regular module was extended to a d-dimensional Fano algebra.

Definition 4. [2] Let R be a d-dimensional Fano algebra. A module $M \in \operatorname{mod} R$ is called d-regular if $\nu_d^i(M) \in \operatorname{mod} R$ for all $i \in \mathbb{Z}$.

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The purpose of this ongoing project is to find an algebraic space $Reg(R)$ parameterizing isomorphism classes of simple d-regular modules over a d-dimensional Fano algebra R.

2. Quantum Beilinson Algebras

Let $A = \oplus_{i=0}^{\infty} A_i$ be a right noetherian graded algebra. We denote by $\operatorname{grmod} A$ the category of finitely generated graded right A-modules. For $M \in \operatorname{grmod} A$ and $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} \in \operatorname{grmod} A$ by $M_{\geq n} = \oplus_{i=n}^{\infty} M_i$, and the shift $M(n) \in \operatorname{grmod} A$ by $M(n)_i = M_{n+i}$. We say that A is connected graded if $A_0 = k$ and, in this case, $k = A/A_{\geq 1} \in \operatorname{grmod} A$.

For a right noetherian connected graded algebra A, we denote by $\operatorname{tors} A$ the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k, and $\operatorname{tails} A := \operatorname{grmod} A/\operatorname{tors} A$ the quotient category. Following [1], $\operatorname{Proj_{nc}} A$ is an imaginary geometric object whose category of "coherent sheaves" is $\operatorname{tails} A$ since if A is commutative and generated in degree 1, then $\operatorname{tails} A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$.

Definition 5. A right noetherian connected graded algebra A is called d-dimensional AS-regular if

(1) $\operatorname{gldim} A = d < \infty$, and

(2) there exists $\ell \in \mathbb{N}^+$ such that $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If A is d-dimensional AS-regular as above, then we define the quantum Beilinson algebra of A by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Remark 6. Every AS-regular algebra as above is a skew Calabi-Yau algebra so that there exists a graded algebra automorphism $\mu \in \operatorname{Aut} A$, called the Nakayama automorphism, such that $\operatorname{Ext}^d_{A^e}(A, A^e) \cong {}_{\mu}A(\ell)$ as graded A-A bimodules where $A^e = A \otimes_k A^{op}$, and ${}_{\mu}A = A$ as a graded vector space with the new bimodule structure $a * x * b = \mu(a)xb$.

Theorem 7. [4] *If A is a d-dimensional AS-regular algebra, then*

(1) *∇A is a $(d-1)$-dimensional Fano algebra,*

(2) *$\operatorname{grmod} A \cong \operatorname{grmod} \Pi(\nabla A)$, and*

(3) *$D^b(\operatorname{tails} A) \cong D^b(\operatorname{mod} \nabla A)$.*

By the above theorem, we call R a d-dimensional quantum Beilinson algebra if there exists a $(d+1)$-dimensional AS-regular algebra A such that $R \cong \nabla A$.

Remark 8. If A is a d-dimensional AS-regular algebra, then $\operatorname{Proj_{nc}} A$ can be viewed as a weighted quantum $\mathbb{P}^{d-1}$ since a commutative d-dimensional AS-regular algebra is exactly a weighted polynomial algebra in d variables. Since a (weighted) quantum projective space is one of the main objects of study in noncommutative algebraic geometry, the above theorem provides strong interactions between noncommutative algebraic geometry and representation theory of algebras.

The main observation in [5] claims that $Reg(\nabla A) = |\operatorname{Proj}_{\mathrm{nc}} A|$ the set of closed points of $\operatorname{Proj}_{\mathrm{nc}} A$, which is expected to have a structure of an algebraic stack. Instead of making this claim more precise, we will give explicit examples below.

3. Hereditary Cases

The results in this section are well-known in representation theory of algebras. We will recover these results using noncommutative algebraic geometry.

If $A = k[x, y]$ is a weighted polynomial algebra with $\deg x = a, \deg y = b \in \mathbb{N}^+$ such that $\gcd(a, b) = 1$, then A is a 2-dimensional AS-regular algebra with $\ell = a + b$, so ∇A is a hereditary algebra of infinite representation type (a 1-dimensional Fano algebra). In fact, $\nabla A = kQ$ is a path algebra where Q is a quiver of type $\widetilde{A_{\ell-1}}$.

Theorem 9. [5] *In the above setting,* $Reg(\nabla A) = \big[(\mathbb{A}^2 \setminus \{(0, 0)\})/\sim\big]$ *the quotient stack where* $(x, y) \sim (\lambda^a x, \lambda^b y)$ *for* $0 \neq \lambda \in k$.

If $a = b = 1$, then $\big[(\mathbb{A}^2 \setminus \{(0, 0)\})/\sim\big] = \mathbb{P}^1$ by the definition of $\mathbb{P}^1$. In general, $\big[(\mathbb{A}^2 \setminus \{(0, 0)\})/\sim\big]$ is almost $\mathbb{P}^1$ but the point $(0, 1) \in \mathbb{P}^1$ splits into a points, and the point $(1, 0) \in \mathbb{P}^1$ splits into b points.

