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Hideto Asashiba
Shizuoka, Japan
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Preface

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Tsunekazu Nishinaka
Kobe, Japan
February, 2016

The 48th Symposium on Ring Theory and Representation Theory (2015)

Program

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Hirotaka KOGA (Tokyo Denki University)

Crossed products for matrix rings

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Convolution algebras and enveloping algebras I

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Almost Gorenstein Rees algebras

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Atom-molecule correspondence in Grothendieck categories

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Tilting theory of preprojective algebras and c-sortable elements

11:10–11:40 Kenta Ueyama (Hirosaki University), Izuru Mori (Shizuoka University)
Tilting objects for noncommutative quotient singularities

THE CLASSIFICATION OF TWO-TERM TILTING COMPLEXES FOR BRAUER GRAPH ALGEBRAS

TAKAHIDE ADACHI

Abstract. The study of derived categories have been one of the central themes in representation theory. From Morita theoretic perspective, tilting complexes play an important role because the endomorphism algebras are derived equivalent to the original algebra [4]. It is well-known that derived equivalences preserve many homological properties. Thus it is important to classify tilting complexes for a given algebra. Our aim of this report is to give a classification of two-term tilting complexes for Brauer graph algebras.

1. Preliminaries

In this section, we collect some results which are necessary in this report. Throughout this report, K is an algebraically closed field. All algebras are assumed to be basic, indecomposable, and finite dimensional over K. We always work with finite dimensional right modules. For an algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of finite dimensional right Λ-modules and by $\mathrm{proj}\Lambda$ the full subcategory of $\mathrm{mod}\Lambda$ consisting of all finite dimensional projective Λ-modules. We sometimes write $\Lambda = KQ/I$, where Q is a quiver with relations I. We denote by P_i an indecomposable projective Λ-module corresponding to a vertex i of Q. An arrow of Q is identified to a map between indecomposable projective Λ-modules. The composition of maps $f : X \to Y$ and $g : Y \to Z$ is denoted as $gf : X \to Z$. For an object X, we denote by $|X|$ the number of isomorphism classes of indecomposable summands of X.

1.1. **Tilting theory.** In this subsection, we recall the definition of tilting complexes. Let Λ be an algebra. We denote by $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ the bounded homotopy category of $\mathrm{proj}\Lambda$.

Definition 1. Let T be a complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.

(1) We say that T is *pretilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(T, T[n]) = 0$ for all non-zero integers n.
(2) We say that T is *tilting* if it is pretilting and generates $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ by taking direct sums, direct summands, mapping cones and shifts.
(3) We say that T is *two-term* if it is of the form $(0 \to T^{-1} \to T^0 \to 0)$, where T^n is the n-th term of T.

We denote by $\mathsf{2\text{-}ptilt}\Lambda$ the set of isomorphism classes of indecomposable two-term pretilting complexes of Λ and by $\mathsf{2\text{-}tilt}\Lambda$ the set of isomorphism classes of basic two-term tilting complexes of Λ.

Proposition 2. [1, 3] *Let Λ be a symmetric algebra and T a two-term pretilting complex of Λ. Then the following hold:*

The detailed version of this paper will be submitted for publication elsewhere.

(1) *T satisfies* $\mathsf{add}T^0 \cap \mathsf{add}T^{-1} = 0$.
(2) *T is two-term tilting if and only if* $|T| = |\Lambda|$.

1.2. **Ribbon graphs and signed walks.** In this subsection, we introduce the notion of signed walks and admissible walks (see [2] for details). Throughout this report, we assume that all graphs contain no loops. A *ribbon graph* is a graph equipped with a cyclic ordering of the edges around each vertex. For a ribbon graph G, we denote by G_0 the set of vertices of G and by G_1 the set of edges of G. The degree $\mathrm{d}(v)$ of a vertex $v \in G_0$ is the number of edges incident to v.

Definition 3. A walk $w = (e_1, e_2, \ldots, e_l)$ (*i.e.*, it is a sequence of edges) of a graph is called a *signed walk* of a (ribbon) graph if it is equipped with a map $\epsilon : \{e_1, e_2, \ldots, e_l\} \to \{+1, -1\}$ such that $\epsilon(e_i) = -\epsilon(e_{i+1})$ for any $i \in \{1, 2, \ldots, l-1\}$. we call e_1, e_l the endpoints of the (signed) walk w. We often notate a signed walk by $(w; \epsilon)$ or $(e_1^{\epsilon(e_1)}, e_2^{\epsilon(e_2)}, \ldots, e_l^{\epsilon(e_l)})$. We denote by $\mathsf{SW}(G)$ the set of signed walks of a ribbon graph G.

To give a combinatorial description of an indecomposable two-term pretilting complex, we introduce a special signed walk, which is called an admissible walk.

Definition 4. We say that a signed walk $w = (e_1, \ldots, e_l; \epsilon)$ *satisfies the sign condition* if $\epsilon(e_1) = \epsilon(e_l)$ whenever the endpoints of w are same vertex. In general, two signed walks w and w' *satisfy the sign condition* if the signatures are same whenever two of four endpoints of w and w' are same vertex.

We will attach some extra data for a signed walk, which are uniquely determined by the signature. A *virtual edge* is an element in the set $\{\mathrm{vr}_-(e), \mathrm{vr}_+(e) \mid e \in G_1\}$. Let $(e_1, e_2, \ldots, e_k)_v$ be the cyclic ordering around a vertex $v \in G_0$. We define the cyclic ordering accounting the virtual edges as

$$(\mathrm{vr}_-(e_1), e_1, \mathrm{vr}_+(e_1), \mathrm{vr}_-(e_2), e_2, \mathrm{vr}_+(e_2), \ldots, \mathrm{vr}_-(e_k), e_k, \mathrm{vr}_+(e_k))_v.$$

For a signed walk $w = (e_1, \ldots, e_l; \epsilon)$, we define the following virtual edges attached to w:

$$e_0 := \mathrm{vr}_{-\epsilon(e_1)}(e_1), \quad e_{l+1} := \mathrm{vr}_{-\epsilon(e_l)}(e_l).$$

We also define $\epsilon(e_0) := -\epsilon(e_1)$ and $\epsilon(e_{l+1}) := -\epsilon(e_l)$. To improve readability of various statements, we only write down the edges required in the cyclic ordering around a vertex. For example, if the edges e, f, g are only important edges incident to a vertex v, then we will write the cyclic ordering $(e, f, g)_v$ instead of $(e, \ldots, f, \ldots, g, \ldots)_v$.

Let $w = (e_1, e_2, \ldots, e_n; \epsilon)$ and $w' = (e'_1, e'_2, \ldots, e'_m; \epsilon')$ be signed walks. Moreover, it is automatically understood what we mean by $e_0, e_{n+1}, e'_0, e'_{m+1}$ from the definition of virtual edges. Assume that a, b, c, d are edges incident to a vertex v given by

$$\{a, b\} := \{e_{i-1}, e_i\}, \quad \{c, d\} := \{e'_{j-1}, e'_j\}$$

for some $i \in \{1, 2, \ldots, n+1\}$ and $j \in \{1, 2, \ldots, m+1\}$. We say that v is an *intersecting vertex* of w and w' if a, b, c, d are pairwise distinct.

Definition 5. We say that w and w' is *non-crossing at the intersecting vertex* v if at most one of a, b, c, d is virtual, and the cyclic ordering around v with the signature is either

$$(a^+, b^-, c^+, d^-)_v \text{ or } (a^+, b^-, c^-, d^+)_v.$$

A *subwalk* of a walk w is consecutive subsequence of w. A *common walk* of two walks w and w' is a subwalk z of both w and w'. Moreover, it is said to be *maximal* if there is no common walk $z'(\neq z)$ of w and w' such that z is a subwalk of z'.

Definition 6. Let $w = (e_1, e_2, \dots, e_n; \epsilon)$ and $w' = (e'_1, e'_2, \dots, e'_m; \epsilon')$ be signed walks, and $z = (t_1, t_2, \dots, t_l)$ a maximal common subwalk of w and w'. Assume that u (respectively, v) is the endpoint of z for t_1 (respectively, t_l), and $t_k = e_{i+k-1} = e'_{j+k-1}$ for all $k \in \{1, 2, \dots, l\}$. We say that w and w' are *non-crossing at* z if the following hold:

- $\epsilon(t_k) = \epsilon'(t_k)$ for each $k \in \{1, 2, \dots, l\}$.
- With the exception of $i = j = 1$ and/or $m + 1 - i - l = n + 1 - j - l = 0$, the cyclic orderings around u and v are either

$$(t_1, e_{i-1}, e'_{j-1})_u \text{ and } (t_l, e'_{j+l}, e_{i+l})_v \text{ respectively,}$$
$$\text{or } (t_1, e'_{j-1}, e_{i-1})_u \text{ and } (t_l, e_{i+l}, e'_{j+l})_v \text{ respectively.}$$

We say that two signed walks w and w' are *non-crossing* if they are non-crossing at all maximal common subwalks and all intersecting vertices. In particular, w is *self-non-crossing* if w itself is non-crossing.

Definition 7. An admissible walk is a self-non-crossing signed walk which satisfies the sign condition. We denote by $\mathsf{AW}(G)$ the set of admissible walks of a ribbon graph G.

At the end of this subsection, we give the following result for finiteness of $\mathsf{AW}(G)$.

Proposition 8. [2, Proposition 2.12] *Let G be a ribbon graph. Then the following are equivalent:*

(1) $\mathsf{AW}(G)$ *is finite.*
(2) G *consists of at most one odd cycle and no even cycle.*

1.3. **Brauer graph algebras.** In this subsection, we recall the definition of Brauer graph algebras. A *Brauer graph* is a ribbon graph equipped with a map $\mathrm{m} : G_0 \to \mathbb{Z}_{>0}$, which is called multiplicity.

Let $G = (G, \mathrm{m})$ be a Brauer graph. Then we define the Brauer graph algebra Λ_G as follows: First, if G is the graph u —— v and $\mathrm{m}(u) = \mathrm{m}(v) = 1$, then $\Lambda_G = K[x]/(x^2)$. Otherwise, $\Lambda_G = KQ_G/I_G$, where

(1) Q_G is the following quiver:
 - There exists a one-to-one correspondence between the vertex of Q_G and the edges of G.
 - For two distinct vertices e and e' in Q_G corresponding to edges e and e' in G, we draw an arrow $\alpha_{e,e'} : e' \to e$ in Q_G if the edge e' is a direct successor of the edge e in the cyclic ordering around a common vertex in G. If the endpoint v of e in G satisfies $\mathrm{d}(v) = 1$ and $\mathrm{m}(v) > 1$, then we draw an arrow $\alpha_{e,e} : e \to e$ in Q_G.

(2) I_G is a two-sided ideal generated by the following relations: Let $(e_1, e_2, \dots, e_{\mathrm{d}(v)})_v$ be the cyclic ordering around $v \in G$. Then we define α_{e_j,e_i} to be the path

$$\alpha_{e_j,e_{j+1}} \cdots \alpha_{e_{i-2},e_{i-1}} \alpha_{e_{i-1},e_i}$$

in Q_G. Let $C_{e_i,v} := \alpha_{e_i,e_i}$.
 - If the edge e in G has endpoints u and v so that e is not a leaf at u with $\mathrm{m}(u) = 1$

or at v with $\mathrm{m}(v) = 1$, then $C_{e,u}^{\mathrm{m}(u)} - C_{e,v}^{\mathrm{m}(v)} \in I_G$.

• If the edge e in G has endpoints u and v so that e is a leaf at u with $\mathrm{m}(u) = 1$, then $C_{e,v}^{\mathrm{m}(v)}\alpha_{e,e'} \in I_G$, where e' is a direct predecessor of e in the cyclic ordering.

• All path $\alpha\beta$ which is not a subpath of any cycle $C_{e,v}$ are in I_G.

It is well-known that each Brauer graph algebra is a symmetric special biserial algebra, and vice versa [5]. In particular, an indecomposable non-projective module is either a string module or a band module [6]. Note that, for an indecomposable two-term complex T, if the 0-th cohomology $\mathrm{H}^0(T)$ is band, then T is not pretilting. Hence we are interested in only string modules in this report.

2. Main results

Let $G = (G, \mathrm{m})$ be a Brauer graph and $\Lambda = \Lambda_G$ the Brauer graph algebra.

Definition 9. An indecomposable two-term complex T is called a *string complex* if the 0-th cohomology $\mathrm{H}^0(T)$ is a string module. We denote by $\mathsf{2\text{-}scx}\Lambda$ the set of indecomposable stalk complexes of projective modules concentrated in degree 0 or -1, and string complexes $T = (T^{-1} \to T^0)$ with $\mathsf{add}T^0 \cap \mathsf{add}T^{-1} = 0$.

Lemma 10. [2, Lemma 4.4] $\mathsf{2\text{-}ptilt}\Lambda$ *is a subset of* $\mathsf{2\text{-}scx}\Lambda$.

For a signed walk $w = (e_1, e_2, \ldots, e_n; \epsilon)$, we define a two-term complex $T_w = (T^{-1} \xrightarrow{d} T^0)$ as follows:

• $T^0 := \bigoplus_{\epsilon(e_i)=+1} P_{e_i}$ and $T^{-1} := \bigoplus_{\epsilon(e_i)=-1} P_{e_i}$.

• $d = (d_{ij})$, where $d_{ij} : P_{e_j} \to P_{e_i}$ given by

$$d_{ij} := \begin{cases} \alpha_{e_i,e_j} & (|i-j| = 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Note that T_w is in $\mathsf{2\text{-}scx}\Lambda$. On the other hand, for a two-term complex $T \in \mathsf{2\text{-}scx}\Lambda$, we can easily construct a signed walk w_T because $\mathrm{H}^0(T)$ is string. The following proposition plays important role in this report.

Proposition 11. [2, Lemma 4.3] *There are mutually inverse bijections*

$$\mathsf{SW}(G) \longleftrightarrow \mathsf{2\text{-}scx}\Lambda$$

given by $w \mapsto T_w$ *and* $T \to w_T$. *Moreover, the restrictions give mutually inverse bijections*

$$\mathsf{AW}(G) \longleftrightarrow \mathsf{2\text{-}ptilt}\Lambda.$$

Using the correspondences, we state our main result. A collection of admissible walks is *admissible* if any pair in the collection is non-crossing and satisfies the sign condition. Moreover, an admissible collection $\mathbb{W}$ called *complete* if any admissible collection containing $\mathbb{W}$ is $\mathbb{W}$ itself. We denote by $\mathsf{CW}(G)$ the set of all complete admissible collections of G.

Theorem 12. [2, Theorem 4.6] *The correspondences in Proposition 11 induce bijections*

$$\mathsf{CW}(G) \longleftrightarrow \mathsf{2\text{-}tilt}\Lambda.$$

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SURVEY ON REPRESENTATION THEORY OF QUIVER HECKE ALGEBRAS

SUSUMU ARIKI (有木　進)

Abstract. This survey article aims at non-experts. We introduce cyclotomic quiver Hecke algebras with focus on finite and affine Lie types. We begin with explaining how they naturally generalize the group algebra of the symmetric group and the finite Hecke algebra, by mentioning Brundan and Kleshchev's theorem. After giving the definition of the cyclotomic quiver Hecke algebra, we state the fundamental theorem by Kang and Kashiwara on the categorification of integrable highest weight modules, we consider finite Lie types and explain construction of irreducible modules by Benkart-Kang-Oh-Park, and standard and costandard modules by Syu Kato and Brundan-Kleshchev-McNamara. Finally, I briefly explain various Fock spaces, which appear in my series of papers with Euiyong Park.

1. Introduction

昔，Lascoux，Leclerc と Thibon の提案した対称群に付随するヘッケ代数の分解係数に関する予想というものがあり，予想を解決するにあたりヘッケ代数のブロック代数が $A_\ell^{(1)}$ 型リー代数の基本加群 $V(\Lambda_0)$ の重み空間の圏化を与えるというアイデアを用いた．その際併せて円分商という概念を導入し，$G(m,1,n)$ 型ヘッケ代数と呼ぶアフィンヘッケ代数の円分商を用いることで基本重み Λ_0 以外の支配的重み Λ に対する最高重み可積分加群 $V(\Lambda)$ も圏化した．$G(m,1,n)$ 型ヘッケ代数は同時期にリー型有限群の非等標数モジュラー表現の研究のため Broué や Malle が複素鏡映群の群代数の変形として導入した円分ヘッケ代数の例になっており，以後 $G(m,1,n)$ 型の円分ヘッケ代数の表現論の研究も欧米の一部研究者の興味を引くことになった．その研究者の中には，Khovanov や Rouquier といった強力な数学者も含まれていて，その後しばらくたって，任意の対称化可能カルタン行列 A に付随するリー代数 $\mathfrak{g}(A)$ の可積分加群 $V(\Lambda)$ をより精密にした量子普遍包絡代数 $U_q(\mathfrak{g}(A))$ の可積分加群 $V_q(\Lambda)$ の圏化のために Khovanov と Lauda により円分箙ヘッケ代数が導入された．Rouquier も独立にアフィン箙ヘッケ代数を導入したのでアフィン箙ヘッケ代数は KLR 代数，円分箙ヘッケ代数は円分 KLR 代数とも呼ばれる．以上の経緯でわかるように，円分箙ヘッケ代数は対称群の群代数やヘッケ代数の広範な一般化である．

この概説論文では，最初に従来対称群特有のヤング図形の言葉で記述されてきた対称群のモジュラー表現論がより一般的な設定でどう記述されるかを説明する．その後，Kang と Kashiwara による基本的な結果を述べる．後半では，有限型カルタン行列に付随するアフィン箙ヘッケ代数の既約表現の構成に関する結果や，Syu Kato の研究を嚆矢とし，Brundan，Kleshchev と McNamara により導入された標準加群・余標準加群を紹介する．

上で述べた $G(m,1,n)$ 型ヘッケ代数の研究では，可積分加群 $V(\Lambda)$ のフォック空間への埋め込みの圏化理論としての解釈が重要な役割を果たし，フォック空間自体ヘッケ代数の遺伝被覆である量子シューア代数を用いて圏化される．著者と Euiyong Park の研究では

The paper is in a final form and no version of it will be submitted for publication elsewhere.

アフィン型量子普遍包絡代数の基本加群 $V_q(\Lambda_0)$ から得られる円分箙ヘッケ代数を扱っており，従来ヘッケ代数の研究で使われてこなかった別種のフォック空間が現れる．最終節ではこのフォック空間を紹介する．

2. Lie theory for the symmetric group

まずリー理論における標準的な用語を復習しておく．

Definition 1. 整数成分行列 $A = (a_{ij})_{i,j\in I}$ が対称化可能一般カルタン行列とは，
(i) $a_{ii} = 2$.
(ii) $i \neq j$ ならば $a_{ij} \leq 0$.
(iii) $a_{ij} = 0$ は $a_{ji} = 0$ と同値.
(iv) 自然数成分対角行列 D が存在して DA は対称行列.
をみたすときをいう．

対称化可能一般カルタン行列 A が与えられると，$\mathbb{C}$ 上の Kac-Moody リー代数 $\mathfrak{g}(A)$ が定義される．$\mathfrak{g}(A)$ にはカルタン部分代数と呼ばれる可換リー部分代数 $\mathfrak{h}(A)$ が存在する．$\mathfrak{h}(A)^*$ を $\mathfrak{h}(A)$ の双対空間とする．

$\Pi = \{\alpha_i | i \in I\}$ と $\Pi^\vee = \{\alpha_i^\vee | i \in I\}$ を $\mathfrak{g}(A)$ のルートの集合と余ルートの集合とする．重み格子 $P \subseteq \mathfrak{h}(A)^*$ とその双対格子 $P^\vee \subseteq \mathfrak{h}(A)$ が存在し，Π は P，$\Pi^\vee$ は $P^\vee$ の一次独立な元の集合である．$\operatorname{rank} P = \dim \mathfrak{h} = 2|I| - \operatorname{rank}(A)$ かつ DA は P 上の二次形式 $(\ |\ )$ を定める．

ルートデータ $(A, \Pi, \Pi^\vee, P, P^\vee)$ に対し量子普遍包絡代数 $U_q(\mathfrak{g}(A))$ が定義される．$U_q(\mathfrak{g}(A))$ の生成元は Chevalley 生成元 e_i, f_i $(i \in I)$ と q^h $(h \in P^\vee)$ である．

Definition 2. $U_q(\mathfrak{g}(A))$-加群 V が可積分加群であるとは，
(i) V は重み分解をもつ．すなわち，

$$V = \bigoplus_{\mu\in P} V_\mu, \quad V_\mu = \{v \in V \mid q^h v = q^{\langle h,\mu\rangle} v \ (\forall h \in P^\vee)\}.$$

(ii) e_i, f_i $(i \in I)$ の作用は局所ベキ零，すなわち V の任意の元に対し作用はベキ零.
をみたすときをいう．$P^+ = \{\Lambda \in P \mid \langle \alpha_i^\vee, \Lambda\rangle \in \mathbb{Z}_{\geq 0}\ (\forall i \in I)\}$ とおくと最高重み加群 $V_q(\Lambda)$ は $\Lambda \in P^+$ のとき（またそのときに限り）可積分加群である．

可積分最高重み加群 $V_q(\Lambda)$ は対応する柏原結晶 $B(\Lambda)$ から復元でき，テンソル積加群の分解則や制限則などを準正規柏原結晶の言葉で記述できることが知られている．

Definition 3. A を対称化可能一般カルタン行列とし，$\mathfrak{g}(A), \mathfrak{h}(A), \Pi, \Pi^\vee, P, P^\vee, U_q(\mathfrak{g}(A))$ を定める．このとき，集合 B と関数 $\mathrm{wt} : B \to P$，$\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\}$ $(i \in I)$ の組 $(B, \mathrm{wt}, \{\tilde{e}_i, \tilde{f}_i\}_{i\in I})$ が準正規柏原結晶とは，

$$\epsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k b \in B\}, \ \ \varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k b \in B\}$$

と定めたとき，次の条件が成り立つときをいう．
(0) $\tilde{e}_i 0 = 0, \tilde{f}_i 0 = 0$.
(1) $b \in B, i \in I$ に対し，$\epsilon_i(b) < \infty$ かつ $\varphi_i(b) < \infty$.
(2) $b \in B, i \in I$ に対し，$\varphi_i(b) = \epsilon_i(b) + \langle \alpha_i^\vee, \mathrm{wt}(b)\rangle$.
(3) $b \in B$ かつ $\tilde{e}_i b \in B$ ならば，$\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$.
(4) $b, b' \in B$ に対し $b' = \tilde{f}_i b$ は $b = \tilde{e}_i b'$ と同値.

詳細は省くが，柏原結晶 $B(\Lambda)$ は可積分加群 $V_q(\Lambda)$ の結晶基底から構成され，いろいろな組合せ論的な実現をもつ．準正規でない柏原結晶もあり $B(\infty)$ が代表的である．

Example 4. $e \geq 2$ とし，カルタン行列 $A^{(1)}_{e-1}$ を考える．このとき，

$$P^\vee = \mathbb{Z}\alpha_0^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_{e-1}^\vee \oplus \mathbb{Z}d$$

と書けて，基本重み $\Lambda_0 \in P^+$ が $\langle \alpha_i^\vee, \Lambda_0 \rangle = \delta_{i0}, \langle d, \Lambda_0 \rangle = 0$ で定まる．このとき，$B(\Lambda_0)$ は次の e-制限的ヤング図形の集合上に実現される．

$$\{\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \mid 0 \leq \lambda_i - \lambda_{i+1} \leq e-1\}$$

これを Misra-Miwa model という．

他方，Littelmann path model という実現もあり，e-core，つまりこれ以上長さ e の hook を抜けないヤング図形，$\kappa^{(i)}$ の列と，0 から始まり 1 で終わる単調増加有理数列 a_i の組であって，ある条件をみたすものの集合

$$\{(\kappa^{(1)}, \ldots, \kappa^{(s)} : a_0, \ldots, a_s)\}$$

の上に実現できる．

Misra-Miwa model は対称群や対称群に付随するヘッケ代数のモジュラー表現論を研究する際に標準的に使われてきた．しかし，この分野の研究者は柏原結晶という概念を知らなかったので，他の実現を用いて定理を記述するということを思いもしなかった．以下では Mullineaux 写像を取り上げ，古典的な Misra-Miwa model での記述と Littelmann path model での記述を比較しよう．

対称群の群代数や対称群に付随するヘッケ代数は cellular 代数であり，James や Murphy の結果により既約加群は e-制限的ヤング図形でラベルされる．ただし，ヘッケ代数は構造定数に非零定数 q（対称群のときは $q=1$）を含んでおり，e は量子標数である．すなわち，

$$e = \min\{k \in \mathbb{Z}_{\geq 2} \mid 1 + q + \cdots + q^{k-1} = 0\}.$$

そこで既約加群の同型類の集合を $\{D^\lambda \mid \lambda : e\text{-制限的}\}$ で表わす．対称群は Coxeter 関係式をみたす生成元を持つが同様に対称群に付随するヘッケ代数も 2 次の関係式 $(T_i - q)(T_i + 1) = 0$ と braid 関係式をみたす生成元 T_i を持ち，さらに $\theta : T_i \mapsto -qT_i^{-1}$ が自己同型を定める．

Definition 5. e-制限的ヤング図形 λ に対し，D^λ を自己同型 θ でひねった表現加群を $D^\lambda \otimes \mathrm{sgn}$ と書くことにすると，e-制限的ヤング図形 $m(\lambda)$ が存在して $D^\lambda \otimes \mathrm{sgn} \simeq D^{m(\lambda)}$ となる．このとき，写像 $\lambda \mapsto m(\lambda)$ を Mullineaux 写像と呼ぶ．

Mullineaux 写像を具体的に記述する規則は Mullineaux 自身により予想され，最終的に Kleshchev と Ford が証明したのはよく知られているが，規則の記述は複雑である．他方，Littelmann path model で記述すると記述は劇的に簡略化される．すなわち，

$$(\kappa^{(1)}, \ldots, \kappa^{(s)} : a_0, \ldots, a_s)$$

において，各 $\kappa^{(i)}$ を転置すればよい．q が 1 のベキ根でなければヘッケ代数は半単純代数になり，Mullineaux 写像は単に λ を転置する写像になるから，Littelmann path model における規則は半単純代数の場合の自然な拡張になっている．

他にもモジュラー分岐則等，対称群や対称群に付随するヘッケ代数のモジュラー表現論における重要な定理が，柏原結晶や $\mathfrak{g}(A^{(1)}_{e-1})$ 上の可積分加群とそのフォック空間への埋め込みを用いた記述をもち，円分箙ヘッケ代数の場合に一般化される．

3. 円分箙ヘッケ代数

K を次数付可換環とするとき，円分箙ヘッケ代数 $R^\Lambda(n)$ が次数付 K-代数として定義されるが，現在表現論を展開できているのは K が体の場合であるので，以下では K が体の場合に定義を述べる．まず，$Q_{ij}(u,v) = Q_{ji}(v,u)$ をみたす次の形の多項式 $Q_{ij}(u,v)$ $(i,j \in I)$ を用意しておく．

$$Q_{ij}(u,v) = \begin{cases} \sum_{p(\alpha_i|\alpha_i)+q(\alpha_j|\alpha_j)+2(\alpha_i|\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

ただし，係数 $t_{i,j;p,q} \in K$ は $t_{i,j;-a_{ij},0} \neq 0$ と仮定する．

Example 6. たとえば，$t_{i,j;p,q} \neq 0$ かつ $t_{i,j;p,q} = t_{j,i;q,p}$ として

$$Q_{ij}(u,v) = \begin{cases} t_{i,j;0,0} & \text{if } a_{ij} = a_{ji} = 0, \\ t_{i,j;1,0}u + t_{i,j;0,1}v & \text{if } a_{ij} = a_{ji} = -1, \\ t_{i,j;2,0}u^2 + t_{i,j;0,1}v & \text{if } a_{ij} = -2, a_{ji} = -1, \\ t_{i,j;1,0}u + t_{i,j;0,2}v^2 & \text{if } a_{ij} = -1, a_{ji} = -2, \\ 0 & \text{if } i = j. \end{cases}$$

円分箙ヘッケ代数の定義はこの多項式に依存するのであるが，実は代数の構造に大きくは影響せず，たとえばカルタン行列 A の定める無向グラフ（ディンキン図形）が閉路をもたなければ定義から得られる代数の同型類は多項式 $Q_{ij}(u,v)$ の取り方によらない．

また，$\nu = (\nu_1, \ldots, \nu_n) \in I^n$ に対し対称群の作用を項の並べ替えで定義する．すなわち，

$$s_k(\nu_1, \ldots, , \nu_k, \nu_{k+1}, \ldots, \nu_n) = (\nu_1, \ldots, \nu_{k+1}, \nu_k, \ldots, \nu_n).$$

