THE STRUCTURE OF PREENVELOPES WITH RESPECT TO MAXIMAL COHEN-MACAULAY MODULES

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Abstract. This article studies the structure of special preenvelopes with respect to maximal Cohen-Macaulay modules. We investigate the structure of them in terms of their kernels and cokernels. Moreover, using this result, we also study the structure of special proper coresolutions with respect to maximal Cohen-Macaulay modules over a Henselian Cohen-Macaulay local ring.

This article is based on [3]. Throughout this article, we assume that R is a d-dimensional Cohen-Macaulay local ring with canonical module ω. All R-modules are assumed to be finitely generated. Denote by $\mathsf{mod}R$ the category of finitely generated R-modules and by MCM the full subcategory of $\mathsf{mod}R$ consisting of maximal Cohen-Macaulay R-modules.

Auslander and Buchweitz showed the following result which plays an important role in the representation theory of commutative rings.

Theorem 1. [1] *For any R-module M, there exists a short exact sequence*

$$0 \to Y \to X \xrightarrow{\pi} M \to 0$$

such that $X \in \mathsf{MCM}$ and $\mathsf{id}_R Y < \infty$.

The morphism π is called a *maximal Cohen-Macalulay approximation* of M.

In this article, we mainly study a special MCM-preenvelope which is a categorical dual notion of a maximal Cohen-Macaulay approximation.

Definition 2. Let $\mu : M \to X$ be an R-homomorphism with $X \in \mathsf{MCM}$.

(1) μ is called an MCM-*preenvelope* of M if

$$\mathsf{Hom}_R(\mu, X') : \mathsf{Hom}_R(X, X') \to \mathsf{Hom}_R(M, X')$$

is an epimorphism for any $X' \in \mathsf{MCM}$.

(2) μ is called a *special* MCM-*preenvelope* of M if μ is an MCM-preenvelope and satisfies $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.

(3) μ is called an MCM-*envelope* of M if μ is an MCM-preenvelope and every $\phi \in \mathsf{End}_R(X)$ that satisfies $\phi\mu = \mu$ is an automorphism.

The notions of MCM-*precover*, *special* MCM-*precover*, and MCM-*cover* are defined dually.

Remark 3. (1) By definition, a maximal Cohen-Macaulay approximation is nothing but a special MCM-precover.

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(2) Owing to Wakamatsu's lemma, an MCM-envelope is a special MCM-preenvelope, and by definition, a special MCM-preenvelope is an MCM-preenvelope.

(3)
- (Auslander-Buchweitz [1])
  Every R-module has a special MCM-precover.
- (Yoshino [4])
  If R is Henselian (e.g. complete), then every R-module has an MCM-cover.
- (Holm [2])
  Every R-module has a special MCM-preenvelope, and if R is Henselian, every R-module has an MCM-envelope.

Since every MCM-precover is an epimorphism, for any R-homomorphism $\pi : X \to M$ with X maximal Cohen-Macaulay, the following are equivalent.

(1) π is a special MCM-precover.
(2) $\mathsf{Coker}\,\pi = 0$ and $\mathsf{Ext}^1_R(\mathsf{MCM}, \mathsf{Ker}\,\pi) = 0$.
(3) $\mathsf{Coker}\,\pi = 0$ and $\mathsf{id}(\mathsf{Ker}\,\pi) < \infty$.

Therefore, MCM-preenvelopes are characterized by using their kernels and cokernels. We consider the following question.

Question 4. *When is a given morphism $\mu : M \to X$ with $X \in \mathsf{MCM}$ a special MCM-preenvelope?*

The following result is our main theorem in this article and which gives an answer to this question.

Theorem 5. *Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*

(1) *μ is a special MCM-preenvelope of M.*
(2) *$\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$ and $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.*
(3) *$\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, and there exists an exact sequence*

$$0 \to S \to \mathsf{Coker}\,\mu \to T \to U \to 0$$

such that
- *$\mathsf{codim}\,S > 1$, $\mathsf{codim}\,U > 2$,*
- *T satisfies Serre's condition (S_2),*
- *$\mathsf{id}_R T^\dagger < \infty$ and $T^\dagger$ satisfies Serre's condition (S_3).*

The condition (3) in Theorem 5 is rather complicated, but which characterizes a special MCM-preenvelope by some numerical conditions, and has an advantage that it does not contain vanishing condition of Ext-module, which is hard to check.

Next, we give some examples of special MCM-preenvelopes.

Example 6. (1) Let M be an R-module with $\mathsf{codim}\,M > 0$. Then $\mu : M \to 0$ is a special MCM-preenvelope.