Recall that if R is a hereditary algebra of infinite representation type (a 1-dimensional Fano algebra), then two simple regular modules $M, N \in \operatorname{mod} R$ are in the same regular component if and only if they are in the same ν_1 orbit, so the regular components of R are parametrized by $Reg(R)/\langle\nu_1\rangle$. In the above setting, the split a points are in the same ν_1 orbit and the split b points are in the same ν_1 orbit, so we have the following result.

Theorem 10. [5] *In the above setting,* $Reg(\nabla A)/\langle\nu_1\rangle = \mathbb{P}^1$.

4. 2-dimensional Beilinson Algebras

Using the techniques in noncommutative algebraic geometry, we can show that 2-dimensional quantum Beilinson algebras can be constructed as follows. Let $g \in k[x, y, z]_3$ be a cubic polynomial, $E = \operatorname{Proj} k[x, y, z]/(g) \subset \mathbb{P}^2$ and $\sigma \in \operatorname{Aut} E$. Define an algebra $R(E, \sigma) = kQ/I$ where Q is the Beilinson quiver

$$\bullet \;\begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{y_1} \\ \xrightarrow{z_1} \end{array}\; \bullet \;\begin{array}{c} \xrightarrow{x_2} \\ \xrightarrow{y_2} \\ \xrightarrow{z_2} \end{array}\; \bullet$$

and

$$I = (\{f \in kQ_2 \mid f(p, \sigma(p)) = 0 \text{ for all } p \in E\}).$$

It can be shown that $R(E, \sigma)$ is generically a 2-dimensional quantum Beilinson algebra. Define

$$||\sigma|| := \inf\{i \in \mathbb{N}^+ \mid \text{ there exists } \tau \in \operatorname{Aut} \mathbb{P}^2 \text{ such that } \sigma^i = \tau\}.$$

Note that $||\sigma|| \leq |\sigma|$ the order of σ, and $||\sigma|| = 1$ if and only if $E = \mathbb{P}^2$.

Proposition 11. *Suppose that* $R(E, \sigma), R(E', \sigma')$ *are 2-dimensional quantum Beilinson algebras. If* $R(E, \sigma) \cong R(E', \sigma')$, *then* $E \cong E'$ *and* $||\sigma|| = ||\sigma'||$.

Since $R(E,\sigma) \cong R(E',\sigma')$ does not imply $|\sigma| = |\sigma'|$, $||\sigma||$ is more important than $|\sigma|$ to study $R(E,\sigma)$.

Theorem 12. [5] *Let $R(E,\sigma)$ be a 2-dimensional quantum Beilinson algebra. If $||\sigma|| = \infty$, then*

(1) *$RegR(E,\sigma) = E$, and*
(2) *$RegR(E,\sigma)/\langle\nu_2\rangle = E/\langle\mu\sigma^3\rangle$ where μ is the Nakayama automorphism.*

In the case of $||\sigma|| < \infty$, we only have a partial result.

Theorem 13. [5] *Let $R(E,\sigma)$ be a 2-dimensional quantum Beilinson algebra such that $E \subset \mathbb{P}^2$ is a triangle. Then $||\sigma|| < \infty$ if and only if $\Pi R(E,\sigma)$ is finite over its center (that is, $R(E,\sigma)$ is 2-representation tame), and, in this case,*

(1) *$RegR(E,\sigma) = E \sqcup (\mathbb{P}^2 \setminus E)$, and*
(2) *$RegR(E,\sigma)/\langle\nu_2\rangle = E/\langle\mu\sigma^3\rangle \sqcup (\mathbb{P}^2 \setminus E)$ where μ is the Nakayama automorphism.*

Example 14. Let $R = kQ/(y_1z_2 - \alpha z_1y_2, z_1x_2 - \beta x_1z_2, x_1y_2 - \gamma y_1x_2)$ where Q is the Beilinson quiver

$$\bullet \;\begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{y_1} \\ \xrightarrow{z_1} \end{array}\; \bullet \;\begin{array}{c} \xrightarrow{x_2} \\ \xrightarrow{y_2} \\ \xrightarrow{z_2} \end{array}\; \bullet\ .$$

If $\alpha\beta\gamma \neq 0, 1$, then $R = R(E,\sigma)$ is a 2-dimensional quantum Beilinson algebra where $E = \operatorname{Proj} k[x,y,z]/(xyz) = V(x) \cup V(y) \cup V(z) \subset \mathbb{P}^2$ is a triangle and $\sigma \in \operatorname{Aut} E$ is given by

$$\sigma|_{V(x)}(0,b,c) = (0,b,\alpha c)$$
$$\sigma|_{V(y)}(a,0,c) = (\beta a,0,c)$$
$$\sigma|_{V(z)}(a,b,0) = (a,\gamma b,0).$$

It is easy to see that

$$||\sigma|| = |\alpha\beta\gamma| \leq \operatorname{lcm}(|\alpha|,|\beta|,|\gamma|) = |\sigma|,$$

so if $|\alpha\beta\gamma| = \infty$, then $Reg(R) = E$, and if $|\alpha\beta\gamma| < \infty$, then $Reg(R) = E \sqcup (\mathbb{P}^2 \setminus E)$.

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