Definition 7. $\Lambda \in P^+$ に対し，円分箙ヘッケ代数 $R^\Lambda(n)$ とは，生成元

$$\{e(\nu) \mid \nu = (\nu_1, \ldots, \nu_n) \in I^n\},\ \{x_k \mid 1 \leq k \leq n\},\ \{\psi_k \mid 1 \leq k \leq n-1\}$$

と次の基本関係で定まる K-代数である．

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu),\ \sum_{\nu \in I^n} e(\nu) = 1,\ x_k e(\nu) = e(\nu)x_k,\ x_k x_l = x_l x_k,$$

$$\psi_k e(\nu) = e(s_k(\nu))\psi_k,\ \psi_k\psi_l = \psi_l\psi_k \text{ if } |k-l| > 1,$$

$$\psi_k^2 e(\nu) = Q_{\nu_k\,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\psi_k x_l - x_{s_k(l)}\psi_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\psi_{k+1}\psi_k\psi_{k+1} - \psi_k\psi_{k+1}\psi_k)e(\nu)$$
$$= \begin{cases} \dfrac{Q_{\nu_k\,\nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k\,\nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise,} \end{cases}$$

$$x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0.$$

$R^\Lambda(n)$ は次のように次数を定めることで次数付代数になる．

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}|\alpha_{\nu_k}), \quad \deg(\psi_k e(\nu)) = -(\alpha_{\nu_k}|\alpha_{\nu_{k+1}}).$$

Example 8. $A = A^{(1)}_{e-1}$ とする．$e = 2$ なら，一般性を失うことなく

$$Q_{01}(u,v) = Q_{10}(u,v) = u^2 + \lambda uv + v^2$$

とできる．ただし，$\lambda \in K$ である．$e \geq 3$ なら，一般性を失うことなく $\lambda \neq 0$ かつ

$$Q_{ij}(u,v) = \begin{cases} u+v & \text{if } j = i+1, 0 \leq i \leq e-2, \\ u+\lambda v & \text{if } i = e-1, j = 0, \\ 1 & \text{if } j \neq i \pm 1 \mod e, \\ 0 & \text{if } i = j. \end{cases}$$

と仮定してよい．$A^{(1)}_{e-1}$ 以外のアフィン型や有限型のカルタン行列のときは，$Q_{ij}(u,v)$ を媒介変数を含まない特定の形に固定してかまわない．

Definition 9. $\beta \in Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i \subseteq P$ に対し

$$I^\beta = \{\nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}$$

と置くと，$e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$ は $R^\Lambda(n)$ の中心元であり，$R^\Lambda(\beta) = R^\Lambda(n)e(\beta)$ は K-代数として $R^\Lambda(n)$ の直和因子になる．$R^\Lambda(\beta)$ も円分箙ヘッケ代数と呼ぶ．また，有限次元次数付 $R^\Lambda(\beta)$-加群のなす圏を $R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}}$ で表わし，$q^i : R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}} \to R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}}$ を各加群の次数付けを一斉に i 次増やす関手とする．

Definition 10. $e(\beta, i) = \sum_{\nu \in I^\beta} e(\nu, i)$ と置き，

$$\begin{aligned} E_i &= e(\beta,i)R^\Lambda(\beta+\alpha_i) \otimes_{R^\Lambda(\beta+\alpha_i)} - &&: R^\Lambda(\beta+\alpha_i)\text{-mod}^{\mathbb{Z}} \to R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}} \\ F_i &= R^\Lambda(\beta+\alpha_i)e(\beta,i) \otimes_{R^\Lambda(\beta)} - &&: R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}} \to R^\Lambda(\beta+\alpha_i)\text{-mod}^{\mathbb{Z}} \end{aligned}$$

をそれぞれ制限関手，誘導関手と呼ぶ．

Remark 11. Kashiwara の定理 [8] により制限関手は誘導関手の両側随伴関手である．

次の Kang と Kashiwara による定理 [7] が円分箙ヘッケ代数の表現論における基本定理である．

Theorem 12. A を対称化可能一般カルタン行列とし，A から定まる量子普遍包絡代数や円分箙ヘッケ代数に関する記号は上記のとおりとする．また，$l_i = \langle \alpha_i^\vee, \Lambda - \beta \rangle$ と置く．

(1) E_i と F_i はともに完全関手である．

(2) $l_i \geq 0$ ならば，関手の同型 $q^{-2d_i}F_iE_i \oplus (\oplus_{k=0}^{l_i-1} q^{2d_ik}) \simeq E_iF_i$ が成立．

(3) $l_i \leq 0$ ならば，関手の同型 $q^{-2d_i}F_iE_i \simeq E_iF_i \oplus (\oplus_{k=0}^{-l_i-1} q^{-2d_i(k+1)})$ が成立．

とくに，$R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}}$ の直和

$$\mathcal{V}(\Lambda) = \bigoplus_{\beta \in Q_+} R^\Lambda(\beta)\text{-mod}^{\mathbb{Z}}$$

と $\mathcal{V}(\Lambda)$ から $\mathcal{V}(\Lambda)$ 自身への関手 $q^{d_i(1-l_i)}E_i, F_i \ (i \in I)$ は $U_q(\mathfrak{g}(A))$-加群 $V_q(\Lambda)$ を圏化する．

$G(e,1,n)$ 型ヘッケ代数 $\mathcal{H}_n(q,\gamma_0,\ldots,\gamma_{e-1})$ を生成元が $T_0,T_1,\ldots,T_{n-1}$ で，関係式が

$$(T_0-q^{\gamma_0})\cdots(T_0-q^{\gamma_{e-1}})=0, \quad (T_i-q)(T_i+1)=0\ (1\le i\le n-1)$$

および B 型 braid 関係式で与えられる K-代数とする．次の Brundan と Kleshchev の定理 [4] により，対称群の群代数や対称群に付随するヘッケ代数は円分箙ヘッケ代数の特別な場合であることがわかる．

Theorem 13. $A=A^{(1)}_{e-1}$ とし，$Q_{ij}(u,v)=-(u-v)^{-a_{ij}}$ $(i\ne j)$ に選ぶ．e が K の標数で割り切れないとすると，$q=\sqrt[e]{1}$, $\langle\alpha_i^\vee,\Lambda\rangle=\gamma_i$ $(0\le i\le e-1)$ として，$G(e,1,n)$ 型ヘッケ代数 $\mathcal{H}_n(q,\gamma_0,\ldots,\gamma_{e-1})$ は円分箙ヘッケ代数 $R^\Lambda(n)$ に K-代数として同型である．

Remark 14. とくに $\Lambda=\Lambda_0$ の場合を有限箙ヘッケ代数と呼ぼう．この定理により，対称群の群代数 KS_n が次数付代数であることがわかり，KS_n の $A^{(1)}_{e-1}$ 型有限箙ヘッケ代数による変形族を考えることができる．

Remark 15. カルタン行列 A の定める無向グラフ（ディンキン図形）がグラフ自己同型 σ をもち，$\sigma(\Lambda)=\Lambda$ ならば，$Q_{ij}(u,v)=Q_{\sigma(i)\sigma(j)}(u,v)$ と定めた円分箙ヘッケ代数 $R^\Lambda(n)$ に対し，σ は $e(\nu)\mapsto e(\sigma\nu), x_k\mapsto x_k, \psi_k\mapsto\psi_k$ により自己同型を誘導する．Mullineaux 写像はその特別な場合であるから，Mullineaux と同じ問題をより一般のカルタン行列と柏原結晶の実現に対して考えることができる．言い換えれば，Mullineaux の問題は対称群のモジュラー表現論特有の問題ではないのである．

次の定理は最近 Webster のアイデアに基づいて Shan, Varagnolo, Vasserot により証明された．

Theorem 16. 円分箙ヘッケ代数 $R^\Lambda(n)$ は対称代数である．また，その非退化跡形式は，

- $\nu\ne\nu'$ ならば，$e(\nu)R^\Lambda(n)e(\nu')\to 0$.
- $\nu=\nu'$ ならば，余単位自然変換 $E_{\nu_1}\cdots E_{\nu_n}F_{\nu_n}\cdots F_{\nu_1}\to \mathrm{Id}$ を用いて
$$e(\nu)R^\Lambda(n)e(\nu)=E_{\nu_1}\cdots E_{\nu_n}F_{\nu_n}\cdots F_{\nu_1}R^\Lambda(0)\longrightarrow R^\Lambda(0)=K$$
を各 ν ごとに適切に定数倍．

として与えられる．

Remark 17. A を体 K 上の有限次元代数，$e\in A$ をベキ等元とし，

$$F=Ae\otimes_K -, \quad E=eA\otimes_A -$$

とすると，F は E の左随伴関手であるが，ここで F が E の右随伴関手でもあるとしよう．すると，余単位自然変換 $\epsilon: EF\to \mathrm{Id}_{K\text{-mod}}$ が線型形式 $t: eAe\to K$ を定める．そこで，$\eta:\mathrm{Id}_{A\text{-mod}}\to FE$ を単位自然変換として，$1_E=\epsilon E\circ E\eta$ を $E(A)\to E(A)$ に対し考えれば，$u_i\in Ae, v_i\in eA$ が存在して，任意の $u\in eA$ に対して $u=\sum t(uu_i)v_i$ である．とくに $t(uu_i)=0$ なら $u=0$ である．$A=R^\Lambda(n)$ ならば Theorem16 のように $t: A\to K$ を定義することで $t(ab)=t(ba)$ $(a,b\in A)$ が証明でき，t は $R^\Lambda(n)$ の非退化跡形式になる．

4. 有限型円分箙ヘッケ代数の既約加群の構成

本節では A を有限型カルタン行列と仮定し，Benkart, Kang, Oh, Park による既約 $R^\Lambda(\beta)$-加群の完全代表系の構成法 [3] を説明する．この構成法では A が定める有限ワイル群 W の最長元 w_0 の適切な最短表示をひとつ固定するが，簡単のため $A=A_{m-1}$，最短表示を

$$w_0=w^{(1)}\cdots w^{(m-1)}=(s_{m-1})(s_{m-2}s_{m-1})\cdots(s_1\cdots s_{m-1})=s_{i_1}\cdots s_{i_N}$$

と限定して説明する．$B(\Lambda)$ が柏原結晶であったことを思い出そう．

Definition 18. $b \in B(\Lambda)$ に対し，$a(b) = (a_1, \ldots, a_N) \in \mathbb{Z}_{\geq 0}^N$ が b の adapted string とは

$$a_1 = \epsilon_{i_1}(b),\ a_2 = \epsilon_{i_2}(\tilde{e}_{i_1}^{a_1} b),\ \cdots,\ a_N = \epsilon_{i_N}(\tilde{e}_{i_{N-1}}^{a_{N-1}} \cdots \tilde{e}_{i_1}^{a_1} b)$$

のときをいう．$S^\Lambda = \{a(b) \mid b \in B(\Lambda)\}$ を adapted string の集合とする．

Remark 19. $(a_1, \ldots, a_N) \mapsto \tilde{f}_{i_1}^{a_1} \cdots \tilde{f}_{i_N}^{a_N} b_\Lambda$ により全単射 $S^\Lambda \simeq B(\Lambda)$ が得られる．また，

$$\tilde{f}_{\mathbf{i}_1}^{\mathbf{a}_1} = \tilde{f}_{m-1}^{a_1},\ \tilde{f}_{\mathbf{i}_2}^{\mathbf{a}_2} = \tilde{f}_{m-2}^{a_2} \tilde{f}_{m-1}^{a_3},\ \tilde{f}_{\mathbf{i}_3}^{\mathbf{a}_3} = \tilde{f}_{m-3}^{a_4} \tilde{f}_{m-2}^{a_5} \tilde{f}_{m-1}^{a_6}, \ldots$$

とすると，最短表示の取り方 $(s_{m-1})(s_{m-2}s_{m-1}) \cdots (s_1 \cdots s_{m-1})$ は

$$\epsilon_i(\tilde{e}_{\mathbf{i}_k}^{\mathbf{a}_k} \cdots \tilde{e}_{\mathbf{i}_1}^{\mathbf{a}_1} b) = 0, \quad (m - k \leq i \leq m-1)$$

を保障する．つまり adapted string は Levi 部分代数の減少列に対する最高重み元の列を与えている．

Definition 20. 円分箙ヘッケ代数 $R^\Lambda(n)$ の定義関係式から $x_1^{\langle h_{\nu_1}, \Lambda \rangle} e(\nu) = 0$ を除いて定義した代数を $R(n)$ と書きアフィン箙ヘッケ代数と呼ぶ．$\beta \in Q_+$ に対し $e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$ とすれば $R(\beta) = R(n)e(\beta)$ も定義される．

$b \in B(\Lambda)$ に対し adapted string を $a(b) = (\mathbf{a}_1, \ldots, \mathbf{a}_{m-1})$ と書き，

$$\beta_k = a_{1+(k-1)k/2} \alpha_{m-k} + a_{2+(k-1)k/2} \alpha_{m-k+1} + \cdots + a_{k(k+1)/2} \alpha_{m-1}$$

とする．このとき，$R(\beta_k)$-加群 $N_k(b)$ を

$$N_k(b) = \tilde{f}_{\mathbf{i}_k}^{\mathbf{a}_k}(R(0))$$

と定める．ただし，$R(\beta)$-加群 M に対し，$\tilde{f}_i(M) = \mathrm{Top}(R(\beta + \alpha_i)e(\beta, i) \otimes_{R(\beta)} M)$ であり，$R(0) = K$ は自明な $R(0)$-加群である．また，$\beta = \sum_{k=1}^{m-1} \beta_k$ として，$R(\beta)$-加群

$$\mathrm{Ind}(\boxtimes_{k=1}^{m-1} N_k(b)) = R(\beta) \otimes_{R(\beta_1) \boxtimes \cdots \boxtimes R(\beta_{m-1})} N_1(b) \boxtimes \cdots \boxtimes N_{m-1}(b)$$

を定める．Benkart, Kang, Oh, Park は円分箙ヘッケ代数およびアフィン箙ヘッケ代数に対し，次のような既約加群の完全代表系を得た．

Theorem 21. 有限型カルタン行列から定まる円分箙ヘッケ代数 $R^\Lambda(\beta)$ に対し，

$$\{\mathrm{Top}(\mathrm{Ind}(\boxtimes_{k=1}^{m-1} N_k(b))) \mid b \in B(\Lambda),\ \mathrm{wt}(b) = \Lambda - \beta\}$$

は既約 $R^\Lambda(\beta)$-加群の完全代表系である．

Theorem 22. 有限型カルタン行列から定まる箙ヘッケ代数 $R(\beta)$ に対し，

$$\{\mathrm{Top}(\mathrm{Ind}(\boxtimes_{k=1}^{m-1} N_k(b))) \mid b \in B(\infty),\ \mathrm{wt}(b) = -\beta\}$$

は既約 $R(\beta)$-加群の完全代表系である．

次節では，加藤周のコストカ系の理論 [9] に触発された McNamara による別の既約加群の構成法 [10] を紹介する．

5. 標準加群と余標準加群

本節でも A を有限型カルタン行列と仮定するが，最長元の最短表示 $w_0 = s_{i_1} \cdots s_{i_N}$ は任意でよい．まず正ルート系の集合 Δ^+ に全順序を以下のように定める．

$$\beta_1 = \alpha_{i_1} < \beta_2 = s_{i_1}\alpha_{i_2} < \cdots < \beta_N = s_{i_1} \cdots s_{i_{N-1}}\alpha_{i_N}.$$

この順序は凸順序である．すなわち，$\beta, \gamma, \beta+\gamma \in \Delta^+$ かつ $\beta < \gamma$ ならば $\beta < \beta+\gamma < \gamma$ が成り立つ．

Definition 23. 量子普遍包絡代数 $U_q(\mathfrak{g}(A))$ の自己同型 T_i が

$$T_i(f_j) = \sum_{r+s=-a_{ij}} (-1)^r q^{d_i r} f_i^{(r)} f_j f_i^{(s)} \quad (j \neq i)$$

等により定義される．そこで，$E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(f_{i_k})$ と定めると

$$\{E_{\beta_N}^{(c_N)} \cdots E_{\beta_1}^{(c_1)} \mid c_1, \ldots, c_N \in \mathbb{Z}\}$$

は量子普遍包絡代数の基底になる．この基底を PBW 基底と呼ぶ．

$R(\alpha_i) = F[x_1]$ は唯一の既約加群 $L(\alpha_i)$ を持つ．また，正ルート $\beta \in \Delta^+$ に対し，$w \in W$ と $i \in I$ が存在して $\beta = w\alpha_i$ と書けることと Chuang と Rouquier による導来圏同値を使えば，$R(\beta)$ も唯一の既約加群 $L(\beta)$ を持つことがわかる．$\{L(\beta) \mid \beta \in \Delta^+\}$ をカスピダル加群と呼ぶ．

Definition 24. 次で定義された加群を固有標準加群と呼ぶ．

$$\overline{\Delta}(c_1, \ldots, c_N) = \mathrm{Ind}\left(L(\beta_N)^{\boxtimes c_N} \boxtimes \cdots \boxtimes L(\beta_1)^{\boxtimes c_1}\right)$$

また，その双対加群を $\overline{\nabla}(c_1, \ldots, c_N)$ で表わし，固有余標準加群と呼ぶ．

Remark 25. 箙ヘッケ代数が量子普遍包絡代数の負ルート部分の圏化をしているという Khovanov と Lauda の描像では，固有標準加群は PBW 基底の双対基底と対応している．

Theorem 26. $L(c_1, \ldots, c_N) = \mathrm{Top}\,\overline{\Delta}(c_1, \ldots, c_N)$ と置くと，

$$\{L(c_1, \ldots, c_N) \mid c_1, \ldots, c_N \in \mathbb{Z}_{\geq 0},\ \sum_{k=1}^{N} c_k\beta_k = \beta\}$$

は既約 $R(\beta)$-加群の完全代表系であり，次数付加群とみると自己双対的である．

[6] において，Brundan, Kleshchev, McNamara は次の完全列をみたす次数付直既約 $R(\alpha)$-加群 $\Delta_n(\alpha)$ の存在を示した．

$$0 \to q^{(n-1)(\alpha|\alpha)}L(\alpha) \to \Delta_n(\alpha) \to \Delta_{n-1}(\alpha) \to 0$$

$$0 \to q^{(\alpha|\alpha)}\Delta_{n-1}(\alpha) \to \Delta_n(\alpha) \to L(\alpha) \to 0.$$

さらに，次が成り立つ．

$$\dim \mathrm{Ext}^1_{R(\alpha)}(\Delta_n(\alpha), L(\alpha)) = 1, \quad \dim \mathrm{Ext}^d_{R(\alpha)}(\Delta_n(\alpha), L(\alpha)) = 0\ (d \geq 2).$$

$\Delta_n(\alpha)$ の射影極限を $\Delta(\alpha)$ で表わし，ルート加群と呼ぶ．$\mathrm{End}_{R(c_k\beta_k)}(\mathrm{Ind}\,\Delta(\beta_k)^{\boxtimes c_k})$ を考えると，階数 c_k のベキ零ヘッケ代数に同型になり，その結果 $\mathrm{Ind}\,\Delta(\beta_k)^{\boxtimes c_k}$ の直和因子として直既約加群 $\Delta(c_k\beta_k)$ が定義される．$\Delta(c_1, \ldots, c_N) = \mathrm{Ind}\,\Delta(c_N\beta_N) \boxtimes \cdots \boxtimes \Delta(c_1\beta_1)$ を標準加群と呼ぶ．

Remark 27. $(c'_1,\dots,c'_N) < (c_1,\dots,c_N)$ を $c'_N = c_N,\dots,c'_{k+1} = c_{k+1}, c'_k < c_k$ により定める．$L(c_1,\dots,c_N)$ の射影被覆を $P(c_1,\dots,c_N)$ とすると，$P(c_1,\dots,c_N)$ の部分加群

$$\sum_{(c'_1,\dots,c'_N)\not\leq(c_1,\dots,c_N)} \sum_{f\in\mathrm{Hom}(P(c'_1,\dots,c'_N),P(c_1,\dots,c_N))} \mathrm{Im}\, f$$

による商加群は $\Delta(c_1,\dots,c_N)$ に同型である．

次の定理は Brundan, Kleshchev, McNamara および Syu Kato による．

Theorem 28.
(1) $\mathrm{Ext}^d_{R(\beta)}(\Delta(c_1,\dots,c_N), \overline{\nabla}(c'_1,\dots,c'_N))$ は，$d=0$ かつ $(c_1,\dots,c_N) = (c'_1,\dots,c'_N)$ のときのみ 1 次元で，それ以外のときは 0 である．
(2) 有限生成 $R(\beta)$-加群 M が $\sum_{k=1}^N c_k\beta_k = \beta$ をみたす任意の $(c_1,\dots,c_N)$ に対して

$$\mathrm{Ext}^1_{R(\beta)}(M, \overline{\nabla}(c_1,\dots,c_N)) = 0$$

をみたすならば M は Δ-*flag* をもつ．

6. 有限簾ヘッケ代数（$\Lambda=\Lambda_0$ の円分簾ヘッケ代数）とフォック空間

$A^{(1)}_\ell$ 型の円分簾ヘッケ代数 $R^{\Lambda_0}(n)$ が対称群の群代数や対称群に付随するヘッケ代数を含む場合であって，色付きヤング図形を基底とするフォック空間を用いる議論により，分解係数の決定，既約加群の柏原結晶による分類，Dipper-James-Murphy 予想の証明等，種々の議論を走らせる場となってきた．たとえば [5] や [11] を参照せよ．Euiyong Park との共同研究では，他のフォック空間を利用することにより，両端に 2 重線をもつアフィン型 $A^{(2)}_{2\ell}, D^{(2)}_{\ell+1}, C^{(1)}_\ell$ に対し円分簾ヘッケ代数 $R^{\Lambda_0}(\beta)$ の次元の明示公式の導出や表現型の決定などを行った．(手法は同じなので参考文献としては [1]，[2] の 2 編のみを挙げてある．)

$$A^{(2)}_{2\ell} : \circ \Leftarrow \cdots \Leftarrow \circ \qquad D^{(2)}_{\ell+1} : \circ \Leftarrow \cdots \Rightarrow \circ$$

$$C^{(1)}_\ell : \circ \Rightarrow \cdots \Leftarrow \circ$$

対称群に付随するヘッケ代数の場合はフォック空間全体が量子シューア代数で圏化されるが，それ以外の場合についてはまだあまり事情がわかっていないので，他のアフィン型でどういうフォック空間が現れるかだけを紹介しておく．まず $A^{(1)}_\ell$ 型の場合を復習すると，次の residue pattern が定める色付きヤング図形が基底となる．見てわかるように対角線には同じ数が並ぶ.

$$\begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & \cdots & \ell & 0 & 1 & \cdots \\ \ell & 0 & 1 & 2 & \cdots & \cdots & \ell & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & & & & \end{array}$$

Example 29. $\ell \geq 3$ のとき，1 行めの長さが 3 で 2 行めの長さが 1 の色付きヤング図形は

$$\begin{array}{ccc} 0 & 1 & 2 \\ \ell & & \end{array}$$

$C_\ell^{(1)}$ 型のときは，やはり色付きヤング図形を基底とするフォック空間を考える．ただし，residue pattern が対角線に関して線対称な形に変わる．

$$\begin{array}{cccccccccc}
0 & 1 & 2 & \cdots & \cdots & \ell & 0 & 1 & \cdots & \cdots \\
1 & 0 & 1 & 2 & \cdots & \cdots & \ell & 0 & 1 & \cdots \\
2 & 1 & \ddots & \ddots & \ddots & & & \ddots & \ddots & \ddots \\
\vdots & \vdots & & & & & & & &
\end{array}$$

$A_{2\ell}^{(2)}$ 型と $D_{\ell+1}^{(2)}$ 型の場合は色付きずらしヤング図形を基底とするフォック空間を考える．ただし，$A_{2\ell}^{(2)}$ 型のときの residue pattern は

$$\begin{array}{ccccccccccc}
0 & 1 & \cdots & \ell & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots \\
 & 0 & 1 & \cdots & \ell & \cdots & 1 & 0 & 0 & 1 & \cdots \\
 & & \ddots & \ddots & & \ddots & & \ddots & \ddots & \ddots & \ddots
\end{array}$$

であり，$D_{\ell+1}^{(2)}$ 型のときの residue pattern は

$$\begin{array}{cccccccccccc}
0 & 1 & \cdots & \ell & \ell & \cdots & 1 & 0 & 0 & 1 & \cdots & \cdots \\
 & 0 & 1 & \cdots & \ell & \ell & \cdots & 1 & 0 & 0 & 1 & \cdots \\
 & & \ddots & \ddots & & \ddots & \ddots & & \ddots & \ddots & \ddots & \ddots
\end{array}$$

である．

Example 30. $A_4^{(2)}$ 型のとき，たとえば下記は色付きずらしヤング図形である．

$$\begin{array}{cccccc}
0 & 1 & 2 & 1 & 0 & 0 \\
 & 0 & 1 & & &
\end{array}$$

ヘッケ代数のときの理論展開を思い出せば，これらのフォック空間の正しい圏化は何かや，Specht 加群理論の構築などが自然な未解決問題として浮上してくる．また他方で，順表現型の計算例からは次の予想も自然である．

Conjecture 31. $R^\Lambda(\beta)$ は直既約代数であり，順標準型なら *Brauer graph* 代数であろう．

Remark 32. 対称群の群代数やヘッケ代数の場合は，Scopes 同値により予想が正しいことがわかる．

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THE GROTHENDIECK GROUPS OF MESH ALGEBRAS

SOTA ASAI

Abstract. This note is written on the Grothendieck groups of the stable categories of finite-dimensional mesh algebras.

1. Introduction

This note is the collection of the results of our calculations of the Grothendieck groups of the stable categories of finite-dimensional mesh algebras.

The concepts of mesh algebras and mesh categories are proposed by Riedtmann, and important because many derived categories are recovered from the mesh categories of their Auslander-Reiten quivers. For example, if Γ is the path algebra of a quiver with its underlying graph a Dynkin diagram Δ, then $D^{\mathrm{b}}(\operatorname{mod}\Gamma)$ is recovered from the mesh category of its AR quiver $\boldsymbol{Z}\Delta$ [3].

Some of the results in this note have been obtained in [1], but this note is based on different methods from the ones in [1]. The detail of the new methods and the calculations will be submitted later.

1.1. **Conventions.** In this note, let K be a field and Λ be a finite-dimensional self-injective K-algebra. $\operatorname{mod}\Lambda$ denotes the category of finitely generated right Λ-modules. $\operatorname{proj}\Lambda$ is the fullsubcategory of $\operatorname{mod}\Lambda$ consisting of all projective Λ-modules, and $\underline{\operatorname{mod}}\Lambda = \operatorname{mod}\Lambda/\operatorname{proj}\Lambda$ is the *stable category* of $\operatorname{mod}\Lambda$. Because Λ is self-injective, $\operatorname{mod}\Lambda$ is an abelian Frobenius category and $\underline{\operatorname{mod}}\Lambda$ has a structure of a triangulated category [3]. The unit 1_Λ is decomposed into primitive orthogonal idempotents $e_1 + \cdots + e_m$. In this case, we put $P_i = e_i\Lambda$, $I_i = \operatorname{Hom}_K(\Lambda e_i, K)$, and $S_i = \operatorname{top} P_i = \operatorname{soc} I_i$. We define *Nakayama permutation* ν as $P_i \cong I_{\nu(i)}$.

2. Preliminary

First, we recall basic properties on Grothendieck groups and mesh algebras. We can refer to [3] for the detail.

Definition 1. Let $\mathcal{C}$ be a triangulated category.

The *Grothendieck group* $K_0(\mathcal{C})$ is defined with its generators all isomorphic classes in $\mathcal{C}$ and its relations $[X] - [Y] + [Z] = 0$ for each triangle $X \to Y \to Z \to X[1]$.

We have the following important proposition to calculate the Grothendieck group of the stable category $\underline{\operatorname{mod}}\Lambda$. The latter part of (2) is deduced by Rickard's famous triangle equivalence $\underline{\operatorname{mod}}\Lambda \cong D^{\mathrm{b}}(\operatorname{mod}\Lambda)/K^{\mathrm{b}}(\operatorname{proj}\Lambda)$ [4, Theorem 2.1].

Proposition 2. *Let Λ be a finite-dimensional self-injective K-algebra.*

The detailed version of this paper will be submitted for publication elsewhere.