(2) Let $\underline{x} = x_1, x_2, \ldots, x_n$ be an R-regular sequence with $n \geq 3$. Consider an exact sequence

$$0 \to M \xrightarrow{\mu} R^{\oplus n} \xrightarrow{(x_1,\ldots,x_n)} R \to R/(\underline{x}) \to 0.$$

Then μ is a special MCM-preenvelope.

(3) Let K and C be R-modules with $\mathsf{codim}\, K > 0, \mathsf{codim}\, C > 1$ and $\sigma \in \mathsf{Ext}^2_R(C, K)$. σ defines an exact sequence

$$0 \to K \to M \xrightarrow{\mu} F \to C \to 0$$

with F a free R-module. Then μ is a special MCM-preenvelope of M.

Using the condition (3) in Theorem 5, we have a result about a special proper MCM-coresolution.

Definition 7. Let M be an R-module, and

$$(*) \qquad 0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots$$

be an R-complex with $X^i \in \mathsf{MCM}$ for each i. Put $\mu^0 := \delta^0$ and let $\mu^i : \mathsf{Coker}\, \delta^{i-1} \to X^i$ be the induced morphism from δ^i for $i > 0$.

- If each μ^i is a special MCM-preenvelope (resp. an MCM-envelope), then we call (*) a special proper MCM-coresolution (resp. a minimal proper MCM-coresolution) of M.
- For a minimal proper MCM-coresolution (*), $\mathsf{Coker}\, \mu^{i-1}$ is called an i-th minimal MCM-cosyzygy of M, and it is denoted by ${\mathsf{Cosyz}_{\mathsf{MCM}}}^i M$.

Remark 8. Suppose R is Henselian. For a special proper MCM-coresolution (*),

- $\mathsf{Coker}\, \mu^i$ are unique up to free summands, and
- $\mathsf{Ker}\, \mu^i$ are unique up to isomorphism.

Theorem 9. *Suppose R is Henselian. Let M be an R-module and*

$$0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots$$

a special proper MCM-coresolution of M. Put $\mu^0 := \delta^0$ and let $\mu^i : \mathsf{Coker}\, \delta^{i-1} \to X^i$ be the induced homomorphisms. Then for each $i \geq 0$, one has

(1) $\mathsf{codim}(\mathsf{Ker}\, \mu^i) > i$,
(2) *there exists an exact sequence*

$$0 \to S^i \to \mathsf{Coker}\, \mu^i \to T^i \to U^i \to 0$$

such that

- $\mathsf{codim}\, S^i > i + 1$, $\mathsf{codim}\, U^i > i + 2$,
- T^i *satisfies* (S_2),
- $\mathsf{id}_R(T^i)^\dagger < \infty$ *and* $(T^i)^\dagger$ *satisfies* (S_{i+3}).

From the above remark, we can show this theorem by constructing such a special proper MCM-coresoltion.

Letting $i = d - 2, d - 1$ in Theorem 9, we have the following corollary.

Corollary 10. *Suppose R is Henselian. For any R-module M,*

- ${\mathsf{Cosyz}_{\mathsf{MCM}}}^d M = 0$ *and*
- ${\mathsf{Cosyz}_{\mathsf{MCM}}}^{d-1} M$ *has finite length.*

In particular, for any R-module M, the minimal proper MCM-coresolution of M has length at most $\min\{0, d - 2\}$.

Remark 11. This corollary refines a theorem due to Holm [2, Theorem C]: For any R-module M, the minimal proper MCM-coresolution of M has length at most $\min\{0, d-1\}$.

For the last of this article, we give another characterization of special MCM-preenvelopes in terms of the existence of certain complexes.

Auslander and Buchweitz also state the following result

Theorem 12. [1] *Let $\pi : X \to M$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*

(1) *π is a special MCM-precover of M.*
(2) *There exists an R-complex*

$$C = (0 \to C_d \xrightarrow{\delta_{d-1}} C_{d-1} \xrightarrow{\delta_{d-2}} \cdots \to C_1 \xrightarrow{\delta_0} C_0 \xrightarrow{\delta_{-1}} C_{-1} \to 0)$$

such that

- *C_i is a finite direct sum of ω for $1 \le i \le d$,*
- *$\delta_{-1} = \pi$,*
- *C is exact.*

The following theorem is dual of this theorem.

Theorem 13. *Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*

(1) *μ is a special MCM-preenvelope of M.*
(2) *There exists an R-complex*

$$C = (0 \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{d-2}} C^{d-1} \to 0)$$

such that

- *C^i is free for $1 \le i \le d-1$,*
- *$\delta^{-1} = \mu$,*
- *$\operatorname{codim} \mathsf{H}^i(C) > i+1$ for any i.*

References

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