(1) [3, III.1.2] *All isomorphic classes of simple Λ-modules $[S_1], \dots, [S_m]$ form a basis of the Grothendieck group of the derived category $K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$.*

(2) *The natural embedding $K^{\mathrm{b}}(\mathrm{proj}\,\Lambda) \to D^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$ canonically induces a morphism $K_0(K^{\mathrm{b}}(\mathrm{proj}\,\Lambda)) \to K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$, and its cokernel is isomorphic to $K_0(\underline{\mathrm{mod}}\,\Lambda)$.*

Definition 3. Let $Q = (Q_0, Q_1)$ be a locally finite quiver and τ be an automorphism on Q_0. We call the pair $Q = (Q, \tau)$ a *stable translation quiver* if the number of arrows from x to y coincide with the one from y to $\tau^{-1}x$ for $x, y \in Q_0$.

It will be seen that a stable translation quiver with multiple arrows does not give a finite-dimensional mesh algebra from Rickard's structure theorem. Thus, in this note, we assume any stable translation quivers do not contain multiple arrows for the convinience.

Definition 4. Let Q be a stable translation quiver.

For a vertex $a \in Q_0$, we denote by a^+ the set of targets of arrows from a^+.

Let $b_1, \dots, b_m$ be all distinct elements of a^+. Then the full subquiver

$$\begin{tikzcd}[row sep=small]
 & b_1 \arrow[dr, "\beta_1"] & \\
a \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_m"'] & \vdots & \tau^{-1}a \\
 & b_m \arrow[ur, "\beta_m"'] &
\end{tikzcd}$$

is called a *mesh* and the corresponding *mesh relation* is $\alpha_1\beta_1 + \cdots + \alpha_m\beta_m = 0$.

We define the *mesh algebra* of Q as the quotient of the path algebra of Q by all mesh relations in Q.

The following example introduces an important way to construct a translation quiver.

Example 5. Let Q be a finite quiver. We define the quiver $\boldsymbol{Z}Q = ((\boldsymbol{Z}Q)_0, (\boldsymbol{Z}Q)_1)$ as follows; the vertices are the elements of $(\boldsymbol{Z}Q)_0 = Q_0 \times \boldsymbol{Z}$, the arrows are the elements of $(\boldsymbol{Z}Q)_1 = \{(i, a) \to (j, a) \mid (i \to j) \in Q_1, a \in \boldsymbol{Z}\} \amalg \{(j, a) \to (i, a+1) \mid (i \to j) \in Q_1, a \in \boldsymbol{Z}\}$, and the translation is given by $\tau(i, a) = (i, a-1)$. Then $\boldsymbol{Z}Q$ is a stable translation quiver.

Remark 6. If the underlying graph of Q is a Dynkin diagram Δ, the translation quiver $\boldsymbol{Z}Q$ does not depend on the orientations of Q up to isomorphism, thus we set $\boldsymbol{Z}\Delta = \boldsymbol{Z}Q$.

Example 7. Let A_4 be oriented as $1 \to 2 \to 3 \to 4$. Then $\boldsymbol{Z}A_4$ is the following quiver

$$\begin{tikzcd}[column sep=tiny]
 & (4,-2) \arrow[dr] & & (4,-1) \arrow[dr] & & (4,0) \arrow[dr] & & (4,1) \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr] & & (3,-1) \arrow[ur] \arrow[dr] & & (3,0) \arrow[ur] \arrow[dr] & & (3,1) \arrow[ur] \arrow[dr] & & \cdots \\
 & (2,-1) \arrow[ur] \arrow[dr] & & (2,0) \arrow[ur] \arrow[dr] & & (2,1) \arrow[ur] \arrow[dr] & & (2,2) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] & & (1,0) \arrow[ur] & & (1,1) \arrow[ur] & & (1,2) \arrow[ur] & & \cdots
\end{tikzcd}$$

with its translation $\tau(i, a) = (i, a-1)$.

Considering mesh relations, the following paths are the longest nonzero paths in $\boldsymbol{Z}A_4$;

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & (4,a) \\
 & & (3,a) \arrow[ur] & \\
 & (2,a) \arrow[ur] & & \\
(1,a) \arrow[ur] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
 & & (4,a) \arrow[dr] & \\
 & (3,a) \arrow[ur] \arrow[dr] & & (3,a+1) \\
(2,a) \arrow[ur] \arrow[dr] & & (2,a+1) \arrow[ur] & \\
 & (1,a+1) \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & (4,a) \arrow[dr] & & \\
(3,a) \arrow[ur] \arrow[dr] & & (3,a+1) \arrow[dr] & \\
 & (2,a+1) \arrow[ur] \arrow[dr] & & (2,a+2) \\
 & & (1,a+2) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small, row sep=small]
(4,a) \arrow[dr] & & & \\
 & (3,a) \arrow[dr] & & \\
 & & (2,a) \arrow[dr] & \\
 & & & (1,a)
\end{tikzcd} .$$

The second figure means all paths from $(2,a)$ to $(3,a+1)$ are the longest nonzero paths. We can see that any of the longest nonzero paths from (i,a) ends at $(5-i, a+i-1)$.

To get a finite-dimensional mesh algebra, we take the quotient of $\boldsymbol{Z}A_4$ by an automorphism τ^3. Then the quiver $\boldsymbol{Z}A_4/\langle\tau^3\rangle$ is the following quiver

$$\begin{tikzcd}[column sep=small, row sep=small]
(4,\bar{1}) \arrow[dr] & & (4,\bar{2}) \arrow[dr] & & (4,\bar{0}) \arrow[dr] & & (4,\bar{1}) \\
 & (3,\bar{2}) \arrow[ur] \arrow[dr] & & (3,\bar{0}) \arrow[ur] \arrow[dr] & & (3,\bar{1}) \arrow[ur] \arrow[dr] & \\
(2,\bar{2}) \arrow[ur] \arrow[dr] & & (2,\bar{0}) \arrow[ur] \arrow[dr] & & (2,\bar{1}) \arrow[ur] \arrow[dr] & & (2,\bar{2}) \\
 & (1,\bar{0}) \arrow[ur] & & (1,\bar{1}) \arrow[ur] & & (1,\bar{2}) \arrow[ur] &
\end{tikzcd}$$

with its relation $\tau(i, a+k\boldsymbol{Z}) = (i, a-1+k\boldsymbol{Z})$. This quiver looks like a cylinder. From the discussion on the longest nonzero paths, we have $\nu(i, a+k\boldsymbol{Z}) = (5-i, a+i-1+k\boldsymbol{Z})$.

We can deduce the following lemma similarly as above.

Lemma 8. *A translation quiver $\boldsymbol{Z}A_n/\langle\tau^k\rangle$ gives a finite-dimensional mesh algebra for integers $n, k \geq 1$. The Nakayama permutation of this mesh algebra is given by $\nu(i, a + k\boldsymbol{Z}) = (n+1-i, a+i-1+k\boldsymbol{Z})$.*

Actually, it is rare for mesh algebras to be finite-dimensional. This is stated in Riedtmann's structure theorem.

Theorem 9. [5] *If a stable translation quiver gives a finite-dimensional mesh algebra, then it has a form of $\boldsymbol{Z}\Delta/G$, where Δ is a Dynkin diagram, and G is an admissible subgroup of* Aut $\boldsymbol{Z}\Delta$. *Namely, it is isomorphic to one of the following translation quivers;*

$\boldsymbol{Z}A_n/\langle\tau^k\rangle$, $\boldsymbol{Z}A_n/\langle\tau^k\psi\rangle$ *with* n *odd*, $\boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle$ *with* n *even*,
$\boldsymbol{Z}D_n/\langle\tau^k\rangle$, $\boldsymbol{Z}D_n/\langle\tau^k\psi\rangle$, $\boldsymbol{Z}D_4/\langle\tau^k\chi\rangle$,
$\boldsymbol{Z}E_6/\langle\tau^k\rangle$, $\boldsymbol{Z}E_6/\langle\tau^k\psi\rangle$, $\boldsymbol{Z}E_7/\langle\tau^k\rangle$, $\boldsymbol{Z}E_8/\langle\tau^k\rangle$;
where ψ,χ,φ *are automorphisms on* $\boldsymbol{Z}\Delta$ *satisfying* $\psi^2=\mathrm{id}$, $\chi^3=\mathrm{id}$, *and* $\varphi^2=\tau^{-1}$.

It is well-known that all finite-dimensional mesh algebras are self-injective.

3. Results

In the previous section, all finite-dimensional mesh algebras are obtained. We can state the following main theorem on the Grothendieck groups of finite-dimensional mesh algebras. This is the collection of our main results.

Theorem 10. *Let* $Q=\boldsymbol{Z}\Delta/G$ *be a stable translation quiver giving a finite-dimensional mesh algebra* Λ. *Then the Grothendieck group* $K_0(\underline{\mathrm{mod}}\,\Lambda)$ *is isomorphic to the following, where* c *be the Coxeter number of* Δ,

$$d=\begin{cases}\gcd(c,2k-1)/2 & (\boldsymbol{Z}\Delta/G=\boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle)\\ \gcd(c,k) & (\text{otherwise})\end{cases}$$

and $r=c/d$;

$$Q=\boldsymbol{Z}A_n/\langle\tau^k\rangle\Rightarrow\begin{cases}\boldsymbol{Z}^{(nd-3d+2)/2}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{d-1} & (r\in 2\boldsymbol{Z})\\ \boldsymbol{Z}^{(nd-2d+2)/2} & (r\notin 2\boldsymbol{Z})\end{cases},$$

$$\begin{matrix}Q=\boldsymbol{Z}A_n/\langle\tau^k\psi\rangle\\ (n\notin 2\boldsymbol{Z})\end{matrix}\Rightarrow\begin{cases}\boldsymbol{Z}^{(nd-3d)/2}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{d-1}\oplus(\boldsymbol{Z}/4\boldsymbol{Z}) & (r\in 4\boldsymbol{Z})\\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-2d+1} & (r\in 2+4\boldsymbol{Z})\\ \boldsymbol{Z}^{(nd-d)/4} & (r\notin 2\boldsymbol{Z})\end{cases},$$

$$\begin{matrix}Q=\boldsymbol{Z}A_n/\langle\tau^k\varphi\rangle\\ (n\in 2\boldsymbol{Z})\end{matrix}\Rightarrow(\boldsymbol{Z}/2\boldsymbol{Z})^{nd-2d+1},$$

$$Q=\boldsymbol{Z}D_n/\langle\tau^k\rangle\Rightarrow\begin{cases}\boldsymbol{Z}^{d-1}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d}\oplus(\boldsymbol{Z}/r\boldsymbol{Z}) & (k\in 2\boldsymbol{Z},\ r\in 2\boldsymbol{Z})\\ \boldsymbol{Z}^{(nd-d-2)/2}\oplus(\boldsymbol{Z}/r\boldsymbol{Z}) & (k\in 2\boldsymbol{Z},\ r\notin 2\boldsymbol{Z})\\ \boldsymbol{Z}^{d}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} & (k\notin 2\boldsymbol{Z},\ r\in 4\boldsymbol{Z})\\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-d-1} & (k\notin 2\boldsymbol{Z},\ r\notin 4\boldsymbol{Z})\end{cases},$$

$$Q=\boldsymbol{Z}D_n/\langle\tau^k\psi\rangle\Rightarrow\begin{cases}\boldsymbol{Z}^{d}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d} & (k\in 2\boldsymbol{Z},\ r\in 4\boldsymbol{Z})\\ (\boldsymbol{Z}/2\boldsymbol{Z})^{nd-d-1} & (k\in 2\boldsymbol{Z},\ r\in 2+4\boldsymbol{Z})\\ \boldsymbol{Z}^{(nd-2d)/2} & (k\in 2\boldsymbol{Z},\ r\notin 2\boldsymbol{Z})\\ \boldsymbol{Z}^{d-1}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{nd-3d}\oplus(\boldsymbol{Z}/r\boldsymbol{Z}) & (k\notin 2\boldsymbol{Z})\end{cases},$$

$$Q=\boldsymbol{Z}D_4/\langle\tau^k\chi\rangle\Rightarrow\begin{cases}\boldsymbol{Z}^4 & (k\in 2\boldsymbol{Z})\\ (\boldsymbol{Z}/2\boldsymbol{Z})^4 & (k\notin 2\boldsymbol{Z})\end{cases},$$

$$Q=\boldsymbol{Z}E_6/\langle\tau^k\rangle\Rightarrow\begin{cases}\boldsymbol{Z}^{d+1}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{d+1}\oplus(\boldsymbol{Z}/4\boldsymbol{Z})^{d-1} & (d=1,3)\\ \boldsymbol{Z}^{(3d+2)/2}\oplus(\boldsymbol{Z}/2\boldsymbol{Z})^{(3d+2)/2} & (d=2,6)\\ \boldsymbol{Z}^{(9d+12)/4} & (d=4,12)\end{cases},$$

$$Q = \boldsymbol{Z}E_6/\langle \tau^k\psi\rangle \Rightarrow \begin{cases} \boldsymbol{Z}^{2d} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d+1} & (d=1,3) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{(9d+6)/2} & (d=2,6) \\ \boldsymbol{Z}^{(3d+4)/2} & (d=4,12) \end{cases},$$

$$Q = \boldsymbol{Z}E_7/\langle \tau^k\rangle \Rightarrow \begin{cases} (\boldsymbol{Z}/2\boldsymbol{Z})^{6} & (d=1) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{6d+2} & (d=3,9) \\ \boldsymbol{Z}^{6} \oplus (\boldsymbol{Z}/3\boldsymbol{Z}) & (d=2) \\ \boldsymbol{Z}^{3d+2} & (d=6,18) \end{cases},$$

$$Q = \boldsymbol{Z}E_8/\langle \tau^k\rangle \Rightarrow \begin{cases} (\boldsymbol{Z}/2\boldsymbol{Z})^{8d} & (d=1,3,5) \\ (\boldsymbol{Z}/2\boldsymbol{Z})^{112} & (d=15) \\ \boldsymbol{Z}^{4d} & (d=2,6,10) \\ \boldsymbol{Z}^{112} & (d=30) \end{cases}.$$

4. Proof for $\boldsymbol{Z}A_n/\langle \tau^k\rangle$

In the rest of this note, we prove the main theorem for $\boldsymbol{Z}A_n/\langle \tau^k\rangle$. We orient A_n as $1 \to 2 \to \cdots \to n$, and set $Q = \boldsymbol{Z}A_n/\langle \tau^k\rangle$. The vertices of Q are the elements of $\{1,\dots,n\} \times (\boldsymbol{Z}/k\boldsymbol{Z})$. The following proposition is crucial to prove the theorem.

Proposition 11. *Let three abelian subgroups $H, H', H'' \subset K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$ be*

$$H = \langle [P_x] \mid x \in Q_0\rangle, \quad H' = \langle [S_x] + [S_{\nu\tau^{-1}x}] \mid x \in Q_0\rangle,$$
$$H'' = \langle [P_x] \mid x \in \{1\} \times (\boldsymbol{Z}/k\boldsymbol{Z})\rangle \subset H.$$

Then we have $H = H' + H''$ and thus $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))/(H' + H'')$.

Proof. Let $x \in Q_0$. A projective resolution of Λ-module S_x has a form of

$$0 \to S_{\nu\tau^{-1}x} \to P_{\tau^{-1}x} \to \bigoplus_{y \in x^+} P_y \to P_x \to S_x \to 0.$$

This is induced by a projective resolution of Λ as Λ-Λ-bimodule given by [2, (4.1)–(4.3), Corollary 4.3].

Now we prove $H' + H'' \subset H$. $H'' \subset H$ is clear. $H' \subset H$ holds because the above projective resolution implies

$$[S_x] + [S_{\nu\tau^{-1}x}] = [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x] \in H.$$

We have $H' + H'' \subset H$.

The remained task is to prove $H \subset H' + H''$. We assume $k = 1$ and $Q_0 = \{1,\dots,n\}$ first. It is enough to show $[P_i] \in H' + H''$. We prove this by induction on i. If $i = 1$, then $[P_1] \in H''$. If $i = 2,\dots,n$, put $x = i-1$. The projective resolution of Λ-module S_x implies

$$\begin{aligned}[S_x] + [S_{\nu\tau^{-1}x}] &= [P_{\tau^{-1}x}] - \sum_{y \in x^+} [P_y] + [P_x] \\ &= [P_{i-1}] - ([P_{i-2}] + [P_i]) + [P_{i-1}],\end{aligned}$$

where we set $P_0 = 0$. Therefore, we have

$$[P_i] = -([S_x] + [S_{\nu\tau^{-1}x}]) + [P_{i-1}] - [P_{i-2}] + [P_{i-1}].$$

From the induction hypothesis, we have $[P_{i-1}]-[P_{i-2}]+[P_{i-1}] \in H'+H''$, and by definition, we have $[S_x] + [S_{\nu\tau^{-1}x}] \in H'$. Now, $[P_i] \in H' + H''$ is proved. The induction has been completed. A similar proof holds even if $k \neq 1$. We have $H = H' + H''$.

The latter assertion is proved by Proposition 2. $\square$

Now our task is moved to express the generators of H' and H'' as linear combinations of the images of simple Λ-modules. For this purpose, we define some matrices.

Definition 12. We define three matrices.

(1) $X_k \in \mathrm{GL}_k(\boldsymbol{Z})$ as the permutation matrix of a cyclic permutation $(1, 2, \ldots, k)$.
(2) $T_n(x) \in \mathrm{Mat}_{n,n}(\boldsymbol{Z}[x])$, $U_n(x) \in \mathrm{Mat}_{n,1}(\boldsymbol{Z}[x])$ as

$$T_n(x) = \begin{pmatrix} & & & x^n \\ & & \cdots & \\ & x^2 & & \\ x & & & \end{pmatrix}, \quad U_n(x) = \begin{pmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{pmatrix}.$$

For example,

$$X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Using these matrices, the Grothendieck group is written in the following way.

Lemma 13. *We have* $K_0(\underline{\mathrm{mod}}\,\Lambda) \cong \mathrm{Cok}\begin{pmatrix}1_{nk} + T_n(X_k) & U_n(X_k)\end{pmatrix}$.

Proof. For $i \in \{1, \ldots, n\}$ and $a \in \{0, \ldots, k-1\}$, we let the $(i-1)k + (a+1)$th row of the matrix in the right-hand side correspond to $[S_{i,a+k\boldsymbol{Z}}]$, the element of the basis of $K_0(D^{\mathrm{b}}(\mathrm{mod}\,\Lambda))$. Then it is easy to see the columns of $1_{nk} + T_n(X_k)$ and $U_n(X_k)$ correspond to the generators of H' and H'', respectively. Using Proposition 11, we have the assertion. $\square$

We consider transformations of $\begin{pmatrix}1_n + T_n(x) & U_n(x)\end{pmatrix}$ in $\mathrm{Mat}_{n+1,n}(\boldsymbol{Z}[x])$.

Example 14. If $n = 7$, $\begin{pmatrix}1 + T_n(x) & U_n(x)\end{pmatrix}$ is

$$\begin{pmatrix} 1 & & & & & & x^7 & 1 \\ & 1 & & & & x^6 & & 1 \\ & & 1 & & x^5 & & & 1 \\ & & & 1+x^4 & & & & 1 \\ & & x^3 & & 1 & & & 1 \\ & x^2 & & & & 1 & & 1 \\ x & & & & & & 1 & 1 \end{pmatrix}.$$

This can be transformed as a matrix on $\boldsymbol{Z}[x]$ as follows;

$$
\begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1+x^4 &&&& 1 \\
&&&& 1-x^8 &&& 1-x^3 \\
&&&&& 1-x^8 && 1-x^2 \\
&&&&&& 1-x^8 & 1-x
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 1+x^4 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&&&&& 1-x^8 & 1-x
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&& -(1-x)(1+x^4) &&& 1-x^8 & 0
\end{pmatrix}
$$

$$
\mapsto \begin{pmatrix}
1 &&&&&&& \\
& 1 &&&&&& \\
&& 1 &&&&& \\
&&& 0 &&&& 1 \\
&&&& 1-x^8 &&& \\
&&&&& 1-x^8 && \\
&&& -(1-x)(1+x^4) &&& 0 & 0
\end{pmatrix}.
$$

Thus we have $\mathrm{Cok}\begin{pmatrix}1_{7k}+T_7(X_k) & U_7(X_k)\end{pmatrix} \cong (\mathrm{Cok}(1-X_k^8))^2 \oplus \mathrm{Cok}((1-X_k)(1+X_k^4))$.

If $n=6$, $\begin{pmatrix}1+T_n(x) & U_n(x)\end{pmatrix}$ is

$$
\begin{pmatrix}
1 &&&&& x^6 & 1 \\
& 1 &&& x^5 && 1 \\
&& 1 & x^4 &&& 1 \\
&& x^3 & 1 &&& 1 \\
& x^2 &&& 1 && 1 \\
x &&&&& 1 & 1
\end{pmatrix}.
$$

This can be transformed as a matrix on $\boldsymbol{Z}[x]$ as follows;

$$
\begin{pmatrix}
1 &&&&&& \\
& 1 &&&&& \\
&& 1 &&&& \\
&&& 1-x^7 &&& 1-x^3 \\
&&&& 1-x^7 && 1-x^2 \\
&&&&& 1-x^7 & 1-x
\end{pmatrix}
$$

$$\mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1-x^7 & & & \\ & & & & 1-x^7 & & \\ & & & & & 1-x^7 & 1-x \end{pmatrix}$$

$$\mapsto \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1-x^7 & & & \\ & & & & 1-x^7 & & \\ & & & & & 0 & 1-x \end{pmatrix}.$$

Thus we have $\mathrm{Cok}\begin{pmatrix} 1_{6k}+T_6(X_k) & U_6(X_k) \end{pmatrix} \cong (\mathrm{Cok}(1-X_k^7))^2 \oplus \mathrm{Cok}(1-X_k)$.

These examples are generalized as follows.

Lemma 15. *$K_0(\underline{\mathrm{mod}}\,\Lambda) \cong \mathrm{Cok}\begin{pmatrix} 1+T_n(X_k) & U_n(X_k) \end{pmatrix}$ is isomorphic to*

$$\begin{cases} (\mathrm{Cok}(1_k - X_k^{n+1}))^{(n-3)/2} \oplus \mathrm{Cok}((1_k - X_k)(1_k + X_k^{(n+1)/2})) & (n \notin 2\boldsymbol{Z}) \\ (\mathrm{Cok}(1_k - X_k^{n+1}))^{(n-2)/2} \oplus \mathrm{Cok}(1_k - X_k) & (n \in 2\boldsymbol{Z}) \end{cases}.$$

Now we only have to calculate the direct summands appeared in the previous lemma. The results are the following, and using these, the part for $\boldsymbol{Z}A_n/\langle\tau^k\rangle$ of the main theorem is proved.

Lemma 16. [1, Lemma 2.8, Lemma 2.12] *We have* $\mathrm{Cok}(1_k - X_k) \cong \boldsymbol{Z}$, $\mathrm{Cok}(1_k - X_k^{n+1}) \cong \boldsymbol{Z}^d$ *and if* $n \notin 2\boldsymbol{Z}$,

$$\mathrm{Cok}((1_k - X_k)(1_k + X_k^{(n+1)/2})) \cong \begin{cases} \boldsymbol{Z} \oplus (\boldsymbol{Z}/2\boldsymbol{Z})^{d-1} & (r \in 2\boldsymbol{Z}) \\ \boldsymbol{Z}^{(d+2)/2} & (r \notin 2\boldsymbol{Z}) \end{cases}.$$

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COHEN-MONTGOMERY DUALITY FOR BIMODULES AND ITS APPLICATIONS TO EQUIVALENCES GIVEN BY BIMODULES

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Abstract. Let G be a group and $\Bbbk$ be a commutative ring. We define a G-invariant bimodule ${}_SM_R$ over G-categories R, S and a G-graded bimodule ${}_BN_A$ over G-graded categories A, B, and introduce the orbit bimodule M/G and the smash product bimodule $N\#G$. We will show that these constructions are inverses to each other. This will be apply to Morita equivalences and stable equivalences of Morita type.

Introduction

We fix a commutative ring $\Bbbk$ and a group G. To include infinite coverings of $\Bbbk$-algebras into consideration we usually regard $\Bbbk$-algebras as locally bounded $\Bbbk$-categories with finite objects, and we will work with small $\Bbbk$-categories. For small $\Bbbk$-categories R and S with G-actions we introduce *G-invariant S-R-bimodules* and their category denoted by $S\text{-Mod}^G\text{-}R$, and denote by R/G the orbit category of R by G, which is a small G-graded $\Bbbk$-category. For small G-graded $\Bbbk$-categories A and B we introduce *G-graded B-A-bimodules* and their category denoted by $B\text{-Mod}_G\text{-}A$, and denote by $A\#G$ the smash product of A and G, which is a small $\Bbbk$-category with G-action. Then the Cohen-Montgomery duality theorem [4, 2] says that we have equivalences $(R/G)\#G \simeq R$ and $(A\#G)/G \simeq A$, by which we identify these pairs (see also [3]). Here we introduce functors $(\text{-})/G : S\text{-Mod}^G\text{-}R \to (S/G)\text{-Mod}_G\text{-}(R/G)$ and $(\text{-})\#G : A\text{-Mod}_G\text{-}B \to (A\#G)\text{-Mod}^G\text{-}(B\#G)$, and show that they are equivalences and quasi-inverses to each other (by applying $A := R/G$, $R := A\#G$, etc.), have good properties with tensor products and preserve projectivity of bimodules. We apply this to equivalences given by bimodules such as Morita equivalences, stable equivalences of Morita type to have theorems such as for stable equivalences of Morita type:

Theorem. (1) *There exists a "G-invariant stable equivalence of Morita type" between R and S if and only if there exists a "G-graded stable equivalence of Morita type" between R/G and S/G.*

(2) *There exists a "G-graded stable equivalence of Morita type" between A and B if and only if there exists a "G-invariant stable equivalence of Morita type" between $A\#G$ and $B\#G$.*

We note that a G-invariant (resp. G-graded) stable equivalence of Morita type is defined to be a usual stable equivalence of Morita type with additional properties, and does not mean an equivalence between stable categories of G-invariant (resp. G-graded) modules (see section 6 for detail).

The detailed version of this paper will be submitted for publication elsewhere.

1. Preliminaries

For a category R we denote the class of objects and morphisms of R by R_0 and R_1, respectively. We sometimes write $x \in R$ for $x \in R_0$. We first recall definitions of G-categories and their 2-category G-**Cat**.

Definition 1.1. (1) A $\Bbbk$-category with a *G-action*, or a *G-category* for short, is a pair (R, X) of a category R and a group homomorphism $X\colon G \to \mathrm{Aut}(R)$. We often write ax for $X(a)x$ for all $a \in G$ and $x \in R_0 \cup R_1$ if there seems to be no confusion.

(2) Let $R = (R, X)$ and $R' = (R', X')$ be G-categories. Then a *G-equivariant* functor from R to R' is a pair (E, ρ) of a $\Bbbk$-functor $E\colon R \to R'$ and a family $\rho = (\rho_a)_{a\in G}$ of natural isomorphisms $\rho_a\colon X'_aE \Rightarrow EX_a$ $(a \in G)$ such that the diagrams

$$\begin{tikzcd} X'_{ba}E = X'_bX'_aE \arrow[r, Rightarrow, "X'_b\rho_a"] \arrow[dr, Rightarrow, "\rho_{ba}"'] & X'_bEX_a \arrow[d, Rightarrow, "\rho_bX_a"] \\ & EX_{ba} = EX_bX_a \end{tikzcd}$$

commute for all $a, b \in G$.

(3) A $\Bbbk$-functor $E\colon R \to R'$ is called a *strictly G-equivariant* functor if $(E, (\mathbb{1}_E)_{a\in G})$ is a G-equivariant functor, i.e., if $X'_aE = EX_a$ for all $a \in G$.

(4) Let $(E, \rho), (E', \rho')\colon R \to R'$ be *G-equivariant* functors. Then a *morphism* from (E, ρ) to (E', ρ') is a natural transformation $\eta\colon E \Rightarrow E'$ such that the diagrams

$$\begin{tikzcd} X'_aE \arrow[r, Rightarrow, "\rho_a"] \arrow[d, Rightarrow, "X'_a\eta"'] & EX_a \arrow[d, Rightarrow, "\eta X_a"] \\ X'_aE' \arrow[r, Rightarrow, "\rho'_a"'] & E'X_a \end{tikzcd}$$

commute for all $a \in G$.

These data define a 2-category G-**Cat** of small G-categories.

Next we recall definitions of G-graded categories and their 2-category G-**GrCat**.

Definition 1.2. (1) A *G-graded* $\Bbbk$-category is a category A together with a family of direct sum decompositions $A(x, y) = \bigoplus_{a\in G} A^a(x, y)$ $(x, y \in A)$ of $\Bbbk$-modules such that $A^b(y, z) \cdot A^a(x, y) \subseteq A^{ba}(x, z)$ for all $x, y \in A$ and $a, b \in G$. It is easy to see that $\mathbb{1}_x \in A^1(x, x)$ for all $x \in A_0$.

(2) A *degree-preserving* functor is a pair (H, r) of a $\Bbbk$-functor $H\colon A \to B$ of G-graded categories and a map $r\colon A_0 \to G$ such that

$$H(A^{r_y a}(x, y)) \subseteq B^{ar_x}(Hx, Hy)$$

(or equivalently $H(A^a(x, y)) \subseteq B^{r_y^{-1}ar_x}(Hx, Hy)$) for all $x, y \in A$ and $a \in G$. This r is called a *degree adjuster* of H.

(3) A $\Bbbk$-functor $H\colon A \to B$ of G-graded categories is called a *strictly* degree-preserving functor if $(H, 1)$ is a degree-preserving functor, where 1 denotes the constant map $A_0 \to G$ with value $1 \in G$, i.e., if $H(A^a(x, y)) \subseteq B^a(Hx, Hy)$ for all $x, y \in A$ and $a \in G$.

(4) Let $(H,r),(I,s)\colon A\to B$ be degree-preserving functors. Then a natural transformation $\theta\colon H\Rightarrow I$ is called a *morphism* of degree-preserving functors if $\theta x\in B^{s_x^{-1}r_x}(Hx,Ix)$ for all $x\in A$.

These data define a 2-category G-**GrCat** of small G-graded categories.

Finally we recall definitions of orbit categories and smash products, and their relationships.

Definition 1.3. Let R be a G-category. Then the *orbit category* R/G of R by G is a category defined as follows.

- $(R/G)_0 := R_0$;
- For any $x,y\in G$, $(R/G)(x,y):=\bigoplus_{a\in G}R(ax,y)$; and
- For any $x\xrightarrow{f}y\xrightarrow{g}z$ in R/G, $gf:=(\sum_{a,b\in G;ba=c}g_b\cdot b(f_a))_{c\in G}$.
- For each $x\in(R/G)_0$ its identity $\mathbb{1}_x:=\mathbb{1}_x^{R/G}$ in R/G is given by $\mathbb{1}_x=(\delta_{a,1}\mathbb{1}_x^R)_{a\in G}$, where $\mathbb{1}_x^R$ is the identity of x in R.

By setting $(R/G)^a(x,y):=R(ax,y)$ for all $x,y\in R_0$ and $a\in G$, the decompositions $(R/G)(x,y)=\bigoplus_{a\in G}(R/G)^a(x,y)$ makes R/G a G-graded category.

Definition 1.4. Let A be a G-graded category. Then the *smash product* $A\#G$ is a category defined as follows.

- $(A\#G)_0:=A_0\times G$, we set $x^{(a)}:=(x,a)$ for all $x\in A$ and $a\in G$.
- $(A\#G)(x^{(a)},y^{(b)}):=A^{b^{-1}a}(x,y)$ for all $x^{(a)},y^{(b)}\in A\#G$.
- For any $x^{(a)},y^{(b)},z^{(c)}\in A\#G$ the composition is given by the following commutative diagram

$$\begin{tikzcd}
(A\#G)(y^{(b)},z^{(c)})\times(A\#G)(x^{(a)},y^{(b)}) \arrow[r] \arrow[d, equal] & (A\#G)(x^{(a)},z^{(c)}) \arrow[d, equal] \\
A^{c^{-1}b}(y,z)\times A^{b^{-1}a}(x,y) \arrow[r] & A^{c^{-1}a}(x,z),
\end{tikzcd}$$

  where the lower horizontal homomorphism is given by the composition of A.
- For each $x^{(a)}\in(A\#G)_0$ its identity $\mathbb{1}_{x^{(a)}}$ in $A\#G$ is given by $\mathbb{1}_x\in A^1(x,x)$.

$A\#G$ has a free G-action defined as follows: For each $c\in G$ and $x^{(a)}\in A\#G$, $cx^{(a)}:=x^{(ca)}$; and for each $f\in(A\#G)(x^{(a)},y^{(b)})=A^{b^{-1}a}(x,y)=(A\#G)(x^{(ca)},y^{(cb)})$, $cf:=f$.

The following two propositions were proved in [1].

Proposition 1.5 ([1, Proposition 5.6])**.** *Let A be a G-graded category. Then there is a strictly degree-preserving equivalence $\omega_A\colon A\to(A\#G)/G$ of G-graded categories.*

Proposition 1.6 ([1, Theorem 5.10])**.** *Let R be a category with a G-action. Then there is a G-equivariant equivalence $\varepsilon_R\colon R\to(R/G)\#G$.*

In fact, the orbit category construction and the smash product construction can be extended to 2-functors $(\text{-})/G\colon G$-**Cat** $\to G$-**GrCat** and $(\text{-})\#G\colon G$-**GrCat** $\to G$-**Cat**, respectively, and they are inverses to each other as stated in the following theorem, where $\omega:=(\omega_A)_A$ and $\varepsilon:=(\varepsilon_R)_R$ are 2-natural isomorhisms.

Theorem 1.7 ([2, Theorem 7.5]). *(-)/G is strictly left 2-adjoint to (-)#G and they are mutual 2-quasi-inverses.*

Remark 1.8. $\omega_A\colon A \to (A\#G)/G$ above is an equivalence in the 2-category G-**GrCat** and $\varepsilon_R\colon R \to (R/G)\#G$ above is an equivalence in the 2-category G-**Cat**. By these equivalences we identify $(A\#G)/G$ with A, and $(R/G)\#G$ with R in the following sections.

2. G-invariant bimodules and G-graded bimodues

Definition 2.1. Let $R = (R, X)$ and $S = (S, Y)$ be small $\Bbbk$-categories with G-actions.

(1) A *G-invariant S-R-bimodule* is a pair (M, ϕ) of an S-R-bimodule M and a family $\phi := (\phi_a)_{a\in G}$ of natural transformations $\phi_a\colon M \to M(X(a)(\text{-}), Y(a)(\text{-}))$, where $\phi_a = (\phi_a(x,y))_{(x,y)\in R_0\times S_0}$, $\phi_a(x,y)\colon M(x,y) \to M(ax,ay)$ is in $\operatorname{Mod}\Bbbk$, such that the following diagram commutes for all $a, b \in G$ and all $(x,y) \in R_0 \times S_0$:

$$\begin{tikzcd} M(x,y) \arrow[r, "\phi_a(x,y)"] \arrow[dr, "\phi_{ba}(x,y)"'] & M(ax,ay) \arrow[d, "\phi_b(ax,ay)"] \\ & M(bax,bay). \end{tikzcd}$$

(2) Let (M, ϕ) and (N, ψ) be G-invariant S-R-bimodules. A *morphism* $(M, \phi) \to (N, \psi)$ is an S-R-bimodule morphism $F\colon M \to N$ such that the following diagram commutes for all $a \in G$ and all $(x,y) \in R_0 \times S_0$:

$$\begin{tikzcd} M(x,y) \arrow[r, "\phi_a(x,y)"] \arrow[d, "F(x,y)"'] & M(ax,ay) \arrow[d, "F(ax,ay)"] \\ N(x,y) \arrow[r, "\psi_a(x,y)"'] & N(ax,ay). \end{tikzcd}$$

(3) The class of all G-invariant S-R-bimodules together with all morphisms between them forms a $\Bbbk$-category denoted by $S\text{-}\operatorname{Mod}^G\text{-}R$.

Remark 2.2. The commutativity of the diagram in (1) above for $a = b = 1$ shows that $\phi_1 = \mathbb{1}_M$, which also shows that $\phi_a(x,y)^{-1} = \phi_{a^{-1}}(ax,ay)$ for all $a \in G$ and all $(x,y) \in R_0 \times S_0$.

Definition 2.3. Let A and B be G-graded small $\Bbbk$-categories.

(1) A G-graded B-A-bimodule is a B-A-bimodule M together with decompositions $M(x,y) = \bigoplus_{a\in G} M^a(x,y)$ in $\operatorname{Mod}\Bbbk$ for all $(x,y) \in A_0 \times B_0$ such that

$$B^c(y,y') \cdot M^a(x,y) \cdot A^b(x',x) \subseteq M^{cab}(x',y')$$

for all $a, b, c \in G$ and all $x, x' \in A_0, y, y' \in B_0$.

(2) Let M and N be G-graded B-A-bimodules. Then a *morphism* $M \to N$ is a B-A-bimodule morphism $F\colon M \to N$ such that $F(M^a(x,y)) \subseteq N^a(Fx, Fy)$ for all $a \in G$ and all $(x,y) \in A_0 \times B_0$.

(3) The class of all G-graded B-A-bimodules together with all morphisms between them forms a $\Bbbk$-category denoted by $B\text{-}\operatorname{Mod}_G\text{-}A$.

Throughout this section $R = (R, X)$ and $S = (S, Y)$ are small $\Bbbk$-categories with G-actions, and $E\colon R \to R/G$ and $F\colon S \to S/G$ the canonical G-covering, respectively.

Definition 3.1. (1) Let $M = (M, \phi)$ be a G-invariant S-R-bimodule. Then we form a G-graded S/G-R/G-bimodule M/G as follows which we call the *orbit bimodule* of M by G:

- For each $(x, y) \in (R/G)_0 \times (S/G)_0 = R_0 \times S_0$ we set
$$(M/G)(x, y) := \bigoplus_{a \in G} M(ax, y). \tag{3.1}$$
- For each $(x, y), (x', y') \in (R/G)_0 \times (S/G)_0 = R_0 \times S_0$ and each $(r, s) \in (R/G)(x', x) \times (S/G)(y, y')$ we define a morphism
$$(M/G)(r, s)\colon (M/G)(x, y) \to (M/G)(x', y')$$
in $\operatorname{Mod} \Bbbk$ by
$$(M/G)(r, s)(m) := s \cdot m \cdot r := \left(\sum_{cba=d} s_c \cdot \phi_c(m_b) \cdot cbr_a \right)_{d \in G} \tag{3.2}$$
for all $r = (r_a)_{a \in G} \in \bigoplus_{a \in G} R(ax', x), m = (m_b)_{b \in G} \in \bigoplus_{b \in G} M(ax, y)$, and $s = (s_c)_{c \in G} \in \bigoplus_{c \in G} S(cy, y')$. By the naturality of ϕ_a $(a \in G)$ we easily see that (3.2) defines an (S/G)-(R/G)-bimodule structure on M/G.
- We set $M^a(x, y) := M(ax, y)$ for all $a \in G$ and all $(x, y) \in R_0 \times S_0$. We easily see that this defines a G-grading on M/G by (3.1) and (3.2).

(2) Let $f\colon M \to N$ be in $S\text{-}\operatorname{Mod}^G\text{-}R$. For each $(x, y) \in R_0 \times S_0$ we set
$$(f/G)(x, y) := \bigoplus_{a \in G} f(ax, y).$$
Then as is easily seen $f/G := (f/G(x, y))_{(x,y) \in R_0 \times S_0}$ turns out to be a morphism $M/G \to N/G$ in $(S/G)\text{-}\operatorname{Mod}_G\text{-}(R/G)$.

(3) It is easy to see that (1) together with (2) above defines a $\Bbbk$-functor
$$(\text{-})/G\colon S\text{-}\operatorname{Mod}^G\text{-}R \to (S/G)\text{-}\operatorname{Mod}_G\text{-}(R/G).$$

Lemma 3.2. *By regarding R/G as a left R-module and a right R-module via the canonical G-covering functor $E\colon R \to R/G$, we have*
$$R/G \otimes_R R/G \cong R/G \otimes_{R/G} R/G \cong R/G$$
as (R/G)-(R/G)-bimodules.

Proposition 3.3. *Let M be a G-invariant S-R-bimodule. Then*

(1) $M \otimes_R (R/G) \cong {}_F M/G$ *as S-(R/G)-bimodules; and*

(2) $(S/G) \otimes_S M \cong M/G_E$ *as (S/G)-R-bimodules.*

Hence in particular we have isomorphisms of S-R-bimodules

(3) $M \otimes_R (R/G)_E \cong {}_F M/G_E \cong {}_F(S/G) \otimes_S M$

and an isomorphism of G-graded (S/G)-(R/G)-bimodules

(4) $(S/G) \otimes_S M \otimes_R (R/G) \cong M/G$.

Proposition 3.4. *Let $T = (T, Z)$ be a small $\Bbbk$-category with G-action, and ${}_SM_R$, ${}_TN_S$ be G-invariant bimodules. Then*

(1) ${}_T(N \otimes_S M)_R$ *is a G-invariant bimodule.*

(2) $(N \otimes_S M)/G \cong (N/G) \otimes_{S/G} (M/G)$ *in* $(T/G)\text{-}\mathrm{Mod}_G\text{-}(R/G)$.

Proposition 3.5. *Let ${}_SP_R$ be a projective bimodule that is G-invariant. Then ${}_{(S/G)}P/G_{(R/G)}$ is a projective bimodule that is G-graded.*

Remark 3.6. In the proof above, note that in general we have

$$(S(w,\text{-}) \otimes_\Bbbk R(\text{-},z))/G \ncong (S/G) \otimes_S S(w,\text{-}) \otimes_\Bbbk R(\text{-},z) \otimes_R (R/G)$$

because $S(w,\text{-}) \otimes_\Bbbk R(\text{-},z)$ is not always G-invariant.

4. SMASH PRODUCT

Throughout this section A and B are G-graded small $\Bbbk$-categories.

Definition 4.1. (1) Let M be a G-graded B-A-bimodule. Then we define a G-invariant $(B\#G)$-$(A\#G)$-bimodule $M\#G$ as follows, which we call the *smash product* of M and G:

- For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ we set
$$(M\#G)(x^{(a)}, y^{(b)}) := M^{b^{-1}a}(x, y).$$
- For each $(x^{(a)}, y^{(b)}), (x'^{(a')}, y'^{(b')}) \in (A\#G)_0 \times (B\#G)_0$ and each $(\alpha, \beta) \in (A\#G)(x'^{(a')}, x^{(a)}) \times (B\#G)(y^{(b)}, y'^{(b')}) = A^{a^{-1}a'}(x', x) \times B^{b'^{-1}b}(y, y')$ we define a morphism $(M\#G)(\alpha, \beta)$ in $\mathrm{Mod}\,\Bbbk$ by the following commutative diagram:
$$\begin{tikzcd}
(M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "(M\#G)(\alpha,\beta)"] \arrow[d, equal] & (M\#G)(x'^{(a')}, y'^{(b')}) \arrow[d, equal] \\
M^{b^{-1}a}(x, y) \arrow[r, "M(\alpha,\beta)"'] & M^{b'^{-1}a'}(x', y').
\end{tikzcd}$$
Since $M(\alpha, \beta)(m) \in M^{(b'^{-1}b)(b^{-1}a)(a^{-1}a')}(x', y') = M^{b'^{-1}a'}(x', y')$ for all $m \in M^{b^{-1}a}(x, y)$, the bottom morphism is well-defined. It is easy to verify that this makes $M\#G$ a $(B\#G)$-$(A\#G)$-bimodule.
- For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ and each $c \in G$ we define $\phi_c(x^{(a)}, y^{(b)})$ by the following commutative diagram:
$$\begin{tikzcd}
(M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "\phi_c(x^{(a)},y^{(b)})"] \arrow[d, equal] & (M\#G)(c \cdot x^{(a)}, c \cdot y^{(b)}) \arrow[d, equal] \\
M^{b^{-1}a}(x, y) \arrow[r, equal] & (M\#G)(x^{(ca)}, y^{(cb)}).
\end{tikzcd}$$
Then by letting $\phi_c := (\phi_c(x^{(a)}, y^{(b)}))_{(x^{(a)},y^{(b)})}$, and $\phi := (\phi_c)_{c\in G}$, we have a G-invariant $(B\#G)$-$(A\#G)$-bimodule $(M\#G, \phi)$.

(2) Let $f\colon M \to N$ be in B-Mod_G-A. For each $(x^{(a)}, y^{(b)}) \in (A\#G)_0 \times (B\#G)_0$ we define $(f\#G)(x^{(a)}, y^{(b)})$ by the commutative diagram

$$\begin{tikzcd}
(M\#G)(x^{(a)}, y^{(b)}) \arrow[r, "(f\#G)(x^{(a)},y^{(b)})"] \arrow[d, equal] & (N\#G)(x^{(a)}, y^{(b)}) \arrow[d, equal] \\
M^{b^{-1}a}(x, y) \arrow[r, "f|_{M^{b^{-1}a}(x,y)}"'] & N^{b^{-1}a}(x, y).
\end{tikzcd}$$

Then as is easily seen $f\#G := ((f\#G)(x^{(a)}, y^{(b)}))_{(x^{(a)},y^{(b)})}$ is a morphism $M\#G \to N\#G$ in the category $(B\#G)$-Mod^G-$(A\#G)$.

(3) It is easy to see that (1) together with (2) above defines a $\Bbbk$-functor

$$(\text{-})\#G\colon B\text{-}\mathrm{Mod}_G\text{-}A \to (B\#G)\text{-}\mathrm{Mod}^G\text{-}(A\#G).$$

Proposition 4.2. *Let C be a G-graded small $\Bbbk$-category, and ${}_BM_A, {}_CN_B$ G-graded bimodules. Then*

1. *$N \otimes_B M$ is a G-graded C-A-bimodule.*
2. *$(N \otimes_B M)\#G \cong (N\#G) \otimes_{B\#G} (M\#G)$ in $(C\#G)$-Mod^G-$(A\#G)$.*

Proposition 4.3. *Let ${}_BP_A$ be a projective bimodule that is G-graded. Then ${}_{B\#G}(P\#G)_{A\#G}$ is a projective bimodule that is G-invariant.*

5. Cohen-Montgomery duality for bimodules

Theorem 5.1. *Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.*

1. *The functor $(\text{-})/G\colon S\text{-}\mathrm{Mod}^G\text{-}R \to (S/G)\text{-}\mathrm{Mod}_G\text{-}(R/G)$ is an equivalence, a quasi-inverse of which is given by the composite*

$$(S/G)\text{-}\mathrm{Mod}_G\text{-}(R/G) \xrightarrow{(\text{-})\#G} ((S/G)\#G)\text{-}\mathrm{Mod}^G\text{-}((R/G)\#G) \xrightarrow{\sim} S\text{-}\mathrm{Mod}^G\text{-}R.$$

2. *The functor $(\text{-})\#G\colon B\text{-}\mathrm{Mod}_G\text{-}A \to (B\#G)\text{-}\mathrm{Mod}^G\text{-}(A\#G)$ is an equivalence, a quasi-invers of which is given by the composite*

$$(B\#G)\text{-}\mathrm{Mod}^G\text{-}(A\#G) \xrightarrow{(\text{-})/G} ((B\#G)/G)\text{-}\mathrm{Mod}_G\text{-}((A\#G)/G) \xrightarrow{\sim} B\text{-}\mathrm{Mod}_G\text{-}A.$$

3. *In particular, for each G-invariant bimodule ${}_RM_S$ we have $(M/G)\#G \cong M$ as S-R-bimodules, and for each G-graded bimodule ${}_BM_A$ we have $(M\#G)/G \cong M$ as B-A-bimodules.*

6. Applications

Definition 6.1. Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.

1. A pair $({}_SM_R, {}_RN_S)$ of bimodules is said to give a *G-invariant stable equivalence of Morita type* between R and S if ${}_SM, M_R, {}_RN, N_S$ are projective modules and ${}_SM_R, {}_RN_S$ are G-invariant bimodules such that $N \otimes_S M \cong R \oplus {}_RP_R$ and

$M \otimes_R N \cong S \oplus {}_SQ_S$ as G-invariant bimodules for some projective bimodules ${}_RP_R, {}_SQ_S$ that are G-invariant.

(1′) A pair $({}_SM_R, {}_RN_S)$ of bimodules is said to give a *G-invariant Morita equivalence* between R and S if it gives a G-invariant stable equivalence of Morita type with $P = 0 = Q$ in (1) above.

(2) A pair $({}_BM_A, {}_AN_B)$ of bimodules is said to give a *G-graded stable equivalence of Morita type* between A and B if ${}_BM, M_A, {}_AN, N_B$ are projective modules and ${}_BM_A, {}_AN_B$ are G-graded bimodules such that $N \otimes_B M \cong A \oplus {}_AP_A$ and $M \otimes_A N \cong B \oplus {}_BQ_B$ as G-graded bimodules for some projective bimodules ${}_AP_A, {}_BQ_B$ that are G-graded.

(2′) A pair $({}_BM_A, {}_AN_B)$ of bimodules is said to give a *G-graded Morita equivalence* between A and B if it gives a G-graded stable equivalence of Morita type with $P = 0 = Q$ in (2) above.

Theorem 6.2. *Let R, S be small $\Bbbk$-categories with G-actions, and A, B be G-graded small $\Bbbk$-categories.*

(1) *A pair $({}_SM_R, {}_RN_S)$ of bimodules gives a G-invariant stable equivalence of Morita type between R and S if and only if the pair $(M/G, N/G)$ gives a G-graded stable equivalence of Morita type between R/G and S/G.*

(1′) *A pair $({}_SM_R, {}_RN_S)$ of bimodules gives a G-invariant Morita equivalence between R and S if and only if the pair $(M/G, N/G)$ gives a G-graded Morita equivalence between R/G and S/G.*

(2) *A pair $({}_BM_A, {}_AN_B)$ of bimodules gives a G-graded stable equivalence of Morita type between A and B if and only if the pair $(M\#G, N\#G)$ gives a G-invariant stable equivalence of Morita type between $A\#G$ and $B\#G$.*

(2′) *A pair $({}_BM_A, {}_AN_B)$ of bimodules gives a G-graded Morita equivalence between A and B if and only if the pair $(M\#G, N\#G)$ gives a G-invariant Morita equivalence between $A\#G$ and $B\#G$.*

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STABLE DEGENERATIONS OF COHEN-MACAULAY MODULES OVER SIMPLE SINGULARITIES OF TYPE (A_n)

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Abstract. We study the stable degeneration problem for Cohen-Macaulay modules over simple singularities of type (A_n). We prove that the stable hom order is actually a partial order over the ring and are able to show that the stable degenerations can be controlled by the stable hom order.

1. Introduction

The concept of degenerations of modules introduced in representation theory for studying the structure of the module variety over a finite dimensional algebra. Classically Bongartz [1] investigated the degeneration problem of modules over an artinian algebra in relation with the Auslander-Reiten quiver. In [11], Zwara gave a complete description of degenerations of modules over representation finite algebras by using some order relations for modules known as the hom order, the degeneration order and the extension order. Now a theory of degenerations is considered for not only module categories, but derived categories [5] or stable categories [10], more generally, triangulated categories [7].

Let R be a commutative Gorenstein local k-algebra which is not necessary finite dimensional. Yoshino [10] introduced a notion of the stable analogue of degenerations of (maximal) Cohen-Macaulay R-module in the stable category $\underline{\mathrm{CM}}(R)$. The author [4] give a complete description of degenerations of Cohen-Macaulay modules over a ring of even dimensional simple singularity of type (A_n) by using the description of stable degenerations over it. Hence it is also important for the study of degeneration problem to investigate the description of stable degenerations.

The purpose of this paper is to describe stable degenerations of Cohen-Macaulay modules over simple singularities of type (A_n).

$$k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2).$$

First we consider an order relation on $\underline{\mathrm{CM}}(R)$ which is the stable analogue of the hom order and show that the order $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is of odd dimensional. By using the stable analogue of the argument over finite dimensional algebras in [11], we can describe stable degenerations of Cohen-Macaulay modules over the ring in terms of the stable hom order.

2. Stable hom order

Throughout the paper R is a commutative Henselian Gorenstein local ring that is k-algebra where k is an algebraically closed field of characteristic 0. For a finitely generated

The detailed version of this paper will be submitted for publication elsewhere.

R-module M, we say that M is a Cohen-Macaulay R-module if

$$\mathrm{Ext}^i_R(M, R) = 0 \quad \text{for any } i > 0.$$

We denote by $\mathrm{CM}(R)$ the category of Cohen-Macaulay R-modules with all R-homomorphisms. And we also denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. The objects of $\underline{\mathrm{CM}}(R)$ are the same as those of $\mathrm{CM}(R)$, and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M, N) = \mathrm{Hom}_R(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(R)$, where $P(M, N)$ denote the set of morphisms from M to N factoring through free R-modules. For a Cohen-Macaulay module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$. For a finitely generated R-module M, take a free resolution

$$\cdots \to F_1 \xrightarrow{d} F_0 \to M \to 0.$$

We denote $\mathrm{Im} d$ by ΩM. We note that this defines the functor giving an auto-equivalence of $\underline{\mathrm{CM}}(R)$. It is known that $\underline{\mathrm{CM}}(R)$ has a structure of a triangulated category with the shift functor defined by the functor Ω^{-1}. We recommend the reader to [2, Chapter 1], [8, Section 4] for the detail. Since R is Henselian, $\mathrm{CM}(R)$, hence $\underline{\mathrm{CM}}(R)$, is a Krull-Schmidt category, namely each object can be decomposed into indecomposable objects up to isomorphism uniquely.

Definition 1. We say that $(R, \mathfrak{m})$ is an isolated singularity if each localization $R_\mathfrak{p}$ is regular for each prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq \mathfrak{m}$.

We say that $\mathrm{CM}(R)$ (resp. $\underline{\mathrm{CM}}(R)$) admits AR sequences (resp. AR triangles) if there exists an AR sequence (reap. AR triangle) ending in X (resp. $\underline{X}$) for each indecomposable Cohen-Macaulay R-module X. If R is an isolated singularity, $\mathrm{CM}(R)$ admits AR sequences (see [8, Theorem 3.2]). Hence, $\underline{\mathrm{CM}}(R)$ also admits AR triangles. We also say that R is of finite representation type if there are only a finite number of isomorphism classes of indecomposable Cohen-Macaulay R-modules. We note that if R is of finite representation type, then R is an isolated singularity (cf. [8, Chapter 3.]).

Lemma 2. [8, Lemma 3.9] *Let M and N be finitely generated R-modules. Then we have a functorial isomorphism*

$$\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Tor}^R_1(\mathrm{Tr}M, N).$$

Here $\mathrm{Tr}M$ *is an Auslander transpose of M.*

According to Lemma 2, $\underline{\mathrm{Hom}}_R(M, N)$ is of finite dimensional as a k-module for M, $N \in \mathrm{CM}(R)$ if R is an isolated singularity. Thus the following definition makes sense.

Definition 3. For M, $N \in \mathrm{CM}(R)$ we define $\underline{M} \leq_{hom} \underline{N}$ if $[\underline{X}, \underline{M}] \leq [\underline{X}, \underline{N}]$ for each $\underline{X} \in \underline{\mathrm{CM}}(R)$. Here $[\underline{X}, \underline{M}]$ is an abbreviation of $\dim_k \underline{\mathrm{Hom}}_R(X, M)$.

Now let us consider the full subcategory of the functor category of $\mathrm{CM}(R)$ which is called the Auslander category. The Auslander category $\mathrm{mod}(\mathrm{CM}(R))$ is the category whose objects are finitely presented contravariant additive functors from $\mathrm{CM}(R)$ to the category of Abelian groups and whose morphisms are natural transformations between functors.

Lemma 4. [8, Theorem 13.7] *A group homomorphism*

$$\gamma : \mathrm{G}(\mathrm{CM}(R)) \to K_0(\mathrm{mod}(\mathrm{CM}(R))),$$

defined by $\gamma(M) = [\mathrm{Hom}_R(\ ,M)]$ *for* $M \in \mathrm{CM}(R)$*, is injective. Here* $\mathrm{G}(\mathrm{CM}(R))$ *is a free Abelian group* $\bigoplus \mathbb{Z} \cdot X$*, where* X *runs through all isomorphism classes of indecomposable objects in* $\mathrm{CM}(R)$*.*

We denote by $\underline{\mathrm{mod}}(\mathrm{CM}(R))$ the full subcategory $\mathrm{mod}(\mathrm{CM}(R))$ consisting of functors F with $F(R) = 0$. Note that every object $F \in \underline{\mathrm{mod}}(\mathrm{CM}(R))$ is obtained from a short exact sequence in $\mathrm{CM}(R)$. Namely we have the short exact sequence $0 \to L \to M \to N \to 0$ such that

$$0 \to \mathrm{Hom}_R(\ ,L) \to \mathrm{Hom}_R(\ ,M) \to \mathrm{Hom}_R(\ ,N) \to F \to 0$$

is exact in $\mathrm{mod}(\mathrm{CM}(R))$. If $0 \to Z \to Y \to X \to 0$ is an AR sequence in $\mathrm{CM}(R)$, then the functor S_X defined by an exact sequence

$$0 \to \mathrm{Hom}_R(\ ,Z) \to \mathrm{Hom}_R(\ ,Y) \to \mathrm{Hom}_R(\ ,X) \to S_X \to 0$$

is a simple object in $\mathrm{mod}(\mathrm{CM}(R))$ and all the simple objects in $\mathrm{mod}(\mathrm{CM}(R))$ are obtained in this way from AR sequences.

Proposition 5. [3, Lemma 2.8] *If* R *is of finite representation type, then we have the equality in* $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$

$$[\underline{\mathrm{Hom}}_R(-,M)] = \sum_{X_i \in \mathrm{indCM}(R)} [\underline{X_i}, \underline{M}] \cdot [S_{X_i}]$$

for each $M \in \mathrm{CM}(R)$*.*

Proof. Since $\underline{\mathrm{Hom}}_R(-,M)$ is an object in $\underline{\mathrm{mod}}(\mathrm{CM}(R))$ we have a filtration by subjects $0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$ such that each F_i/F_{i-1} is a simple object S_X in $\mathrm{mod}(\mathrm{CM}(R))$ (cf. [8, (13.7.4)]). Hence we have the equality in $K_0(\underline{\mathrm{mod}}(\mathrm{CM}(R)))$:

$$[\underline{\mathrm{Hom}}_R(-,M)] = \sum_{X_i \in \mathrm{indCM}(R)} c_i \cdot [S_{X_i}].$$

By using a property of AR sequences, one can show that $c_i = [\underline{X_i}, \underline{M}]$, so that we obtain the equation. ☐

Combining Proposition 5 with Lemma 4, we have

Theorem 6. [3, Theorem 2.9] *Let* R *be of finite representation type and* M *and* N *be Cohen-Macaulay* R*-modules. Suppose that* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$*. Then* $\underline{M} \oplus \underline{\Omega M} \cong \underline{N} \oplus \underline{\Omega N}$*.*

It immediately follows from the theorem that

Corollary 7. *Let* R *be of finite representation type. Suppose that* $\underline{U} \cong \underline{U}[-1]$ *for each indecomposable Cohen-Macaulay* R*-module* U*. Then* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$ *if and only if* $\underline{M} \cong \underline{N}$*. Particularly,* $\leq_{\underline{hom}}$ *is a partial order on* $\underline{\mathrm{CM}}(R)$*.*

Example 8. Let R be a one dimensional simple singularity of type (A_n), that is $R = k[[x,y]]/(x^{n+1}+y^2)$. If n is an even integer, one can show that X is isomorphic to ΩX up to free summed for each $X \in \mathrm{CM}(R)$. See [8, Proposition 5.11]. Thus $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ if n is an even integer.

If n is an odd integer, we have indecomposable modules $X \in \mathrm{CM}(R)$ such that $\underline{X} \ncong \underline{X}[-1]$. In fact, let $N_{\pm} = R/(x^{(n+1)/2} \pm \sqrt{-1}y)$. Then N_+ (resp. N_-) is a Cohen-Macaulay R-module which is isomorphic to ΩN_- (resp. ΩN_+), so that $\underline{N_+} \ncong \underline{N_+}[-1]$ (resp. $\underline{N_-} \ncong \underline{N_-}[-1]$). Although we can also show that $\leq_{\underline{hom}}$ is a partial order on $\underline{\mathrm{CM}}(R)$ even if n is an odd integer.

Proposition 9. [3, Proposition 2.12] *Let* $R = k[[x,y]]/(x^{n+1}+y^2)$. *Then* $[\underline{X}, \underline{M}] = [\underline{X}, \underline{N}]$ *for each* $\underline{X} \in \underline{\mathrm{CM}}(R)$ *if and only if* $\underline{M} \cong \underline{N}$.

Remark 10. The stable hom order $\leq_{\underline{hom}}$ is not always a partial order on $\underline{\mathrm{CM}}(R)$ even if the base ring R is a simple singularity of type (A_n). See [3, Remark 2.13].

3. Stable degeneration of Cohen-Macaulay modules

In this section, we shall describe the stable degenerations of Cohen-Macaulay modules over simple singularities over type (A_n), namely

$$k[[x_0, x_1, x_2, \cdots, x_d]]/(x_0^{n+1} + x_1^2 + x_2^2 + \cdots + x_d^2). \tag{3.1}$$

First let us recall the definition of stable degenerations of Cohen-Macaulay modules.

Definition 11. [10, Definition 4.1] Let $\underline{M}, \underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ stably degenerates to $\underline{N}$ if there exists a Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\underline{Q[1/t]} \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\underline{Q \otimes_V V/tV} \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

It is known that the ring (3.1) is of finite representation type, so that it is an isolated singularity. If a ring is an isolated singularity, there is a nice characterization of stable degenerations.

Theorem 12. [10, Theorem 5.1, 6.1] *Consider the following three conditions for Cohen-Macaulay R-modules M and N:*

1. *$M \oplus P$ degenerates to $N \oplus Q$ for some free R-modules P, Q.*
2. *$\underline{M}$ stably degenerates to $\underline{N}$.*
3. *There is a triangle*

$$\underline{Z} \longrightarrow \underline{M} \oplus \underline{Z} \longrightarrow \underline{N} \longrightarrow \underline{Z}[1]$$

in $\underline{\mathrm{CM}}(R)$.

If R is an isolated singularity, then (2) *and* (3) *are equivalent. Moreover, if R is artinian, the conditions* (1), (2) *and* (3) *are equivalent.*

We state order relations with respect to stable degenerations and triangles.

Definition 13. [4, Definition 3.2., 3.3.] Let M and $N \in \mathrm{CM}(R)$.

1. We denote by $\underline{M} \leq_{st} \underline{N}$ if $\underline{N}$ is obtained from $\underline{M}$ by iterative stable degenerations, i.e. there is a sequence of Cohen-Macaulay R-modules $\underline{L_0}, \underline{L_1}, \ldots, \underline{L_r}$ such that $\underline{M} \cong \underline{L_0}$, $\underline{N} \cong \underline{L_r}$ and each $\underline{L_i}$ stably degenerates to $\underline{L_{i+1}}$ for $0 \leq i < r$.

(2) We say that $\underline{M}$ stably degenerates by a triangle to $\underline{N}$, if there is a triangle of the form $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$ in $\underline{\mathrm{CM}}(R)$ such that $\underline{U} \oplus \underline{V} \cong \underline{N}$. We also denote by $\underline{M} \leq_{tri} \underline{N}$ if $\underline{N}$ is obtain from $\underline{M}$ by iterative stable degenerations by a triangle.

Remark 14. It has shown in [10] that the stable degeneration order is a partial order. Moreover if there is a triangle $\underline{U} \to \underline{M} \to \underline{V} \to \underline{U}[1]$, then we can show that $\underline{M}$ stably degenerates to $\underline{U} \oplus \underline{V}$ (cf. [4, Remark 3.4. (2)]). Hence $\underline{M} \leq_{tri} \underline{N}$ induces $\underline{M} \leq_{st} \underline{N}$. It also follows from Theorem 12 that $\underline{M} \leq_{st} \underline{N}$ induces that $\underline{M} \leq_{\underline{hom}} \underline{N}$.

Let R be an one dimensional simple singularity of type (A_n). That is $R = k[[x,y]]/(x^{n+1}+y^2)$. As stated in [8, Proposition 5.11], if n is an even integer, the set of ideals of R

$$\{ \ I_i = (x^i, y) \mid 1 \leq i \leq n/2 \ \}$$

is a complete list of isomorphic indecomposable non free Cohen-Macaulay R-modules. On the other hand, if n is an odd integer, then

$$\{ \ I_i = (x^i, y) \mid 1 \leq i \leq (n-1)/2 \ \} \cup \{ \ N_+ = R/(x^{(n+1)/2} + \sqrt{-1}y), \ N_- = R/(x^{(n+1)/2} - \sqrt{-1}y) \ \}$$

is a complete list of the ones (cf. [8, Paragraph (9.9)]).

In this section, we shall show

Theorem 15. [3, Theorem 4.6] *Let $R = k[[x,y]]/(x^{n+1}+y^2)$. Then the stable hom order coincides with the stable degeneration order. Particularly, we have the following.*

(1) *If n is an even integer,*

$$\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \cdots \leq_{st} \underline{I_{n/2}}.$$

(2) *If n is an odd integer,*

$$\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \cdots \leq_{st} \underline{I_{(n-1)/2}} \leq_{st} \underline{N_+} \oplus \underline{N_-}.$$

and

$$\underline{N_\pm} \leq_{st} \underline{N_\pm} \oplus \underline{I_1} \leq_{st} \cdots \leq_{st} \underline{N_\pm} \oplus \underline{I_{(n-1)/2}} \leq_{st} \underline{N_\pm} \oplus \underline{N_+} \oplus \underline{N_-} \quad \textit{(double sign corresponds).}$$

To show this, we use the stable analogue of the argument over finite dimensional algebras in [11] The lemma below is well known for the case in an Abelian category (cf. [11, Lemma 2.6]). The same statement follows in an arbitrary k-linear triangulated category.

Lemma 16. [3, Lemma 4. 7] *Let*

$$\Sigma_1 : N_1 \xrightarrow{\binom{f_1}{v}} L_1 \oplus N_2 \xrightarrow{(u,g_1)} L_2 \longrightarrow N_1[1]$$

and

$$\Sigma_2 : M_1 \xrightarrow{\binom{f_2}{w}} N_1 \oplus M_2 \xrightarrow{(v,g_2)} N_2 \longrightarrow M_1[1]$$

be triangles in a k-linear triangulated category. Then we also have the following triangle.

$$M_1 \to L_1 \oplus M_2 \to L_2 \to M_1[1].$$

Definition 17. Let M and N be Cohen-Macaulay R-modules. We define a function $\delta_{\underline{M},\underline{N}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{M},\underline{N}}(-) = [-,\underline{N}] - [-,\underline{M}].$$

For a triangle $\underline{\Sigma} : \underline{L} \to \underline{M} \to \underline{N} \to \underline{L}[1]$, we also define a function $\delta_{\underline{\Sigma}}(-)$ on $\underline{\mathrm{CM}}(R)$ by

$$\delta_{\underline{\Sigma}}(-) = [-,\underline{L}] + [-,\underline{N}] - [-,\underline{M}].$$

Instead of giving the proof of Theorem 15, we give the concrete construction of the stable degenerations.

Example 18. Let $R = k[[x,y]]/(x^7 + y^2)$. Then $\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \underline{I_3}$.

Proof. We show $\underline{I_1} \leq_{st} \underline{I_2}$. We construct the triangle Σ such that

$$\delta_\Sigma = \delta_{\underline{I_1},\underline{I_2}}.$$

(This Σ induces the triangle which gives the stable degenerations.)

Note that the table of dimension of $\underline{\mathrm{Hom}}_R$ and AR triangles on $\underline{\mathrm{CM}}(R)$ are the following:

$[-,-]$	$\underline{I_1}$	$\underline{I_2}$	$\underline{I_3}$
$\underline{I_1}$	2	2	2
$\underline{I_2}$	2	4	4
$\underline{I_3}$	2	4	6

$$\begin{aligned} \Sigma_{\underline{I_1}} &: \underline{I_1} \to \underline{I_2} \to \underline{I_1}, \\ \Sigma_{\underline{I_2}} &: \underline{I_2} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}, \\ \Sigma_{\underline{I_3}} &: \underline{I_3} \to \underline{I_2} \oplus \underline{I_3} \to \underline{I_3}. \end{aligned}$$

We also note that $\delta_{\Sigma_{\underline{I_i}}}(\underline{X}) = 2$ if $\underline{X} = \underline{I_i}$, or 0 if not.

First, since $\delta_{\underline{I_1},\underline{I_2}}(\underline{X}) = \begin{cases} 0 & \text{if } \underline{X} = \underline{I_1}, \\ 2 & \text{if } \underline{X} = \underline{I_2}, \\ 2 & \text{if } \underline{X} = \underline{I_3} \end{cases}$ and $\mu(\underline{I_2}, \underline{I_i}) = 0$ if $i \neq 2$, we take the AR triangle $\Sigma_{\underline{I_2}} : \underline{I_2} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}$. Then we see that $\delta_{\Sigma_{\underline{I_2}}} < \delta_{\underline{I_1},\underline{I_2}}$.

Now $\delta_{\Sigma_{\underline{I_2}}}(\underline{I_3}) = 0$ but $\delta_{\underline{I_1},\underline{I_2}}(\underline{I_3}) = 2$. Apply Lemma 16 to $\Sigma_{\underline{I_2}}$ and $\Sigma_{\underline{I_3}}$, We obtain the triangle $\Sigma : \underline{I_3} \to \underline{I_1} \oplus \underline{I_3} \to \underline{I_2}$ such that $\delta_\Sigma = \delta_{\Sigma_{\underline{I_2}}} + \delta_{\Sigma_{\underline{I_3}}} = \delta_{\underline{I_1},\underline{I_2}}$. Therefore $\underline{I_1} \leq_{st} \underline{I_2}$. $\square$

Remark 19. By applying Lemma 16 to $\Sigma_{\underline{I_1}}$, $\Sigma_{\underline{I_2}}$ and $\Sigma_{\underline{I_3}}$, we have the diagram below:

$$\begin{array}{ccccccc} \underline{I_3} & \longrightarrow & \underline{I_2} & \longrightarrow & \underline{I_1} & \longrightarrow & \underline{0} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{I_3} & \longrightarrow & \underline{I_3} & \longrightarrow & \underline{I_2} & \longrightarrow & \underline{I_1}. \end{array}$$

Thus we obtain $\underline{0} \leq_{st} \underline{I_1} \leq_{st} \underline{I_2} \leq_{st} \underline{I_3}$.

Now we consider the following condition which is the necessary condition to make the stable hom order a partial order over hypersurface rings:

($*$) For an AR triangle $\underline{Z} \to \underline{Y} \to \underline{X} \to \underline{Z}[1]$, $[\underline{X}] + [\underline{Z}] - [\underline{Y}] = [\underline{X}[-1]] + [\underline{Z}[-1]] - [\underline{Y}[-1]]$ in $\mathrm{G}(\underline{\mathrm{CM}}(R))$.

More generally, on Theorem 15, we have the following result.

Theorem 20. [11, Paragraph (3.3)][3, Theorem 4.12] *Let R be a simple hypersurface singularity which satisfies ($*$). Then $\underline{M} \leq_{hom} \underline{N}$ if and only if $\underline{M} \leq_{st} \underline{N}$ for Cohen-Macaulay R-modules M and N with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$.*

The following lemma is known as the Knörrer's periodicity (cf. [8, Theorem 12.10]).

Lemma 21. *Let $S = k[[x_0, x_1, \cdots, x_n]]$ be a formal power series ring. For a non-zero element $f \in (x_0, x_1, \cdots, x_n)S$, we consider the two rings $R = S/(f)$ and $R^\sharp = S[[y, z]]/(f + y^2 + z^2)$. Then the stable categories $\underline{\mathrm{CM}}(R)$ and $\underline{\mathrm{CM}}(R^\sharp)$ are equivalent as triangulated categories.*

At the end of this note, we state the result on the stable degenerations over simple hypersurface singularities of type (A_n) of arbitrary dimension.

Theorem 22. *Let R be a simple hypersurface singularities of type (A_n). The following statements hold for Cohen-Macaulay R-modules M and N with $[\underline{M}] = [\underline{N}]$ in $K_0(\underline{\mathrm{CM}}(R))$.*

(1) *If R is of odd dimension then $\underline{M} \leq_{st} \underline{N}$ if and only if $\underline{M} \leq_{\underline{hom}} \underline{N}$.*
(2) *If R is of even dimension then $\underline{M} \leq_{st} \underline{N}$ if and only if $\underline{M} \leq_{tri} \underline{N}$.*

Proof. Since stable degenerations are preserved by stable categorical equivalences (cf. [10, Corollary 6.6]), by virtue of Knörrer's periodicity (Lemma 21), we have only to deal with the case $\dim R = 1$ to show (1) and the case $\dim R = 0$ to show (2). In the case of dimension 0, we know the stable degeneration order coincides with the triangle order on $\underline{\mathrm{CM}}(R)$ (see [4, Corollary 2.12., Proposition 3.10.]). Hence, by Theorem 15, we obtain the assertion. $\square$

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FINITE CONDITION (FG) FOR SELF-INJECTIVE KOSZUL ALGEBRAS

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Abstract. We consider a finite-dimensional algebra over an algebraically closed field k. For a relationship between a cogeometric pair (E, σ) and the finiteness condition denoted by (Fg), the following conjecture is proposed by Mori. Let A be a finite dimensional cogeometric self-injective Koszul algebra such that the complexity of $A/\mathrm{rad}\, A$ is finite. Then A satisfies the condition (Fg) if and only if the order of σ is finite. In this article, we show that if A is cogeometric and satisfies the condition (Fg), then the order of σ is finite. Also, in the case of $E = \mathbb{P}^{n-1}$, we show that this conjecture holds. Moreover, if A satisfies $(\mathrm{rad}\, A)^4 = 0$, then we show that this conjecture holds.

1. Introduction

For a finite-dimensional algebra Λ over an algebraically closed field k, Erdmann, Holloway, Taillefer, Snashall and Solberg [3] introduced certain finiteness condition, denoted by (Fg), by using the Ext algebra of Λ and the Hochschild cohomology ring of Λ. Moreover, Erdmann et al. showed that if Λ satisfies the finiteness condition (Fg), then the support varieties defined by Snashall and Solberg [8] have many properties analogous to those for finite group algebras.

Let A be a graded algebra finitely generated in degree 1 over a field k. Artin, Tate and Van den Bergh [2] introduced a point-module over A which play an important role in studying A in noncommutative algebraic geometry. Mori [6] defined a co-point module over A which is a dual notion of point module introduced by Artin, Tate and Van den Bergh in terms of Koszul duality. A co-point module is parameterized by a subset E of a projective space $\mathbb{P}^{n-1}$. If M_p is a co-point module corresponding to a point $p \in E$, then ΩM_p is also a co-point module. Therefore, there exists a map $\sigma : E \longrightarrow E$ such that $\Omega M_p = M_{\sigma(p)}$. This pair (E, σ) is called a cogeometric pair and, when E is a projective scheme and σ is an automorphism of E, A is called a cogeometric algebra ([6]).

In this article, we consider a finite-dimensional algebra over an algebraically closed field of characteristic 0. For a relationship between a cogeometric pair (E, σ) and the finiteness condition (Fg), the following conjecture is proposed by Mori:

Conjecture by Mori Let A be a cogeometric self-injective Koszul algebra such that the complexity of k is finite. Then A satisfies the condition (Fg) if and only if the order of σ is finite.

The detailed version of this paper will be submitted for publication elsewhere in [5].

If this Mori's conjecture is true, we only need to calculate the order of σ to check whether A satisfies the finiteness condition (Fg) or not without calculating the Ext algebra of A and the Hochschild cohomology ring of A.

In this article, our main results are to give a partial answer to this Mori's conjecture. That is, we show that if A is cogeometric and satisfies the condition (Fg), then the order of σ is finite (see Theorem 9). Also, in the case of $E = \mathbb{P}^{n-1}$, we show that this conjecture is true (see Theorem 10). Moreover, if A satisfies $(\operatorname{rad} A)^4 = 0$, then we show that this conjecture is true (see Theorem 12).

2. Finiteness condition (Fg)

In this section, we recall the definitions of the finiteness condition (Fg) by [3] and the complexity of a module.

For any finite-dimensional algebra Λ, Erdmann, Holloway, Taillefer, Snashall and Solberg [3] have introduced the finiteness condition (Fg) as follows.

Definition 1. ([3]) A finite-dimensional algebra Λ satisfies the finiteness condition (Fg) if there is a graded subalgebra H of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ such that the following two conditions (Fg1) and (Fg2) hold:

(Fg1): H is a commutative Noetherian ring with $H^0 = \mathrm{HH}^0(\Lambda)(= \mathrm{Z}(\Lambda))$.

(Fg2): The Ext algebra of Λ

$$\mathrm{E}(\Lambda) := \mathrm{Ext}^*_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda) = \bigoplus_{r=0}^{\infty} \mathrm{Ext}^r_\Lambda(\Lambda/\operatorname{rad}\Lambda, \Lambda/\operatorname{rad}\Lambda)$$

is a finitely generated H-module.

Here, the Hochschild cohomology ring of Λ is defined to be a graded ring

$$\mathrm{HH}^*(\Lambda) := \mathrm{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda) = \bigoplus_{r=0}^{\infty} \mathrm{Ext}^r_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda),$$

where $\Lambda^{\mathrm{e}} := \Lambda \otimes_k \Lambda^{\mathrm{op}}$ is the enveloping algebra of Λ.

For example, a group algebra kG satisfies (Fg), where k is a field and G is a finite group. A quantum exterior algebra $k\langle x_1, x_2, \dots, x_n\rangle/(x_i^2, x_ix_j - \alpha_{i,j}x_jx_i)$ ($0 \le i, j \le n$, $\alpha_{i,j} \in k\backslash\{0\}$) satisfies (Fg) when $\alpha_{i,j}$ is a root of unity ([4]).

Remark 2. Note that if a finite-dimensional algebra Λ satisfies (Fg), then the Hochschild cohomology rings $\mathrm{HH}^*(\Lambda)/\mathcal{N}_\Lambda$ of Λ modulo nilpotence and the Ext algebra $\mathrm{E}(\Lambda)$ of Λ are finitely generated as algebras.

Now, we recall the definition of the complexity of a left Λ-module M (see [9], for example).

Definition 3. Let $\cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective resolution of M. Then the *complexity* of M is

$$\mathrm{cx}(M) := \min\{b \in \mathbb{N}^0 \mid \exists a \in \mathbb{R}; \dim P_n = an^{b-1}, \forall n \gg 0\}.$$

This definition leads immediately to the following two remarks:

Remark 4. (1) If a left Λ-module M has the complexity zero, then M has finite projective dimension. (2) If a module M satisfies $\mathrm{cx}(M) \leq 1$, then M has bounded Betti numbers, $\beta_n = \dim_k P_n$ for $n \geq 0$.

3. Cogeometric pair and cogeometric algebra

In this section, we will summarize the definitions of a co-point module, a cogeometric pair and a cogeometric algebra from [6]. Mori defined a co-point module over A which is a dual notion of a point module introduced by Artin, Tate and Van den Bergh [2] in terms of Koszul duality. The reader is referred to [6] in details.

Let A be a graded algebra finitely generated in degree 1 over an algebraically closed field k of characteristic 0. That is, $A = k\langle x_1, x_2, \ldots, x_n\rangle/I$, where I is an ideal of A. For a point $p = (a_1, a_2, \ldots, a_n) \in \mathbb{P}^{n-1}$, we define a left A-homomorphism $p : A \longrightarrow A$ by $p(1) := a_1x_1 + a_2x_2 + \cdots + a_nx_n$ $(1 \in A)$. That is, for all $f \in A$, we have $p(f) = f \cdot (a_1x_1 + a_2x_2 + \cdots + a_nx_n)$. Also, $\operatorname{Coker} p$ is denoted by M_p.

Definition 5. ([6]) A left A-module M is called a *co-point module* if, for every $i \in \mathbb{N}$, there exists a point p_i in $\mathbb{P}^{n-1}$ such that a minimal free resolution of M is as follows:

$$\cdots \xrightarrow{p_i} A \xrightarrow{p_{i-1}} \cdots \xrightarrow{p_2} A \xrightarrow{p_1} A \xrightarrow{p_0} A \xrightarrow{\varepsilon} M \longrightarrow 0.$$

By the definition, we have that $M = M_{p_0}$ and the i-th syzygy $\Omega^i M$ of M is M_{p_i} for $i \geq 1$. Note that a co-point module M has Betti numbers $\beta_n = 1$ $(\forall n \in \mathbb{N})$.

A co-point module is parameterized by a subset E of a projective space $\mathbb{P}^{n-1}$. If M_p is a co-point module corresponding to a point $p \in E$, then ΩM_p is also a co-point module. Therefore, there exists a map $\sigma : E \longrightarrow E$ such that $\Omega M_p = M_{\sigma(p)}$.

Definition 6. ([6]) When E is a projective scheme and σ is an automorphism of E, the pair (E, σ) is called a *cogeometric pair* of A and A is called a *cogeometric algebra*. In this case, we write $\mathcal{P}^!(A) = (E, \sigma)$ and $A = \mathcal{A}^!(E, \sigma)$, respectively.

Using [2], [7] and [6], we have the following theorem.

Theorem 7. *If a graded k-algebra A is self-injective Koszul such that the complexity of $A/\operatorname{rad} A$ is finite and $(\operatorname{rad} A)^4 = 0$, then the complexity of $A/\operatorname{rad} A$ is less than or equal to three and A is a cogeometric algebra.*

Example 8. Let A be a graded k-algebra

$$k\langle x, y\rangle/(x^2, \alpha xy + yx, y^2) \quad (\alpha \in k\backslash\{0\}).$$

This algebra A is self-injective Koszul with $\mathrm{cx}(A/\operatorname{rad} A) = 2$. By Theorem 7, A is cogeometric. Therefore, we have that the cogeometric pair $\mathcal{P}^!(A)$ of A is $(\mathbb{P}^1, \sigma)$, where

$$\sigma := \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \in \operatorname{Aut} \mathbb{P}^1 = \mathrm{PGL}_2(k).$$

4. Main results

This section describes our main results in this article about the conjecture proposed by Mori. Recall the Mori's conjecture as follows:

Conjecture by Mori Let A be a cogeometric self-injective Koszul algebra such that the complexity of k is finite. Then A satisfies the condition (Fg) if and only if the order of σ is finite.

As stated in Introduction, if this Mori's conjecture is true, we only need to calculate the order of σ to check whether A satisfies the finiteness condition (Fg) or not without calculating the Ext algebra of A and the Hochschild cohomology ring of A. Our main results are to give a partial answer to this Mori's conjecture.

Theorem 9. ([5]) *If a finite-dimensional k-algebra $A = \mathcal{A}^!(E, \sigma)$ is cogeometric and satisfies the condition* (Fg)*, then the order of σ is finite.*

Theorem 10. ([5]) *Let A be a cogeometric self-injective Koszul algebra such that the complexity of k is finite. If $A = \mathcal{A}^!(\mathbb{P}^{n-1}, \sigma)$, then A satisfies the condition* (Fg) *if and only if the order of σ is finite.*

Using [1], [7] and [6], we have the following classification of self-injective Koszul algebras A of $\operatorname{cx}(A/\operatorname{rad} A) < \infty$:

(i) $\operatorname{rad} A = 0 \rightsquigarrow A \cong k$ (as a graded k-algebra), $\mathcal{P}^!(A) = (\phi, \mathrm{id})$;
(ii) $(\operatorname{rad} A)^2 = 0 \rightsquigarrow A \cong k[x]/(x^2)$ (as a graded k-algebra), $\mathcal{P}^!(A) = (\mathbb{P}^0, \mathrm{id})$;
(iii) $(\operatorname{rad} A)^3 = 0 \rightsquigarrow$

$$\begin{cases} A \cong k\langle x, y\rangle/(x^2, \alpha xy + yx, y^2)(\alpha \in k\backslash\{0\}, \text{ as a graded } k\text{-algebra}), \mathcal{P}^!(A) = (\mathbb{P}^1, \sigma_1), \\ A \cong k\langle x, y\rangle/(-x^2 + xy, xy + yx, y^2)(\text{as a graded } k\text{-algebra}), \mathcal{P}^!(A) = (\mathbb{P}^1, \sigma_2), \end{cases}$$

where $\sigma_1 = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

In the case of $(\operatorname{rad} A)^3 = 0$, we see that E giving a cogeometric pair is a projective space. By the above classification and Theorem 10, we have the following corollary.

Corollary 11. ([5]) *Let A be a cogeometric self-injective Koszul algebra such that the complexity of k is finite. If A satisfies $(\operatorname{rad} A)^3 = 0$, then A satisfies the condition* (Fg) *if and only if the order of σ is finite.*

Theorem 12. ([5]) *Let A be a cogeometric self-injective Koszul algebra such that the complexity of k is finite. If A satisfies $(\operatorname{rad} A)^4 = 0$, then A satisfies the condition* (Fg) *if and only if the order of σ is finite.*

Remark 13. Note that if A satisfies $(\operatorname{rad} A)^3 \neq 0$ and $(\operatorname{rad} A)^4 = 0$, then E giving a cogeometric pair is not always a projective space.

We conclude this article by giving an example for our main results.

Example 14. We consider a graded k-algebra

$$A = k\langle x, y\rangle/(ax^2 + byx, cx^2 + axy + dyx + by^2, cxy + dy^2) \quad (a, b, c, d \in k).$$

Then A is a self-injective Koszul algebra if and only if $ad - bc \neq 0$ holds. Hence, A is cogeometric, so we have

$$\mathcal{P}^!(A) = (\mathbb{P}^1, \sigma) \quad (\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(k) = \operatorname{Aut} \mathbb{P}^1).$$

Moreover, using Corollary 11, A satisfies (Fg) if and only if there exists a natural number $n \in \mathbb{N}$ such that $\sigma^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

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CROSSED PRODUCTS FOR MATRIX RINGS

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Abstract. Let R be a ring and $n \geq 2$ an integer. We provide a systematic way to define new multiplications on $\mathrm{M}_n(R)$, the ring of $n \times n$ full matrices with entries in R. The obtained new rings Λ are Auslander-Gorenstein if and only if so is R.

Introduction

Auslander-Gorenstein rings (see Definition 2) appear in various fields of current research in mathematics. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [3], [4] and [12], respectively). The class of Auslander-Gorenstein rings contains two important particular classes of rings. One is the class of quasi-Frobenius rings and the other is the class of commutative Gorenstein rings. In [1, Section 3] and [9, Section 4] we have provided various constructions of Auslander-Gorenstein rings. In this note, we will provide a systematic construction of Auslander-Gorenstein rings starting from an arbitrary Auslander-Gorenstein ring.

Let us recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [10, 11] which we modify as follows (cf. [1, Section 1]). We use the notation A/R to denote that a ring A contains a ring R as a subring. We say that A/R is a Frobenius extension if the following conditions are satisfied: (F1) A is finitely generated as a left R-module; (F2) A is finitely generated projective as a right R-module; (F3) there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A. Note that ϕ induces a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism of R. Note that a Frobenius extension of first kind is a special case of a Frobenius extension of second kind. Let A/R be a Frobenius extension. Then A is an Auslander-Gorenstein ring if so is R, and the converse holds true if A is projective as a left R-module, and if A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules. It should be noted that A is projective as a left R-module if A/R is of second kind.

Fix a set of integers $I = \{0, 1, \dots, n-1\}$ with $n \geq 2$ arbitrary. To begin with, starting from an arbitrary ring R, we will construct an I-graded ring A so that A/R is a split Frobenius extension of second kind. Namely, we will define an appropriate multiplication on a free right R-module A with a basis $\{u_i\}_{i \in I}$ using the following two data: a certain pair (q, χ) of an integer q and a mapping $\chi : \mathbb{Z}_+ \to \mathbb{Z}$; a certain triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$

The detailed version of this paper will be submitted for publication elsewhere.

and $c, t \in R^\sigma$, the fixed subring of R under σ. Then we will define an appropriate multiplication on a free right A-module Λ with a basis $\{v_i\}_{i \in I}$ so that Λ/A is a Frobenius extension of first kind. To do so, we need the group structure of I. Since we have to distinguish the addition in I and that in $\mathbb{Z}_+$, we fix a cyclic permutation of I

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

and make I a cyclic group with 0 the unit element by the law of composition $I \times I \to I, (i, j) \mapsto \pi^j(i)$. Then, as a right R-module, Λ has a basis $\{e_{ij}\}_{i,j \in I}$ such that $e_{ij}e_{kl} = 0$ unless $j = k$, $e_{ij}e_{jk} = e_{ik}c_{ijk}$ with $c_{ijk} \in R$ for all $i, j, k \in R$ and $xe_{ij} = e_{ij}\sigma_{ij}(x)$ with $\sigma_{ij} \in \mathrm{Aut}(R)$ for all $x \in R$ and $i, j \in I$. Using the above two data, we will provide a concrete construction of such families $\{c_{ijk}\}_{i,j,k \in I}$ and $\{\sigma_{ij}\}_{i,j \in I}$. However, except very simple cases, it would be a rather hard task to find out such families independently. This is the reason why we divide the construction into two steps.

1. Preliminaries

For a ring R we denote by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R, by $\mathrm{Aut}(R)$ the group of ring automorphisms of R, and for $\sigma \in \mathrm{Aut}(R)$ by R^σ the subring of R consisting of all $x \in R$ with $\sigma(x) = x$. The identity element of a ring is simply denoted by 1. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}).

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander). *Let R be a right and left noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

(1) *In a minimal injective resolution $I^\bullet$ of R in* Mod-R, flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.

(2) *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}, flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.

(3) *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M, R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X, R) = 0$ *for all* $0 \leq j < i$.

(4) *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X, R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M, R) = 0$ *for all* $0 \leq j < i$.

Definition 2 ([4]). A right and left noetherian ring R is said to satisfy the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and to be an Auslander-Gorenstein ring if it satisfies the Auslander condition and inj dim $R =$ inj dim $R^{\mathrm{op}} < \infty$.

It should be noted that for a right and left noetherian ring R we have inj dim $R =$ inj dim R^{op} whenever inj dim $R < \infty$ and inj dim $R^{\mathrm{op}} < \infty$ (see [13, Lemma A]).

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [10, 11], which we modify as follows (cf. [1, 7]).

Definition 3 ([7]). A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:
(F1) A is finitely generated as a left R-module;
(F2) A is finitely generated projective as a right R-module;
(F3) $A \cong \mathrm{Hom}_R(A, R)$ as right A-modules.

Proposition 4 ([7, Proposition 1.4]). *Let A/R be a ring extension and $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ an isomorphism in* Mod-A*. Then the following hold.*

(1) *There exists a unique ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$.*
(2) *If $\phi' : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ is another isomorphism in* Mod-A*, then there exists $u \in A^\times$ such that $\phi'(1) = \phi(1)u$ and $\theta'(x) = u^{-1}\theta(x)u$ for all $x \in R$.*
(3) *ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*

Definition 5 (cf. [10, 11]). A Frobenius extension A/R is said to be of first kind if $A \cong \mathrm{Hom}_R(A, R)$ as R-A-bimodules, and to be of second kind if there exists an isomorphism $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$ in Mod-A such that the associated ring homomorphism $\theta : R \to A$ induces a ring automorphism $\theta : R \xrightarrow{\sim} R$.

Proposition 6 ([7, Proposition 1.6]). *If A/R is a Frobenius extension of second kind, then A is projective as a left R-module.*

Proposition 7 ([7, Proposition 1.7]). *For any Frobenius extensions Λ/A, A/R the following hold.*

(1) *Λ/R is a Frobenius extension.*
(2) *Assume Λ/A is of first kind. If A/R is of second (resp., first) kind, then so is Λ/R.*

Definition 8 ([1]). A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 9 ([7, Proposition 1.9]). *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring, then so is A with* inj dim $A \le$ inj dim R.
(2) *Assume A is projective as a left R-module and A/R is split. If A is an Auslander-Gorenstein ring, then so is R with* inj dim $R =$ inj dim A.

2. Construction

Throughout the rest of this note, we fix a ring R and an integer $n \ge 2$. We will provide a systematic way to define new multiplications on $\mathrm{M}_n(R)$, the ring of $n \times n$ full matrices with entries in R (cf. Theorem 15 below). However, we divide the construction into two steps, i.e., we will construct two ring extensions A/R and Λ/A such that $\Lambda \cong \mathrm{M}_n(R)$ as right R-modules. Except very simple cases, the direct construction would be a rather hard task (cf. [1] and [6]).

We need the following two data. Let $I = \{0, 1, \ldots, n-1\}$ be a set of integers and $\mathbb{Z}_+$ the set of non-negative integers. We fix a pair (q, χ) of an integer $q \in \mathbb{Z}$ and a mapping $\chi : \mathbb{Z}_+ \to \mathbb{Z}$ satisfying the following conditions:

(X0) $\chi(0) = 0$;

(X1) $\chi(i + kn) = \chi(i) + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$;

(X2) $\chi(i) + \chi(j) \geq \chi(i + j)$ for all $i, j \in \mathbb{Z}_+$.

Also, we fix a triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R^\sigma$ satisfying the following condition:

$$(*) \quad xc = c\sigma(x), xt = t\sigma^q(x) \text{ for all } x \in R.$$

It should be noted that $ct = tc$.

Example 10. For any pair of integers (p, q) with $np \geq q$, setting

$$\chi(i + kn) = ip + kq$$

for $(i, k) \in I \times \mathbb{Z}_+$, we have a pair (q, χ) satisfying (X0), (X1) and (X2) and, setting

$$\varrho = \sigma^p \in \mathrm{Aut}(R) \quad \text{and} \quad s = tc^{np-q} \in R,$$

we have $s \in R^\varrho$ and $xs = s\varrho^n(x)$ for all $x \in R$.

Example 11. For any $c \in R^\times$ and $s \in \mathrm{Z}(R)$, setting $\sigma(x) = c^{-1}xc$ for $x \in R$ and $t = sc^q$ with $q \in \mathbb{Z}$ arbitrary, we have a triple (σ, c, t) of $\sigma \in \mathrm{Aut}(R)$ and $c, t \in R^\sigma$ satisfying the condition (*).

At first, we will construct a split ring extension A/R. Let A be a free right R-module with a basis $\{u_i\}_{i \in I}$. We set

$$u_{i+kn} = u_i t^k$$

for $(i, k) \in I \times \mathbb{Z}_+$ and

$$\omega(i, j) = \chi(i) + \chi(j) - \chi(i + j)$$

for $i, j \in \mathbb{Z}_+$. Note that ω is symmetric, i.e., $\omega(i, j) = \omega(j, i)$ for all $i, j \in \mathbb{Z}_+$ and that $\omega(i + kn, j + ln) = \omega(i, j)$ for all $(i, k), (j, l) \in I \times \mathbb{Z}_+$. Since by (X2) $\omega(i, j) \geq 0$ for all $i, j \in \mathbb{Z}_+$, we can define a multiplication on A subject to the following axioms:

(A1) $u_i u_j = u_{i+j} c^{\omega(i,j)}$ for all $i, j \in \mathbb{Z}_+$;

(A2) $x u_i = u_i \sigma^{\chi(i)}(x)$ for all $x \in R$ and $i \in \mathbb{Z}_+$,

where as usual we require $a^0 = 1$ for $a \in R$ even if $a = 0$. We denote by $\{\delta_i\}_{i \in I}$ the dual basis of $\{u_i\}_{i \in I}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, i.e., $a = \sum_{i \in I} u_i \delta_i(a)$ for all $a \in A$. Then for any $a, b \in A$ we have

$$ab = \sum_{i,j \in I} u_{i+j} c^{\omega(i,j)} \sigma^{\chi(j)}(\delta_i(a)) \delta_j(b).$$

Proposition 12 ([8, Proposition 2.3(1)]). *A is an associative ring with $1 = u_0$ and contains R as a subring via the injective ring homomorphism $R \to A, x \mapsto u_0 x$.*

Example 13. Consider the case where $q = n - 1$ and $\chi(i + kn) = i + kq$ for all $(i, k) \in I \times \mathbb{Z}_+$. Then $\omega(i, j) = 0$ if $i + j < n$ and $\omega(i, j) = 1$ otherwise for $i, j \in I$. Also, $x(tc) = (tc)\sigma^n(x)$ for all $x \in R$. Let $R[X; \sigma]$ be a right skew polynomial ring with trivial derivation, i.e., $R[X; \sigma]$ consists of all polynomials in an indeterminate X with the right-hand coefficients in R and the multiplication is defined subject to the following rule: $xX = X\sigma(x)$ for all $x \in R$. It then follows that $(X^n - tc) = (X^n - tc)R[X;\sigma]$ is a two-sided ideal of $R[X;\sigma]$ and $A \cong R[X;\sigma]/(X^n - tc)$ as extension rings of R.

Next, we will specialize the construction given in [7, Section 2] and construct a ring extension Λ/A. To do so, we need the group structure of I. In order to distinguish the addition in I and that in $\mathbb{Z}_+$, we fix a cyclic permutation of I

$$\pi = \begin{pmatrix} 0 & 1 & \cdots & n-1 \\ 1 & 2 & \cdots & 0 \end{pmatrix}$$

and make I a cyclic group with 0 the unit element by the law of composition $I \times I \to I, (i,j) \mapsto \pi^j(i)$. It should be noted that

$$i + j = \begin{cases} \pi^j(i) & \text{if } i+j < n, \\ \pi^j(i) + n & \text{if } i + j \geq n \end{cases}$$

for all $i, j \in I$. Thus, setting

$$\epsilon(i,j) = \begin{cases} 0 & \text{if } i+j < n, \\ 1 & \text{if } i + j \geq n \end{cases}$$

for $i, j \in I$, we have $i + j = \pi^j(i) + \epsilon(i,j)n$ and hence

$$\chi(i+j) = \chi(\pi^j(i)) + \epsilon(i,j)q \quad \text{and} \quad u_{i+j} = u_{\pi^j(i)} t^{\epsilon(i,j)}$$

for all $i, j \in I$.

Setting $A_i = u_i R$ for $i \in I$, $A = \oplus_{i \in I} A_i$ yields an I-graded ring with $A_0 = R$. Note however that in the above A is constructed as a residue ring of a positively graded ring (cf. Example 13 above).

We denote by $\varepsilon_i : A \to A_i, a \mapsto u_i \delta_i(a)$ the projection for each $i \in I$. Then the following conditions are satisfied:

(E1) $\varepsilon_i \varepsilon_j = 0$ unless $i = j$ and $\sum_{i \in I} \varepsilon_i = \mathrm{id}_A$;

(E2) $\varepsilon_i(a)\varepsilon_j(b) = \varepsilon_{\pi^j(i)}(\varepsilon_i(a)b)$ for all $a, b \in A$ and $i, j \in I$.

Let Λ be a free right A-module with a basis $\{v_i\}_{i \in I}$ and define a multiplication on Λ subject to the following axioms:

(L1) $v_i v_j = 0$ unless $i = j$ and $v_i^2 = v_i$ for all $i \in I$;

(L2) $a v_i = \sum_{j \in I} v_j \varepsilon_{\pi^{-i}(j)}(a)$ for all $a \in A$ and $i \in I$.

Let us denote by $\{\gamma_i\}_{i \in I}$ the dual basis of $\{v_i\}_{i \in I}$ for the free left A-module $\mathrm{Hom}_A(\Lambda, A)$, i.e., $\lambda = \sum_{i \in I} v_i \gamma_i(\lambda)$ for all $\lambda \in \Lambda$. It is not difficult to see that

$$\lambda\mu = \sum_{i,j \in I} v_i \varepsilon_{\pi^{-j}(i)}(\gamma_i(\lambda))\gamma_j(\mu)$$

for all $\lambda, \mu \in \Lambda$.

Proposition 14. *Λ is an associative ring with $1 = \sum_{i \in I} v_i$ and contains A as a subring via the injective ring homomorphism $A \to \Lambda, a \mapsto \sum_{i \in I} v_i a$.*

Note that $\{v_i u_j\}_{i,j \in I}$ is a basis for the free right R-module Λ with $\{\delta_j \gamma_i\}_{i,j \in I}$ the dual basis for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, and that for any $i \in I$ by (L2) we have $x v_i = v_i x$ for all $x \in R$ and hence Λv_i is a Λ-R-bimodule. Similarly, every $v_i \Lambda$ is an

R-Λ-bimodule. Also, by (L2) $u_k v_j = v_{\pi^k(j)} u_k$ for all $j, k \in I$, so that $v_i \Lambda v_j = v_i u_{\pi^{-j}(i)} R$ and

$$\mathrm{Hom}_\Lambda(v_j\Lambda, v_i\Lambda) \xrightarrow{\sim} R, f \mapsto \delta_{\pi^{-j}(i)}(\gamma_i(f(v_j)))$$

as R-R-bimodules for all $i, j \in I$. In particular,

$$\mathrm{End}_\Lambda(v_i\Lambda) \xrightarrow{\sim} R, f \mapsto \delta_0(\gamma_i(f(v_i)))$$

as rings for all $i \in I$.

Now, setting $e_{ij} = v_i u_{\pi^{-j}(i)}$ for $i, j \in I$, we have a basis $\{e_{ij}\}_{i,j\in I}$ for the free right R-module Λ. Then, as remarked above, we have $v_i \Lambda v_j = e_{ij} R$ for all $i, j \in I$ and $\{\delta_{\pi^{-j}(i)}\gamma_i\}_{i,j\in I}$ is the duel basis of $\{e_{ij}\}_{i,j\in I}$ for the free left R-module $\mathrm{Hom}_R(\Lambda, R)$, i.e.,

$$\lambda = \sum_{i,j\in I} e_{ij}\delta_{\pi^{-j}(i)}(\gamma_i(\lambda))$$

for all $\lambda \in \Lambda$. In particular,

$$\rho : \Lambda \xrightarrow{\sim} \mathrm{M}_n(R), \lambda \mapsto (\delta_{\pi^{-j}(i)}(\gamma_i(\lambda)))_{i,j\in I}$$

as right R-modules.

Theorem 15. *The multiplication in Λ is subject to the following axioms:*
(M1) $e_{ij}e_{kl} = 0$ unless $j = k$;
(M2) $e_{ij}e_{jk} = e_{ik}t^{\epsilon(\pi^{-j}(i),\pi^{-k}(j))}c^{\omega(\pi^{-j}(i),\pi^{-k}(j))}$ for all $i, j, k \in I$;
(M3) $xe_{ij} = e_{ij}\sigma^{\chi(\pi^{-j}(i))}(x)$ for all $x \in R$ and $i, j \in I$.

It should be noted that the axioms above induce another multiplication on $\mathrm{M}_n(R)$ such that ρ is a ring isomorphism.

In the following, setting

$$\Delta_k = \{(i, j) \in I \times I \mid \pi^{-j}(i) = k\}$$

for $k \in I$, we decompose $I \times I$ into a disjoint unioun $I \times I = \cup_{k\in I}\Delta_k$. Note that the Δ_k are π-stable, i.e., $(\pi(i), \pi(j)) \in \Delta_k$ for all $k \in I$ and $(i, j) \in \Delta_k$.

Lemma 16. *We have $v_k = e_{kk}$ and $u_k = \sum_{(i,j)\in\Delta_k} e_{ij}$ for all $k \in I$.*

It follows by Lemma 16 that the axioms (M1), (M2) and (M3) imply the axioms (A1), (A2), (L1) and (L2). Unfortunately, it would be rather hard to check directly that the multiplication defined on Λ subject to the axioms (M1), (M2) and (M3) is actually associative. This is the reason why we divided the construction into two steps. However, we notice the following which could be of use for a direct verification.

Lemma 17. *For any $i, j, k \in I$ the following hold.*
1. *$\epsilon(\pi^{-j}(i), \pi^{-k}(j)) = 0$ if and only if one of the following cases occurs: $i < k < j$, $j < i < k$, $k < j < i$, $i = j$ or $j = k$.*
2. *$\epsilon(\pi^{-j}(i), \pi^{-k}(j)) = 1$ if and only if one of the following cases occurs: $i < j < k$, $k < i < j$, $j < k < i$ or $i = k \neq j$.*
3. *$\chi(\pi^{-j}(i) + \pi^{-k}(j)) = \chi(\pi^{-k}(i)) + \epsilon(\pi^{-j}(i), \pi^{-k}(j))q$.*

Example 18. Let (q, χ) and (ϱ, s) be as in Example 10. Then we have $e_{ij}e_{jk} = e_{ik}s^{\epsilon(\pi^{-j}(i),\pi^{-k}(j))}$ for all $i, j, k \in I$ and $xe_{ij} = e_{ij}\varrho^{\pi^{-j}(i)}(x)$ for all $x \in R$ and $i, j \in I$.

According to Theorem 15, it is easy to see that there exists $\eta \in \mathrm{Aut}(\Lambda)$ such that $\eta(e_{ij}) = e_{\pi(i),\pi(j)}$ for all $i, j \in I$ and $\eta(x) = x$ for all $x \in R$.

Proposition 19. *We have* $\Lambda^\eta = A$.

Definition 20. Let J be a non-empty subset of I and $\Lambda_J = \oplus_{i,j\in J} e_{ij} R$. We define a multiplication on Λ_J subject to the axioms (M1), (M2) and (M3). Then $\mathrm{End}_\Lambda(\oplus_{i\in J} v_i\Lambda) \cong \oplus_{i,j\in J} v_i \Lambda v_j = \Lambda_J$ as rings.

3. Frobenius extensions

In this section, we will study the structure of the ring extension Λ/R.

In the following, we set $\gamma = \sum_{i\in I} \gamma_i \in \mathrm{Hom}_A(\Lambda, A)$.

Proposition 21. *There exists an isomorphism* $\psi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_A(\Lambda, A), \lambda \mapsto \gamma\lambda$ *of* A-Λ-*bimodules, i.e.,* Λ/A *is a Frobenius extension of first kind.*

Proposition 22 ([7, Proposition 2.3(2)])**.** *There exists an isomorphism* $\phi : A \xrightarrow{\sim} \mathrm{Hom}_R(A, R), a \mapsto \delta_{n-1} a$ *in* Mod-A *such that* $x\phi(1) = \phi(1)\sigma^{-\chi(n-1)}(x)$ *for all* $x \in R$, *i.e.,* A/R *is a split Frobenius extension of second kind.*

Corollary 23. *A is an Auslander-Gorenstein ring if and only if so is R.*

Theorem 24. *The following hold.*

(1) $v_i\Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda v_{\pi(i)}, R), \lambda \mapsto \delta_{n-1}\gamma_i\lambda$ *in* Mod-Λ *for all* $i \in I$.
(2) Λ/R *is a split Frobenius extension of second kind.*
(3) Λ *is an Auslander-Gorenstein ring if and only if so is* R.

Remark 25. It follows by Corollary 23 and Theorem 24(3) that Λ is an Auslander-Gorenstein ring if and only if so is A. If $n \cdot 1 \in R^\times$, then Λ/A is split. But we do not know whether or not Λ/A is always split.

It follows by Propositions 21, 22 that we have an isomorphism in Mod-Λ

$$\xi : \Lambda \xrightarrow{\sim} \mathrm{Hom}_R(\Lambda, R), \lambda \mapsto \delta_{n-1}\gamma\lambda$$

such that $x\xi(1) = \xi(1)\sigma^{-\chi(n-1)}(x)$ for all $x \in R$.

Proposition 26. *We have* $\xi(e_{ij}) = \delta_{\pi^{-\pi(i)}(j)}\gamma_j$ *for all* $i, j \in I$.

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ATOM-MOLECULE CORRESPONDENCE IN GROTHENDIECK CATEGORIES

RYO KANDA

Abstract. For a one-sided noetherian ring, Gabriel constructed two maps between the isomorphism classes of indecomposable injective modules and the two-sided prime ideals. In this paper, we give a categorical reformulation of these maps using the notion of Grothendieck category. Gabriel's maps become maps between the atom spectrum and the molecule spectrum in our setting, and these two spectra have structures of partially ordered sets. The main result in this paper shows that the two maps induce a bijection between the minimal elements of the atom spectrum and those of the molecule spectrum.

Key Words: Grothendieck category, Atom spectrum, Molecule spectrum, Prime ideal, Indecomposable injective object.

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1. Introduction

For a one-sided noetherian ring, Gabriel [1] described the relationship between indecomposable injective modules and two-sided prime ideals as follows.

Theorem 1 ([1]). *Let Λ be a right noetherian ring. Then we have two maps*

$$\frac{\{\,\text{indecomposable injective right } \Lambda\text{-modules}\,\}}{\cong} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\,\text{two-sided prime ideals of } \Lambda\,\}$$

characterized by the following properties.

(1) *For each indecomposable injective right Λ-module I, the only associated (two-sided) prime of I is $\varphi(I)$.*
(2) *For each two-sided prime ideal P of Λ, the injective envelope $E(\Lambda/P)$ of the right Λ-module Λ/P is the direct sum of a finite number of copies of the indecomposable injective Λ-module $\psi(P)$.*

Moreover, the composite $\varphi\psi$ is the identity map.

If the ring is commutative, then these maps are bijective ([7, Proposition 3.1]). In general, these maps are far from being bijective as the following example shows.

Example 2. The ring $\Lambda := \mathbb{C}\langle x, y\rangle/(xy - yx - 1)$ is a simple domain which is left and right noetherian. The only two-sided prime ideal of Λ is 0, while there exist infinitely many isomorphism classes of indecomposable injective right Λ-modules.

This is not in final form. The detailed version of this paper will be submitted for publication elsewhere.

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In this paper, we generalize Theorem 1 to a certain class of Grothendieck categories as maps between the *atom spectrum* and the *molecule spectrum* of a Grothendieck category. Moreover, by using naturally defined partial orders on these spectra, we establish a bijection between the minimal elements of the atom spectrum and those of the molecule spectrum. This result would have been unknown even in the case of the category $\operatorname{Mod} \Lambda$ of right modules over a right noetherian ring Λ.

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2. Atom spectrum

Throughout this paper, let $\mathcal{A}$ be a Grothendieck category. In this section, we recall the definition of the atom spectrum of $\mathcal{A}$ and related notions.

The atom spectrum is defined by using monoform objects and an equivalence relation between them.

Definition 3.

(1) A nonzero object H in $\mathcal{A}$ is called *monoform* if for each nonzero subobject L of H, no nonzero subobject of H is isomorphic to a subobject of H/L.

(2) For monoform objects H_1 and H_2 in $\mathcal{A}$, we say that H_1 is *atom-equivalent* to H_2 if there exists a nonzero subobject of H_1 which is isomorphic to a subobject of H_2.

The following result is fundamental.

Proposition 4 ([3, Proposition 2.2])**.** *Every nonzero subobject of a monoform object is monoform.*

For a commutative ring R, all monoform objects in $\operatorname{Mod} R$ can be described in the following sense.

Proposition 5 ([9, p. 626])**.** *Let R be a commutative ring. Then a nonzero object H in $\operatorname{Mod} R$ is monoform if and only if there exist $\mathfrak{p} \in \operatorname{Spec} R$ and a monomorphism $H \to k(\mathfrak{p})$ in $\operatorname{Mod} R$, where $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$.*

The atom equivalence is in fact an equivalence relation between the monoform objects in $\mathcal{A}$.

Definition 6. The *atom spectrum* $\operatorname{ASpec} \mathcal{A}$ of $\mathcal{A}$ is the quotient set of the set of monoform objects in $\mathcal{A}$ by the atom equivalence. Each element of $\operatorname{ASpec} \mathcal{A}$ is called an *atom* in $\mathcal{A}$. For each monoform object H in $\mathcal{A}$, the equivalence class of H is denoted by $\overline{H}$.

The notion of atoms was originally introduced by Storrer [9] for module categories and generalized to arbitrary abelian categories in [3].

The next result shows that the atom spectrum of a Grothendieck category is a generalization of the prime spectrum of a commutative ring.

Proposition 7 ([9, p. 631])**.** *Let R be a commutative ring. Then the map $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ given by $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ is bijective.*

We can also generalize the notions of associated primes and support.

Definition 8. Let M be an object in $\mathcal{A}$.

(1) Define the subset $\operatorname{AAss} M$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{AAss} M = \{ \alpha \in \operatorname{ASpec} \mathcal{A} \mid \alpha = \overline{H} \text{ for some monoform subobject } H \text{ of } M \}.$$

We call each element of $\operatorname{AAss} M$ an *associated atom* of M.

(2) Define the subset $\operatorname{ASupp} M$ of $\operatorname{ASpec} \mathcal{A}$ by

$$\operatorname{ASupp} M = \{ \alpha \in \operatorname{ASpec} \mathcal{A} \mid \alpha = \overline{H} \text{ for some monoform subquotient } H \text{ of } M \}.$$

We call it the *atom support* of M.

Proposition 9. *Let R be a commutative ring, and let M be an R-module. Then the bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *in Proposition 7 induces bijections* $\operatorname{Ass} M \to \operatorname{AAss} M$ *and* $\operatorname{Supp} M \to \operatorname{ASupp} M$.

We introduce a partial order on the atom spectrum, which plays an crucial role in this paper.

Definition 10. For $\alpha, \beta \in \operatorname{ASpec} \mathcal{A}$, we write $\alpha \leq \beta$ if $\beta \in \operatorname{ASupp} H$ holds for each monoform object H in $\mathcal{A}$ with $\overline{H} = \alpha$.

The partial order on the atom spectrum is a generalization of the inclusion relation between prime ideals of a commutative ring.

Proposition 11. *Let R be a commutative ring. Then the bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$ *in Proposition 7 is an isomorphism between the partially ordered sets* $(\operatorname{Spec} R, \subset)$ *and* $(\operatorname{ASpec}(\operatorname{Mod} R), \leq)$.

We can generalize Matlis' correspondence in commutative ring theory.

Theorem 12 ([3, Theorem 5.9])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then we have a bijection*

$$\operatorname{ASpec} \mathcal{A} \xrightarrow{\sim} \frac{\{\, \textit{indecomposable injective objects in } \mathcal{A} \,\}}{\cong}$$

given by $\overline{H} \mapsto E(H)$.

For a locally noetherian Grothendieck category $\mathcal{A}$, the localizing subcategories of $\mathcal{A}$ can be classified by the localizing subsets of $\operatorname{ASpec} \mathcal{A}$.

Definition 13. A subset Φ of $\operatorname{ASpec} \mathcal{A}$ is called a *localizing subset* if there exists an object M in $\mathcal{A}$ such that $\Phi = \operatorname{ASupp} M$.

Theorem 14 ([2, Theorem 3.8], [6, Corollary 4.3], and [3, Theorem 5.5])**.** *Let $\mathcal{A}$ be a locally noetherian Grothendieck category. Then we have a bijection*

$$\{\, \textit{localizing subcategories of } \mathcal{A} \,\} \xrightarrow{\sim} \{\, \textit{localizing subsets of } \operatorname{ASpec} \mathcal{A} \,\}$$

given by $\mathcal{X} \mapsto \operatorname{ASupp} \mathcal{X} := \bigcup_{M \in \mathcal{X}} \operatorname{ASupp} M$. The inverse map is given by $\Phi \mapsto \operatorname{ASupp}^{-1} \Phi$, where

$$\operatorname{ASupp}^{-1} \Phi = \{\, M \in \mathcal{A} \mid \operatorname{ASupp} \subset \Phi \,\}.$$

Moreover, the atom spectrum of the quotient category by a localizing subcategory can be described as follows.

Theorem 15 ([5, Theorem 5.17])**.** *Let $\mathcal{A}$ be a Grothendieck category, and let $\mathcal{X}$ be a localizing subcategory of $\mathcal{A}$. Denote by $F\colon \mathcal{A} \to \mathcal{A}/\mathcal{X}$ the canonical functor. Then we have a bijection*

$$\operatorname{ASpec}\mathcal{A} \setminus \operatorname{ASupp}\mathcal{X} \xrightarrow{\sim} \operatorname{ASpec}\frac{\mathcal{A}}{\mathcal{X}}$$

given by $\overline{H} \mapsto \overline{F(H)}$.

If the Grothendieck category $\mathcal{A}$ has a noetherian generator, then the set of minimal atoms in $\mathcal{A}$ has significant properties. Denote by $\operatorname{AMin}\mathcal{A}$ the set of minimal atoms in $\mathcal{A}$.

Theorem 16. *Let $\mathcal{A}$ be a Grothendieck category having a noetherian generator.*

1. ([5, Proposition 4.7]) *For each $\alpha \in \operatorname{ASpec}\mathcal{A}$, there exists $\beta \in \operatorname{AMin}\mathcal{A}$ satisfying $\beta \leq \alpha$.*
2. ([4, Theorem 4.4]) *$\operatorname{AMin}\mathcal{A}$ is a finite set.*
3. *$\operatorname{ASpec}\mathcal{A} \setminus \operatorname{AMin}\mathcal{A}$ is a localizing subset of $\mathcal{A}$.*

Definition 17. Let $\mathcal{A}$ be a Grothendieck category having a noetherian generator. Define the *artinianization* $\mathcal{A}_{\mathrm{artin}}$ of $\mathcal{A}$ as the quotient category of $\mathcal{A}$ by the localizing subcategory $\operatorname{ASupp}^{-1}(\operatorname{ASpec}\mathcal{A} \setminus \operatorname{AMin}\mathcal{A})$.

It is easy to deduce that the artinianization has a generator of finite length. Moreover, the following result ensures that it is the module category of some right artinian ring.

Theorem 18 (Năstăsescu [8])**.** *Let $\mathcal{A}$ be a Grothendieck category. Then the following assertions are equivalent.*

1. *$\mathcal{A}$ has an artinian generator.*
2. *$\mathcal{A}$ has a generator of finite length.*
3. *There exists a right artinian ring Λ satisfying $\mathcal{A} \cong \operatorname{Mod}\Lambda$.*

3. Molecule spectrum

In this section, we introduce a new spectrum of a Grothendieck category, which we call the *molecule spectrum*. It is a generalization of the set of two-sided prime ideals of a ring. The definition uses the notion of closed subcategory.

Definition 19.

1. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called *closed* if $\mathcal{C}$ is closed under subobjects, quotient objects, arbitrary direct sums, and arbitrary direct products.
2. Let $\mathcal{C}$ and $\mathcal{D}$ be closed subcategories of $\mathcal{A}$. Denote by $\mathcal{C} * \mathcal{D}$ the full subcategory of $\mathcal{A}$ consisting of all objects M in $\mathcal{A}$ such that there exists an exact sequence

$$0 \to L \to M \to N \to 0$$

with $L \in \mathcal{C}$ and $N \in \mathcal{D}$.

For each family of objects $\{M_i\}_{i\in I}$ in $\mathcal{A}$, we have the canonical monomorphism $\bigoplus_{i\in I} M_i \to \prod_{i\in I} M_i$ since $\mathcal{A}$ is a Grothendieck category. Therefore the closedness under arbitrary direct sums can be dropped from the definition of closed subcategory.

The following well-known result shows that closed subcategories of a Grothendieck category is a generalization of two-sided ideals of a ring.

Proposition 20. *Let Λ be a ring.*

(1) *We have a poset isomorphism*

$$(\{\,\textit{two-sided ideals of } \Lambda\,\}, \subset) \xrightarrow{\sim} (\{\,\textit{closed subcategories of } \operatorname{Mod}\Lambda\,\}, \supset)$$

given by $I \mapsto \operatorname{Mod}(\Lambda/I)$, where $\operatorname{Mod}(\Lambda/I)$ is identified with the full subcategory

$$\{\, M \in \operatorname{Mod}\Lambda \mid MI = 0\,\}$$

of $\operatorname{Mod}\Lambda$.

(2) *Let I and J be two-sided ideals of Λ. Then we have*

$$\operatorname{Mod}\frac{\Lambda}{IJ} = \operatorname{Mod}\frac{\Lambda}{J} * \operatorname{Mod}\frac{\Lambda}{I}$$

as a full subcategory of $\operatorname{Mod}\Lambda$, *that is, the isomorphism in (1) induces an isomorphism*

$$(\{\,\textit{two-sided ideals of } \Lambda\,\}, \cdot) \xrightarrow{\sim} (\{\,\textit{closed subcategories of } \operatorname{Mod}\Lambda\,\}, *)$$

of monoids.

(3) *Let M be a right Λ-module. Then the two-sided ideal $\operatorname{Ann}_\Lambda(M)$ corresponds to the smallest closed subcategory $\langle M\rangle_{\mathrm{closed}}$ of $\mathcal{A}$ containing M by the isomorphism in (1).*

We can generalize the notion of two-sided prime ideals of a ring to a Grothendieck category.

Definition 21. A nonzero closed subcategory $\mathcal{P}$ of $\mathcal{A}$ is called *prime* if for each closed subcategories $\mathcal{C}$ and $\mathcal{D}$ satisfying $\mathcal{P} \subset \mathcal{C} * \mathcal{D}$, we have $\mathcal{P} \subset \mathcal{C}$ or $\mathcal{P} \subset \mathcal{D}$.

Proposition 22. *Let Λ be a ring. Then the isomorphism in Proposition 20 (1) induces a poset isomorphism*

$$(\{\,\textit{two-sided prime ideals of } \Lambda\,\}, \subset) \xrightarrow{\sim} (\{\,\textit{prime closed subcategories of } \operatorname{Mod}\Lambda\,\}, \supset).$$

Although the set of prime closed subcategories can be used as the definition of the molecule spectrum, we use the notion of prime object instead, in order to clarify the similarity between the atom spectrum and the molecule spectrum.

Definition 23.

(1) A nonzero object H in $\mathcal{A}$ is called *prime* if for each nonzero subobject L of H, it holds that $\langle L\rangle_{\mathrm{closed}} = \langle H\rangle_{\mathrm{closed}}$.

(2) For prime objects H_1 and H_2 in $\mathcal{A}$, we say that H_1 is *molecule-equivalent* to H_2 if $\langle H_1\rangle_{\mathrm{closed}} = \langle H_2\rangle_{\mathrm{closed}}$.

Definition 24. The *molecule spectrum* $\operatorname{MSpec}\mathcal{A}$ of $\mathcal{A}$ is the quotient set of the set of prime objects in $\mathcal{A}$ by the molecule equivalence. Each element of $\operatorname{MSpec}\mathcal{A}$ is called a *molecule* in $\mathcal{A}$. For each prime object H in $\mathcal{A}$, the equivalence class of H is denoted by $\widetilde{H}$.

The following result shows that the molecule spectrum is also regarded as a generalization of the set of two-sided prime ideals of a ring. Although we assume the existence of a noetherian generator, this result can be also shown for the category $\operatorname{Mod}\Lambda$ of right modules over an arbitrary ring Λ by using classical ring-theoretic argument.

Proposition 25. *Let $\mathcal{A}$ be a Grothendieck category with a noetherian generator. Then we have a bijection*

$$\operatorname{MSpec}\mathcal{A} \xrightarrow{\sim} \{\, \textit{prime closed subcategories of } \mathcal{A} \,\}$$

given by $\widetilde{H} \mapsto \langle H \rangle_{\mathrm{closed}}$. For each $\rho = \widetilde{H} \in \operatorname{MSpec}\mathcal{A}$, the prime closed subcategory $\langle H \rangle_{\mathrm{closed}}$ corresponding to ρ is denoted by $\langle \rho \rangle_{\mathrm{closed}}$.

$\operatorname{MSpec}\mathcal{A}$ has a partial order induced from the set of prime closed subcategories.

Definition 26. Let $\mathcal{A}$ be a Grothendieck category with a noetherian generator. For $\rho, \sigma \in \operatorname{MSpec}\mathcal{A}$, we write $\rho \leq \sigma$ if $\langle \rho \rangle_{\mathrm{closed}} \supset \langle \sigma \rangle_{\mathrm{closed}}$ holds.

The partial order on $\operatorname{MSpec}\mathcal{A}$ can be also defined for $\operatorname{Mod}\Lambda$, where Λ is an arbitrary ring, and we can show the following proposition.

Proposition 27. *Let Λ be a ring. Then we have a poset isomorphism*

$$(\{\, \textit{two-sided prime ideals of } \Lambda \,\}, \subset) \xrightarrow{\sim} (\operatorname{MSpec}\Lambda, \leq)$$

given by $P \mapsto \widetilde{\Lambda/P}$.

Denote by $\operatorname{MMin}\mathcal{A}$ the set of minimal elements of $\operatorname{MSpec}\mathcal{A}$.

4. Atom-molecule correspondence

From now on, let $\mathcal{A}$ be a Grothendieck category having a noetherian generator and satisfying the Ab4* condition, that is, direct product preserves exactness. For a right noetherian ring Λ, the category $\operatorname{Mod}\Lambda$ satisfies this assumption. The following theorem is our main result.

Theorem 28.

(1) *We have a surjective poset homomorphism*

$$\varphi\colon \operatorname{ASpec}\mathcal{A} \to \operatorname{MSpec}\mathcal{A}$$

given by $\overline{H} \mapsto \widetilde{H}$, where H is taken to be a prime monoform object in $\mathcal{A}$.

(2) *The map φ induces a poset isomorphism*

$$\operatorname{AMin}\mathcal{A} \xrightarrow{\sim} \operatorname{MMin}\mathcal{A}.$$

(3) *There exists an injective poset homomorphism*

$$\psi\colon \operatorname{MSpec}\mathcal{A} \to \operatorname{ASpec}\mathcal{A}$$

satisfying the following properties.

(a) *For each* $\rho \in \operatorname{MSpec} \mathcal{A}$*, the image* $\psi(\rho)$ *is the smallest element in the set*

$$\{ \alpha \in \operatorname{ASpec} \mathcal{A} \mid \rho \leq \varphi(\alpha) \}.$$

(b) *The composite* $\varphi\psi$ *is the identity map on* $\operatorname{MSpec} \mathcal{A}$*.*
(c) *The map* ψ *induces a poset isomorphism* $\operatorname{MSpec} \mathcal{A} \xrightarrow{\sim} \Im\psi$*.*
(d) *For each* $\alpha \in \operatorname{ASpec} \mathcal{A}$ *and* $\rho \in \operatorname{MSpec} \mathcal{A}$*, we have*

$$\psi(\rho) \leq \alpha \iff \rho \leq \varphi(\alpha).$$

This theorem is proved by using the next two lemmas. Recall that an object H in $\mathcal{A}$ is called *compressible* if every nonzero subobject of H has a subobject which is isomorphic to H. In our setting, every compressible object in $\mathcal{A}$ is monoform.

Lemma 29. *For every* $\alpha \in \operatorname{AMin} \mathcal{A}$*, there exists a compressible object* H *satisfying* $\alpha = \overline{H}$*.*

A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called *weakly closed* if $\mathcal{X}$ is closed under subobjects, quotient objects, and arbitrary direct sums. The following lemma explains why minimal elements of $\operatorname{ASpec} \mathcal{A}$ behave nicely.

Lemma 30. *Let* M *be an object in* $\mathcal{A}$ *satisfying* $\operatorname{AAss} M \subset \operatorname{AMin} \mathcal{A}$*. Let* $\mathcal{X}$ *be the smallest weakly closed subcategory containing* M*. Then* $\mathcal{X}$ *is closed under arbitrary direct products, that is,* $\mathcal{X} = \langle M \rangle_{\text{closed}}$*.*

The Ab4* condition is used in the proof of Lemma 30. The proof also depends on the fact that $\operatorname{AMin} \mathcal{A}$ is a finite set.

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TILTING THEORY OF PREPROJECTIVE ALGEBRAS AND c-SORTABLE ELEMENTS

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ABSTRACT. For a finite acyclic quiver Q and the corresponding preprojective algebra Π, the quotient algebra Π_w of Π associated with an element w in the Coxeter group of Q was introduced by Buan-Iyama-Reiten-Scott [6]. The algebra Π_w is Iwanaga-Gorenstein and has the natural $\mathbb{Z}$-grading. Recently, the author showed that the stable category of the category of graded Cohen-Macaulay Π_w-modules has tilting objects when w is a c-sortable element. In this paper, we study the endomorphism algebra of a tilting object.

1. INTRODUCTION

The preprojective algebra Π of Q has been introduced by Gelfand-Ponomarev to study representation theory of the path algebra of Q. Preprojective algebras play important roles in many areas of mathematics. One of them is that preprojective algebras provide 2-Calabi-Yau triangulated categories (2-CY, for short) with cluster tilting objects which have been studied in the view point of categorification of cluster algebras.

In the case when Q is a Dynkin quiver, the preprojective algebra Π of Q is a finite dimensional selfinjective algebra. In this case, Geiss-Leclerc-Schröer showed that the stable category $\underline{\mathsf{mod}}\,\Pi$ is a 2-CY category and $\underline{\mathsf{mod}}\,\Pi$ has cluster tilting objects [7]. In the case when Q is non-Dynkin quiver, Π is an infinite dimensional algebra. In this case, Buan-Iyama-Reiten-Scott introduced and studied the factor algebra Π_w associated with an element w in the Coxeter group of Q [6]. They showed that Π_w is a finite dimensional Iwanaga-Gorenstein algebra of dimension at most one and the stable category of $\mathsf{Sub}\,\Pi_w$ is a 2-CY category and has cluster tilting objects, where $\mathsf{Sub}\,\Pi_w$ is the full subcategory of $\mathsf{mod}\,\Pi_w$ of submodules of finitely generated free Π_w-modules.

There are other classes of 2-CY triangulated categories. Amiot introduced the generalized cluster category $\mathcal{C}_A$ for a finite dimensional algebra A of finite global dimension [1]. If $\mathcal{C}_A$ is Hom-finite, then $\mathcal{C}_A$ is a 2-CY category and has cluster tiling objects. There are close connections between 2-CY categories $\underline{\mathsf{Sub}}\,\Pi_w$ and $\mathcal{C}_A$. Amiot-Reiten-Todorov [3] showed that for any finite acyclic quiver Q and any element w of the Coxeter group, there is a triangle equivalence

$$\underline{\mathsf{Sub}}\,\Pi_w \simeq \mathcal{C}_{A_w}$$

for some finite dimensional algebra A_w of global dimension at most two.

The aim of this paper is to construct a derived category version of this equivalence. We regard Π_w as a $\mathbb{Z}$-graded algebra and consider the stable category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ of graded Π_w-submodules of graded free Π_w-modules. We construct a tilting object M in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ and calculate the endomorphism algebra of M.

The detailed version of this paper will be submitted for publication elsewhere.

Notation. Through out this paper, let k be an algebraically closed field and Q a finite acyclic quiver. By a module, we mean a left module. For a ring A, we denote by $\mathsf{mod}A$ the category of finitely generated A-modules and by $\mathsf{proj}A$ the category of finitely generated projective A-modules. For $X \in \mathsf{mod}A$, we denote by $\mathsf{Sub}X$ the full subcategory of $\mathsf{mod}A$ whose objects are submodules of finite direct sums of copies of X. For $X \in \mathsf{mod}A$, we denote by $\mathsf{add}X$ the full subcategory of $\mathsf{mod}A$ whose objects are direct summands of finite direct sums of copies of X. For two arrows α, β of a quiver such that the target point of α is the start point of β, we denote by $\alpha\beta$ the composition of α and β.

2. Preliminaries

In this section, we give definitions used in this paper and recall some result of [6]. We first define preprojective algebras and Coxeter groups of Q.

Definition 1. Let Q be a finite acyclic quiver.

(1) The *double quiver* $\overline{Q}$ of Q is a quiver obtained from Q by adding an arrow $\alpha^* : v \to u$ for each arrow $\alpha : u \to v$ of Q.
(2) We define the *preprojective algebra* Π of Q by

$$\Pi := k\overline{Q}/\langle \sum_{\alpha \in Q_1} \alpha\alpha^* - \alpha^*\alpha \rangle.$$

Let Q be a connected quiver. It is known that the preprojective algebra Π of Q is does not depend on the orientation of Q and that Π is finite dimensional and selfinjective if and only if Q is a Dynkin quiver.

Definition 2. The *Coxeter group* $W = W_Q$ of a quiver Q is the group generated by the set $\{s_u \mid u \in Q_0\}$ with relations

- $s_u^2 = 1$,
- $s_v s_u = s_u s_v$ if there exist no arrows between u and v,
- $s_u s_v s_u = s_v s_u s_v$ if there exists exactly one arrow between u and v.

An expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$ is *reduced* if for any other expression $w = s_{v_1} s_{v_2} \cdots s_{v_m}$, we have $l \leq m$. For a reduced expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$, let $\mathsf{Supp}(w) = \{u_1, \cdots, u_l\}$.

Note that, $\mathsf{Supp}(w)$ is independent of the choice of a reduced expression of w. Let Q be a connected quiver. It is known that W_Q is a finite group if and only if Q is a Dynkin quiver.

Next we define a two-sided ideal of Π and recall some result of [6]. For a vertex $u \in Q_0$, we define a two-sided ideal I_u of Π by

$$I_u = \Pi(1 - e_u)\Pi,$$

where e_u is the idempotent of Π for u. For a reduced expression $w = s_{u_1} s_{u_2} \dots s_{u_l}$, we define a two-sided ideal I_w of Π by

$$I_w := I_{u_1} I_{u_2} \cdots I_{u_l}.$$

Note that, an ideal I_w is independent of the choice of a reduced expression of w by [6].

A finite dimensional algebra A is said to be *Iwanaga-Gorenstein of dimension at most one* if $\mathsf{inj.dim}({}_A A) \leq 1$. In this case, it is known that the category $\mathsf{Sub}A$ is a Frobenius category.

Proposition 3. [6] *For any element $w \in W_Q$, the algebra Π_w is finite dimensional and Iwanaga-Gorenstein of dimension at most one.*

Next we introduce a grading of a preprojective algebra. The path algebra $k\overline{Q}$ is regarded as a $\mathbb{Z}$-graded algebra by the following grading:

$$\deg \beta = \begin{cases} 1 & \beta = \alpha^*, \alpha \in Q_1 \\ 0 & \beta = \alpha, \alpha \in Q_1. \end{cases}$$

Since the element $\sum_{\alpha \in Q_1} (\alpha\alpha^* - \alpha^*\alpha)$ in $k\overline{Q}$ is homogeneous of degree one, the grading of $k\overline{Q}$ naturally gives a grading on the preprojective algebra $\Pi = \bigoplus_{i \geq 0} \Pi_i$. Since Π_0 is spanned by all paths of degree zero, we have $\Pi_0 = kQ$. For any $w \in W$ the ideal I_w of Π is a graded ideal of Π since so is each I_u. In particular, the quotient algebra Π_w is a graded algebra.

For a graded module $X = \bigoplus_{i \in \mathbb{Z}} X_i$ and an integer j, we define a new graded module $X(j)$ by $(X(j))_i = X_{i+j}$. For any integer j, we define a graded submodule $X_{\geq j}$ of M by

$$(X_{\geq j})_i = \begin{cases} X_i & i \geq j \\ 0 & \text{else} \end{cases}$$

and a graded quotient module of X by $X_{\leq j} = X / X_{\geq j+1}$.

Let $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ be the category of finitely generated $\mathbb{Z}$-graded Π_w-modules with degree zero morphisms. We denote by $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ the full subcategory of $\mathsf{mod}^{\mathbb{Z}}\Pi_w$ of submodules of graded free Π_w-modules, that is,

$$\mathsf{Sub}^{\mathbb{Z}}\Pi_w = \left\{ X \in \mathsf{mod}^{\mathbb{Z}}\Pi_w \mid X \subset \bigoplus_{j=1}^{m} \Pi_w(i_j),\ i_j \in \mathbb{Z} \right\}.$$

Since Proposition 3, $\mathsf{Sub}^{\mathbb{Z}}\Pi_w$ is also a Frobenius category. Therefore we have a triangulated category $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. In this paper, we get a tilting object in this category and calculate the endomorphism algebra of it.

3. c-sortable elements and tilting modules

In this section, we define c-sortable elements. Throughout this section, we denote by W the Coxeter group of Q.

Definition 4. Let Q be a finite acyclic quiver with $Q_0 = \{1, 2, \ldots, n\}$.

(1) An element c in W is called a *Coxeter element* if c has an expression $c = s_{u_1}s_{u_2}\ldots s_{u_n}$, where $u_1, \ldots, u_n$ is a permutation of $1, \ldots, n$.

(2) A Coxeter element $c = s_{u_1}s_{u_2}\ldots s_{u_n}$ in W is said to be *admissible with respect to the orientation of* Q if c satisfies $e_{u_j}(kQ)e_{u_i} = 0$ for $i < j$.

Since Q is acyclic, W has a Coxeter element c admissible with respect to the orientation of Q. There are several expressions of $c = s_{u_1}s_{u_2}\ldots s_{u_n}$ satisfying $\{u_1, \ldots, u_n\} = \{1, \ldots, n\}$ and $e_{u_j}(kQ)e_{u_i} = 0$ for $i < j$. However, it is shown that c is uniquely determined as an element of W. ¿From now on, we call a Coxeter element admissible with respect to the orientation of Q simply a Coxeter element. We define a c-sortable elements.

Definition 5. Let c be a Coxeter element of W. An element $w \in W$ is said to be *c-sortable* if there is a reduced expression $w = s_{u_1} \cdots s_{u_l} = c^{(0)}c^{(1)} \cdots c^{(m)}$, where each $c^{(i)}$ is subsequence of c and

$$\mathsf{Supp}(c^{(m)}) \subset \mathsf{Supp}(c^{(m-1)}) \subset \cdots \subset \mathsf{Supp}(c^{(0)}) \subset Q_0.$$

Example 6. Let $Q = \begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. A Coxeter element is $c = s_1s_2s_3$. Then an element $w = s_1s_2s_3s_1s_2s_1$ is a c-sortable element. Actually, $c^{(0)} = s_1s_2s_3$, $c^{(1)} = s_1s_2$, and $c^{(2)} = s_1$. The element $w' = s_1s_2s_3s_1s_3$ is also a c-sortable element. Actually, $c^{(0)} = s_1s_2s_3$ and $c^{(1)} = s_1s_3$.

4. A tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$

In this section, we construct a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. Let $\mathcal{T}$ be a triangulated category. An object M in $\mathcal{T}$ is called a *tilting object* if the following holds.

- $\mathsf{Hom}_{\mathcal{T}}(M, M[j]) = 0$ for any $j \neq 0$,
- $\mathsf{thick}M = \mathcal{T}$, where $\mathsf{thick}M$ is the smallest triangulated full subcategory of $\mathcal{T}$ containing M and closed under direct summands.

Let $\mathcal{T}$ be the stable category of a Frobenius category, and assume that $\mathcal{T}$ is Krull-Schmidt. If there is a tilting object M in $\mathcal{T}$, then it follows from [8, (4.3)] that there exists a triangle equivalence

$$\mathcal{T} \simeq \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\underline{\mathsf{End}}_{\mathcal{T}}(M)),$$

where $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\mathsf{End}_{\mathcal{T}}(M))$ is the homotopy category of bounded complexes of projective $\mathsf{End}_{\mathcal{T}}(M)$-modules.

For a reduced expression $w = s_{u_1} \cdots s_{u_l}$ and $1 \leq i \leq l$, let m_i be the number of elements in $\{1 \leq j \leq i-1 \mid u_j = u_i\}$, that is,

$$m_i = \sharp\{1 \leq j \leq i-1 \mid u_j = u_i\}, \qquad \text{for } 1 \leq i \leq l.$$

Moreover, for $1 \leq i \leq l$, put

$$M^i = (\Pi/I_{u_1 \ldots u_i})e_{u_i}(m_i),$$
$$M = \bigoplus_{i=1}^{l} M^i.$$

Then we have the following theorem.

Theorem 7. [9] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the object M is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$.*

5. The endomorphism algebra of a tilting object

In this section, we calculate the endomorphism algebra of a tilting object which is constructed in Section 4. Throughout this section, for simplicity, assume that a c-sortable element w satisfies $\mathsf{Supp}(w) = Q_0$. Since $\Pi_0 = kQ$, for any graded Π_w-module X, X_0 is a kQ-module. The following theorem is one of the main theorem of [2].

Theorem 8. [2] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element and M be a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ which is constructed in Section 4. Then there exists an unique tilting kQ-module T_w which satisfies $\mathsf{add}M_0 = \mathsf{Sub}T_w$.*

Using Theorem 8, we calculate the endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$. We have the following morphism of algebras:

$$F : \mathsf{End}^{\mathbb{Z}}_{\Pi_w}(M) \to \mathsf{End}_{kQ}(M_0) \quad f \mapsto f|_{M_0}.$$

Theorem 9. [9] *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the morphism F induces an isomorphism of algebras:*

$$\underline{F} : \underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M) \xrightarrow{\sim} \mathsf{End}_{kQ}(M_0)/[T_w],$$

where $[T_w]$ is an ideal consisting of morphisms which factors through $\mathsf{add}T_w$.

We can show that the global dimension of the algebra $\mathsf{End}_{kQ}(M_0)/[T_w]$ is at most two. Actually, we can show the following theorem. Let A be a finite dimensional algebra and T a cotilting A-module of finite injective dimension. We denote by ${}^{\perp_{>0}}T$ the full subcategory of $\mathsf{mod}A$ consisting of modules X satisfying $\mathsf{Ext}^i_A(X,T) = 0$ for any $i > 0$.

Theorem 10. [9] *Assume that the global dimension of A is at most n and that ${}^{\perp_{>0}}T$ has an additive generator N. Then the global dimension of $\mathsf{End}_A(N)/[T]$ is at most $3n-1$.*

Note that $\mathsf{End}_A(M)$ and $\mathsf{End}_A(M)/[T]$ are relative version of Auslander algebras and stable Auslander algebras. It is known that Auslander algebras have global dimension at most two [5], and that stable Auslander algebras have global dimension at most $3n-1$ [4, Proposition 10.2]. We apply Theorem 10 to our endomorphism algebra.

Corollary 11. *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then the global dimension of $\mathsf{End}_{kQ}(M_0)/[T_w]$ is at most two.*

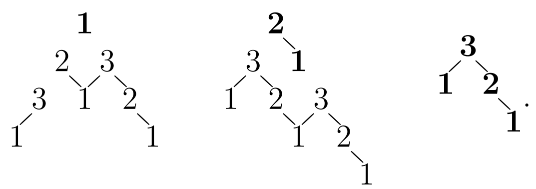
Finally, we have the following theorem.

Theorem 12. *Let $w = s_{u_1} \cdots s_{u_l}$ be a c-sortable element. Then we have a triangle equivalence $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M))$.*

6. Examples

In this section, we calculate some examples.

Example 13. Let Q be a quiver $\begin{tikzcd}[column sep=small] & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[rr] & & 3 \end{tikzcd}$. Let $w = s_1s_2s_3s_1s_2s_1$. This is a c-sortable element. Then we have a graded algebra $\Pi_w = \Pi_w e_1 \oplus \Pi_w e_2 \oplus \Pi_w e_3$,



and a tilting module

$$M = \mathbf{1} \oplus \begin{smallmatrix} \mathbf{2} & \\ & \mathbf{1} \end{smallmatrix} \oplus \left(\begin{smallmatrix} & \mathbf{1} & & & \\ 2 & & 3 & & \\ & 1 & & 2 & \\ & & & & 1 \end{smallmatrix} (1) \right)$$

in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed, and the degree zero parts are denoted by bold numbers. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \qquad ab = 0.$$

We can describe the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$. Let K be the kernel of the canonical epimorphism $\Pi_w e_2 \to S_2$, where S_2 is a simple module associated with the vertex 2, and N be the cokernel of an inclusion $(\Pi_w e_1)_1 \to \Pi_w e_2$:

$$K = \begin{smallmatrix} & 3 & & \mathbf{1} & & & \\ 1 & & 2 & & 3 & & \\ & & & 1 & & 2 & \\ & & & & & & 1 \end{smallmatrix}, \qquad N = \begin{smallmatrix} & & \mathbf{2} & \\ & 3 & & \mathbf{1} \\ 1 & & & \end{smallmatrix}.$$

Then the Auslander-Reiten quiver of $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$ is the following one:

$$\begin{tikzcd}[column sep=tiny]
\cdots \arrow[dr] & & (\Pi_w e_2)_1 \arrow[dr] & & \boxed{(\Pi_w e_1)_0} \arrow[dr] & & (\Pi_w e_1)_{[1,2]}(1) \arrow[dr] & & \boxed{(\Pi_w e_1)_{[0,1]}(1)} \arrow[dr] & \\
& (\Pi_w e_1)_1 \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[dr] & & N \arrow[ur] \arrow[dr] & & (\Pi_w e_1)(1) \arrow[ur] \arrow[dr] & & \cdots, \\
\cdots \arrow[ur] & & (\Pi_w e_1)_{[0,1]} \arrow[ur] & & (\Pi_w e_1)_2(1) \arrow[ur] & & \boxed{N_0} \arrow[ur] & & (\Pi_w e_2)_1(1) \arrow[ur] &
\end{tikzcd}$$

where $M = (\Pi_w e_1)_0 \oplus N_0 \oplus (\Pi_w e_1)_{[0,1]}(1)$.

Example 14. Let Q be a quiver $1 \rightrightarrows 2$. Then we have a graded algebra $\Pi = \Pi e_1 \oplus \Pi e_2$, and these are represented by their radical filtrations as follows:

$$\begin{smallmatrix} \mathbf{1} \\ 2 \quad 2 \\ 1 \quad 1 \quad 1 \\ 2 \quad 2 \quad 2 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \\ 2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 \\ \diagup\diagdown\diagup\diagdown\diagup\diagdown\diagup\diagdown\diagup\diagdown\diagup\diagdown \end{smallmatrix} \qquad \begin{smallmatrix} \mathbf{2} \\ \mathbf{1} \quad \mathbf{1} \\ 2 \quad 2 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \\ 2 \quad 2 \quad 2 \quad 2 \quad 2 \\ 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \end{smallmatrix},$$

where the degree zero parts are denoted by bold numbers. Let $c = s_1 s_2$. This is a Coxeter element. Let $w = c^{n+1} = s_1 s_2 s_1 \cdots s_1 s_2$. This is a c-sortable element. We have $(\Pi/I_{c^i})e_1 = (\Pi/J^{2i-1})e_1$, and $(\Pi/I_{c^i})e_2 = (\Pi/J^{2i})e_2$, where J is the Jacobson radical of Π. The object $M = \bigoplus_{i=1}^{n}(\Pi/I_{c^i})(i-1)$ is a tilting object in $\underline{\mathsf{Sub}}^{\mathbb{Z}}\Pi_w$, where graded projective Π_w-modules are removed. The endomorphism algebra $\underline{\mathsf{End}}^{\mathbb{Z}}_{\Pi_w}(M)$ of M is given by the following quiver with relations

$$\Delta = \ 1 \overset{a}{\underset{b}{\rightrightarrows}} 2 \overset{a}{\underset{b}{\rightrightarrows}} 3 \overset{a}{\underset{b}{\rightrightarrows}} \quad \cdots\cdots \quad \overset{a}{\underset{b}{\rightrightarrows}} 2n-1 \overset{a}{\underset{b}{\rightrightarrows}} 2n\ , \quad aa = bb.$$

The algebra $k\Delta/\langle aa - bb \rangle$ has global dimension two.

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FILTERED CATEGORIES AND REPRESENTATIONS OF BOXES

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Abstract. Boxes are generalisations of algebras; their representations form exact, but in general not abelian categories. Filtered categories that occur naturally in the context of quasi-hereditary algebras, can be described as representations of certain boxes. This context and how boxes are applied here is described.

1. Standardisable sets and quasi-hereditary algebras

Let k be a field and $\mathcal{C}$ an abelian k-category. Morphisms are taken in $\mathcal{C}$ unless specified otherwise. By D we denote k-duality $Hom_k(-, k)$.

The main objects in the story to be told here are the following:

Definition 1.1. Finitely many objects $\Delta(1), \ldots, \Delta(n)$ in $\mathcal{C}$ form a *standardisable set* Δ if and only if the following conditions are satisfied:
(a) For each i: $End(\Delta(i)) = k$.
(b) For all i, j: $Hom(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i \leq j$.
(c) For all i, j: $Ext(\Delta(i), \Delta(j)) \neq 0 \Rightarrow i < j$.
Moreover, all morphism and extension spaces occuring here are supposed to be finite dimensional over k.

Standardisable sets occur rather frequently in algebra and in geometry. Here are some examples:

(1) Exceptional collections in algebraic geometry are standardisable sets in the category of coherent sheaves.
(2) Weyl modules of algebraic groups or of Schur algebras of algebraic groups form standardisable sets and so do Verma modules of semisimple complex Lie algebras.
(3) When A is a finite dimensional algebra, its simple modules form a standardisable set provided there is an ordering such that condition (c) is satisfied. This happens exactly when the quiver of A has no oriented cycles (that is, when A is *directed*).
(4) The standard modules of a quasi-hereditary algebra form a standardisable set:

Definition 1.2. (Cline, Parshall and Scott [4, 5]) A finite dimensional algebra A together with a partial ordering $\leq$ on its set of isomorphism classes of simple modules $S(1), \ldots, S(n)$ is called a *quasi-hereditary algebra* if and only if the following conditions are satisfied:

This is a short introduction to motivation and main results of the joint paper [12] with Julian Külshammer and Sergiy Ovsienko.

Let $P(1), \ldots, P(n)$ be projective covers of the simple modules $S(1), \ldots, S(n)$, respectively and $\Delta(i)$ the largest quotient of $P(i)$such that $[\Delta(i) : S(j)] \neq 0$ implies $i \geq j$. Then:
For each i: $[\Delta(i) : S(i)] = 1$.
The kernel of the surjection $P(i) \to \Delta(i)$ has a filtration with subquotients $\Delta(j)$ where the indices occuring satisfy $j > i$.

This standardisable set $\Delta(1), \ldots, \Delta(n)$ of A often is denoted by Δ_A.

Although the definition only uses a partial order, we write it as a total order. This can be done without loss of generality. The standard modules $\Delta(i)$ are relative projective; in fact, they are projective objects in the category $A - mod[\leq i]$ whose objects have composition factors with indices not bigger than i.

An algebra may be quasi-hereditary for many choices of partial orderings; therefore, a quasi-hereditary algebra more precisely is a pair $(A, \leq)$. Hereditary algebras, for instance path algebras of directed quivers, are quasi-hereditary with any choice of orderings, and they are characterised by this property. Directed algebras are quasi-hereditary, and for a particular ordering the standard modules are simple.

Here is an explicit example of a quasi-hereditary algebra: The algebra A is given by quiver and relations; we also depict the Loewy series of its projective and standard modules.

$$\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet \Big/ \alpha\beta = 0 \qquad P(1) = \begin{matrix}1\\2\\1\end{matrix} \qquad \Delta(1) = 1 \qquad P(2) = \Delta(2) = \begin{matrix}2\\1\end{matrix}$$

This algebra occurs, up to Morita equivalence, rather frequently, for instance over the complex numbers as principal block of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of the simple Lie algebra $\mathfrak{sl}(2)$, and over infinite fields of characteristic two as Schur algebra $S(2, 2)$.

In algebraic Lie theory, one often considers categories $\mathcal{C}$ with infinitely many simple objects and infinitely many standard objects. Examples are rational or polynomial modules of reductive algebraic groups - here, Weyl modules are standard modules - and the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ of semisimple complex Lie algebras - here, Verma modules are standard modules - and various quantisations and further generalisations. Cline, Parshall and Scott [4] have developed the concept of highest weight categories to deal with these situations . Such categories are built up in a precisely described way from (module categories of) quasi-hereditary algebras.

When Δ is a standardisable set, the full subcategory $\mathcal{F}(\Delta)$ of $\mathcal{C}$ whose objects have finite filtrations with subquotients being objects $\Delta(j)$, is called the *Δ-filtered* (or just filtered) *category.*

The standard modules of quasi-hereditary algebras look like a special example, but this is in fact the most general class of examples, by Dlab and Ringel's standardisation theorem (presented in Kyoto in 1990, at the workshop preceding ICRA at Tsukuba):

Theorem 1.3. *(Dlab and Ringel* [7]*) Let $\Delta = \Delta(1), \ldots, \Delta(n)$ be a standardisable set in an abelian k-category $\mathcal{C}$. Then there exists a quasi-hereditary algebra $(A, \leq)$ (unique up to Morita equivalence) such that $\mathcal{F}(\Delta) \simeq \mathcal{F}(\Delta_A)$.*

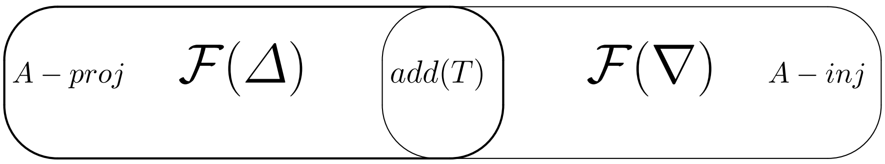
The category $\mathcal{F}(\Delta)$ rarely is abelian - for instance, cokernels of non-trivial maps between standard modules of quasi-hereditary algebras are not Δ-filtered. But it inherits the exact structure of the abelian category $\mathcal{C}$. The equivalence in the standardisation theorem respects the exact structures.

2. Filtered categories and Ringel duality

Let A be a quasi-hereditary algebra. Then the opposite algebra A^{op} is known to be quasi-hereditary, too. Thus, there are modules $\Delta(i, A^{op})$ filtering the projective A^{op}-modules, which are k-dual to the injective A-modules. Therefore, the injective A-modules are filtered by modules $\nabla(i) := D\Delta(i, A^{op})$. They have simple socle $S(i)$ and other properties dual to those of the standard modules Δ. Thus, turning around the partial ordering, the set ∇ turns out to be standardisable as well. Hence, by Dlab and Ringel's standardisation theorem, there must be a quasi-hereditary algebra $R = R(A)$ such that $\mathcal{F}(\nabla) \simeq \mathcal{F}(\Delta, R)$. This algebra now is called the *Ringel dual* of the algebra A; it is unique up to Morita equivalence. At the Tsukuba ICRA in 1990, Ringel presented the following result:

Theorem 2.1. *Let $(A, \leq)$ be a quasi-hereditary algebra. Then there exists a tilting module T such that $\mathcal{F}(\nabla) \cap \mathcal{F}(\Delta) = add(T)$. The module T is called the* characteristic tilting module *of A. Its endomorphism algebra $End_A(T)$ is quasi-hereditary again and it is the Ringel dual R of A.*

The tilting module T is a full injective object in $\mathcal{F}(\Delta)$ and a full projective object in $\mathcal{F}(\nabla)$. It has finite (but arbitrarily large) projective dimension. Taking the Ringel dual of R produces an algebra that is Morita equivalent to A itself.



This picture illustrates the central role of T in the two filtered categories.The category $add(T)$ coincides with the injectives in $\mathcal{F}(\Delta)$ as well as with the projectives in $\mathcal{F}(\nabla)$ and, up to equivalence, also with the projectives of the Ringel dual R.

Ringel duality has become very popular and useful in applications of quasi-hereditary algebras and highest weight categories. Soergel [15] has shown that BGG-category $\mathcal{O}$ is Ringel self-dual. Donkin [8] has shown that classical Schur algebras $S(n, r)$ with $n \geq r$ are Ringel self-dual; in this situation, the characteristic tilting module is a direct sum of tensor products of exterior powers of the natural module over GL_n - a description that yields direct applications to invariant theory [9].

In our example, the characteristic tilting module is a direct sum of the projective-injective module $P(1) = I(1)$ and the simple module $1 = \Delta(1) = \nabla(1)$.

$$\bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet \Big/ \alpha\beta = 0 \qquad T = \begin{smallmatrix}1\\2\\1\end{smallmatrix} \oplus 1 \qquad \nabla(1) = 1 \qquad I(2) = \nabla(2) = \begin{smallmatrix}1\\2\end{smallmatrix}$$

The Ringel dual $R = End_A(T)$ is isomorphic to A itself, so the algebra A is Ringel self-dual. The category $\mathcal{F}(\Delta)$ has three indecomposable objects, up to isomorphism, and its Auslander-Reiten quiver looks as follows:

$$\begin{tikzcd}
 & P(1) = I(1) = \begin{smallmatrix}1\\2\\1\end{smallmatrix} \arrow[dr] & \\
P(2) = \Delta(2) = \begin{smallmatrix}2\\1\end{smallmatrix} \arrow[ur] & & \Delta(1) = 1 \arrow[ll]
\end{tikzcd}$$

The shape of this Auslander-Reiten quiver is very similar to the Auslander-Reiten quiver of the path algebra of type A_2, whose quiver is just one arrow. The difference is that here, there is a map going from right to left, the inclusion $\Delta(1) \to \Delta(2)$. The cokernel of this map is not in $\mathcal{F}(\Delta)$, which just inherits an exact structure from $A-mod$, but not an abelian one. The standard modules $\Delta(1)$ and $\Delta(2)$ look like simple objects in this category, but there are non-trivialmorphisms relating them; so, Schur's lemma is not valid in this situation.

As we have seen, the category $\mathcal{F}(\Delta)$ has more structure than just being exact. It has enough projective objects and enough injective objects. Ringel [14] proved that it has almost split sequences and that it is functorially finite in $A - mod$. It also can be shown that it is closed under kernels of epimorphisms; but it is usually not closed under cokernels of monomorphisms. So, it cannot be a module category, although it has many properties of a module category.

Questions. *(a) What is the structure of the exact category $\mathcal{F}(\Delta)$?*
(b) How is this category related to the algebra A, or rather to its Morita equivalence class?
(c) How to interpret Ringel duality in terms of the standardisable system Δ?

The main result to be reported below answers the first question by interpreting $\mathcal{F}(\Delta)$ as a generalised module category in a precise sense. The second question is, of course, addressed by Dlab and Ringel's standardisation theorem. The main result will provide another, rather different answer, and at the same time an answer to the third question.

3. Boxes

Boxes - originally called bocses (for bimodule over category with coalgebra structure) - have been introduced by the Kiev school around Roiter, Drozd and Ovsienko, in various versions. Alternative approaches to similar concepts are called corings [2] or ditalgebras [1]. The version used in [12] is more general than in the original applications of boxes. The most prominent result based on the theory of boxes is Drozd's tame and wild theorem [10], reproved by Crawley-Boevey [6]. Boxes also have beenused in several attempts to prove Brauer-Thrall type conjectures.

Boxes can be seen as generalisations of algebras. They also have representation categories, which can be studied using methods of representation theory of algebras.

Definition 3.1. A *box* $\mathcal{B}$ is a quadruple (B, W, μ, ϵ), where B is a category, W is a $B-B$-bimodule, $\mu : W \to W \otimes_B W$ is a B-bimodule map that is a coassociative comultiplication, and $\epsilon : W \to B$ is a B-bilinear map that is a counit for μ.

In our context, the category B is just a finite dimensional basic algebra given by quiver and relations. A box $\mathcal{B}$ is called *directed* when the algebra B is directed, which is the same as quasi-hereditary with simple standard modules, and moreover W is a sum of 'directed' projective bimodules (see [12]). Another condition in our context is that the kernel of ϵ has to be a finitely generated projective bimodule.

Modules over a box $\mathcal{B}$ are, by definition, B-modules (which we always assume to be finite dimensional); morphisms are, however, different:

Definition 3.2. Let $\mathcal{B} = (B, W, \mu, \epsilon)$ be a box. A representation of $\mathcal{B}$ is a B-module. Let X and Y be representations of $\mathcal{B}$. Then the morphism space is defined to be $Hom_{\mathcal{B}}(X, Y) := Hom_{B \otimes B^{op}}(W, Hom_k(X, Y))$. Composition of $f : X \to Y$ with $g : Y \to Z$ is defined as follows: f is given by a map $f : W \to Hom_k(X, Y)$ and g is given by a map $g : W \to Hom_k(Y, Z)$. The composition $g \circ f$ is given by a map $W \to Hom_k(X, Z)$ which is the composition $W \xrightarrow{\mu} W \otimes_B W \xrightarrow{g \otimes f} Hom_k(Y, Z) \otimes_B Hom_k(X, Y) \xrightarrow{composition} Hom_k(X, Z)$.

Our definition of homomorphisms is not the usual one, but related to that by adjointness.

The category of representations of a box is not, in general, an abelian category any more. It can, however, be given an exact structure, at least under some assumptions (triangularity of the box).

A very interesting point in changing the definition of morphisms is that also endomorphisms are changing. In particular, since the algebra B is a $\mathcal{B}$-module, it has an endomorphism ring that in general is quite different from Bitself. This is at the basis of a theory developed by Burt and Butler and presented at the Tsukuba ICRA in 1990.

Definition 3.3. Let $\mathcal{B} = (B, W, \mu, \epsilon)$ be a box. The algebras $R_{\mathcal{B}} := End_{\mathcal{B}}(B)^{op} \simeq Hom_B({}_BW, {}_BB)$ and $L_{\mathcal{B}} := End_{\mathcal{B}^{op}}(B) \simeq Hom_B(W_B, B_B)$ are called the *left and right Burt-Butler algebras* of $\mathcal{B}$, respectively.

Here is a brief account of Burt-Butler theory: The bimodule W allows to defineinduction and coinduction functors and hence the following commutative diagram of functors:

$$\begin{tikzcd}
\text{Ind(B,R)} \subset \text{R-Mod} \arrow[rr, shift left, leftarrow, "W \otimes_R -"] \arrow[rr, shift right, "Hom_L(W{,} -)"'] & & \text{CoInd(B,L)} \subset \text{L-Mod} \\
& \text{B-Mod} \arrow[ul, "\text{Ind} = R \otimes_B -"] \arrow[ur, "\text{CoInd}=Hom_B(L{,} -)"'] &
\end{tikzcd}$$

Then there are equivalences of exact categories $rep(\mathcal{B}) \simeq Ind(B,R) \simeq CoInd(B,L)$ provided by restricting the above functors and defining the categories $Ind(B,R)$ and $CoInd(B,L)$ by the images of the respective functors.

As a consequence, the representation category $rep(\mathcal{B})$ has almost split sequences [3]. Moreover, there are double centraliser properties $L_{\mathcal{B}} \simeq End_{R^{op}}(W)$ and $R^{op} \simeq End_L(W)$.

Here is our example again, now from a new point of view. Denote by B the quiver algebra of an A_2 quiver. So, representations are pairs of vector spaces, related by a linear map. For a certain choice of W (revealed in Appendix A.1 in [12], where full details are given), one gets a box (B,W,μ,ϵ) whose representations are the B-representations. The maps between B-representations are triples of linear maps $f = (f_1,f_2,g)$. In the following example, $V : V_1 \xrightarrow{\alpha} V_2$ and $W : W_1 \xrightarrow{\beta} W_2$ are representations of the quiver. The morphism $V \to W$ is given by the triple (f_1,f_2,g). The linear maps f_1 and f_2 make the diagram (without g) commutative, as for ordinary quiver representations. The additional map g makes the difference; in this example it can be chosen freely.

$$\begin{tikzcd}
V_1 \arrow[r, "f_1"] \arrow[d, "\alpha"'] \arrow[dr, "g"] & W_1 \arrow[d, "\beta"] \\
V_2 \arrow[r, "f_2"'] & W_2
\end{tikzcd}$$

Now we specify the representations V and W and get the following homomorphisms form V to W and back:

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "0"'] \arrow[dr, "0"] & k \arrow[r, "0"] \arrow[d, "1"] \arrow[dr, "1"] & 0 \arrow[d, "0"] \\
k \arrow[r, "1"'] & k \arrow[r, "0"'] & k
\end{tikzcd}$$

Because of g, there are now morphisms in both directions, the one on the righthand side not existing on the level of quiver representations, while the left hand morphism is the same as for quiver representations. In fact, this new morphism corresponds exactly to the 'additional' morphism occuring in $\mathcal{F}(\Delta)$, that is, the morphism $\Delta(1) \to \Delta(2)$. This correspondence is a special case of the main result to be formulated next. The quasi-hereditary algebra A occuring here is exactly the algebra of our example.

4. Results in [12]

The main result in [12] characterises quasi-hereditary algebras and at the same time their filtered categories.

Theorem 4.1. *(1) For a finite dimensional algebra A, the following are equivalent:*
(a) A is quasi-hereditary for some partial order $\leq$.

(b) A is Morita equivalent to $L_{\mathcal{B}'}$, the left Burt-Butler algebra of a directed box $\mathcal{B}'$.
(c) A is Morita equivalent to $R_{\mathcal{B}}$, the right Burt-Butler algebraof a directed box $\mathcal{B}$.
In this case, $\mathcal{F}(\Delta) \simeq rep(\mathcal{B})$.

(2) Let $\mathcal{B}$ be a directed box. Then $L_{\mathcal{B}}$ and $R_{\mathcal{B}}$ are Ringel dual to each other.

The Theorem is wrong if we replace 'Morita equivalent' by 'isomorphic'.

The second part implies that Ringel duality is a special case of Burt-Butler duality that relates $L_{\mathcal{B}}$ and $R_{\mathcal{B}}$; the latter also can be formulated for boxes that are not directed.

The proof of the theorem uses Keller's description of filtered categories in terms of A_∞-structures, the machinery of twisted stalks and the Maurer-Cartan equation, a connection of A_∞-structures with differential graded algebras and then a connection between differential graded algebras and boxes.

Finally, here is an application that motivated the whole development. Write $\mathcal{F}(\Delta) \simeq rep(\mathcal{B})$. As explained above in the context of Burt-Butler theory, this means $\mathcal{F}(\Delta) = Ind(B, R_B)$. Write $A = R_B$. Then the following are true:

(a) The induction functor $A \otimes_B -$ is an exact functor, that is, A_B isprojective.

(b) For each i, $A \otimes_B S_B(i) \simeq \Delta(i)$, where S_B denotes simple B-modules.

(c) The algebra B is directed.

This means exactly that B is an *exact Borel subalgebra* of the quasi-hereditary algebra A, in the sense of [11]; hence, the problem of existence of exact Borel subalgebras for quasi-hereditary algebras (up to Morita equivalence) has been solved.

Taking this point of view, the quasi-hereditary algebra A (or rather an algebra Morita equivalent to it) satisfies an analogue of the PBW-theorem, which for semisimple complex Lie algebras states a bimodule isomorphism: $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}_+) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{h}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{n}_-)$. This bimodule isomorphism implies that the Lie theoretic Borel subalgebra $\mathcal{U}(\mathfrak{n}_+ \oplus \mathfrak{h}) \simeq \mathcal{U}(\mathfrak{n}_+) \otimes_{\mathbf{C}} \mathcal{U}(\mathfrak{h})$ satisfies (a); condition (b) is the definition of Verma modules, which are the standard modules in this context, and condition (c) is satisfied by definition. Blocks of the BGG-category $\mathcal{O}$ of $\mathfrak{g}$ are Morita equivalent to quasi-hereditary algebras. For these particular algebras, existence of exact Borel subalgebras had been shown already in [11], but in general it had been an open problem.

For proofs and further details, see [12]. For more on the context and for some recent developments see [13].

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DERIVED EQUIVALENCES AND GORENSTEIN PROJECTIVE DIMENSION

HOSHINO MITSUO AND HIROTAKA KOGA

Abstract. In this note, we introduce the notion of complexes of finite Gorenstein projective dimension and show that a derived equivalence induces an equivalence between the full triangulated subcategories consisting of complexes of finite Gorenstein projective dimension provided that the equivalence satisfies a certain condition.

1. Introduction

Derived equivalences appear in various fields of current research in mathematics. For instance, in [3] Beilinson showed that there exists an algebra A such that the derived category of A is triangle equivalent to the derived category of coherent sheaves on $\mathbb{P}^n$, in [5] Broué conjectured abelian defect group conjecture and in [12] Kontsevich formulated mirror symmetry in terms of derived equivalences. So it is more and more important to study derived equivalences. It is natural to ask when two abelian categories are derived equivalent. A way to answer this question is to compare invariants under derived equivalences. It is well-known that for derived equivalent rings finiteness of selfinjective dimension is an invariant (see e.g. [11]). Finiteness of selfinjective dimension is closely related to Gorenstein projective dimension (see [9, 10]). So one can expect that there are some invariants associated with Gorenstein projective dimension.

In this note, we introduce the notion of complexes of finite Gorenstein projective dimension and show that a derived equivalence induces an equivalence between the full triangulated subcategories consisting of complexes of finite Gorenstein projective dimension provided that the equivalence satisfies a certain condition. Let $\mathcal{A}, \mathcal{B}$ be abelian categories with enough projectives. Denote by $\mathcal{P}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of projective objects and by $\mathcal{GP}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is said to have finite Gorenstein projective dimension if it is isomorphic to a bounded complex of Gorenstein projective objects in $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ (see Definition 11). We denote by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ consisting of complexes of finite Gorenstein projective dimension. Let $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ be a triangle equivalence. Assume that there exists an integer $a > 0$ such that

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_\mathcal{A}$ and $Q \in \mathcal{P}_\mathcal{B}$ unless $-a \leq i \leq a$. Then our main result states that F induces an equivalence between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$ (see Theorem 18). Note that in case $\mathcal{A}$ and $\mathcal{B}$ are module categories then such an integer a always exists for any derived equivalence F. As corollaries we have the following: the equivalence F induces a triangle

The detailed version of this paper will be submitted for publication elsewhere.

equivalence between $\mathcal{GP}_\mathcal{A}/\mathcal{P}_\mathcal{A}$ and $\mathcal{GP}_\mathcal{B}/\mathcal{P}_\mathcal{B}$ (see Corollary 19); $\mathcal{GP}_\mathcal{A} = \mathcal{P}_\mathcal{A}$ if and only if $\mathcal{GP}_\mathcal{B} = \mathcal{P}_\mathcal{B}$, $\widehat{\mathcal{GP}}_\mathcal{A} = \mathcal{GP}_\mathcal{A}$ if and only if $\widehat{\mathcal{GP}}_\mathcal{B} = \mathcal{GP}_\mathcal{B}$, and $\widehat{\mathcal{GP}}_\mathcal{A} = \mathcal{P}_\mathcal{A}$ if and only if $\widehat{\mathcal{GP}}_\mathcal{B} = \mathcal{P}_\mathcal{B}$ where $\widehat{\mathcal{GP}}_\mathcal{A}$ stands for the full subcategory of $\mathcal{A}$ consisting of objects $X \in \mathcal{A}$ with $\mathrm{Ext}^i_\mathcal{A}(X, \mathcal{P}_\mathcal{A}) = 0$ for $i > 0$ (see Corollary 20); and letting A, B be rings, A and B are derived equivalent if and only if $\mathcal{D}^{\mathrm{b}}(\text{Mod-}A)_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{Mod-}B)_{\mathrm{fGpd}}$ are equivalent as triangulated categories (see Corollary 22).

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2. Preliminaries

In this note, complexes are cochain complexes and objects are considered as complexes concentrated in degree zero. Let $\mathcal{A}$ be an abelian category with enough projectives. We denote by $\mathcal{P}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of all projective objects in $\mathcal{A}$. We denote by $\mathcal{D}(\mathcal{A})$ the derived category of complexes over $\mathcal{A}$ and by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ consisting of complexes with bounded cohomology. Also, we denote by $\mathrm{Hom}^\bullet_\mathcal{A}(-,-)$ the associated single complex of the double hom complex.

For an additive category $\mathcal{X}$ we denote by $\mathcal{K}(\mathcal{X})$ the homotopy category of cochain complexes over $\mathcal{X}$ and by $\mathcal{K}^+(\mathcal{X})$ and $\mathcal{K}^{\mathrm{b}}(\mathcal{X})$ the full triangulated subcategories of $\mathcal{K}(\mathcal{X})$ consisting of bounded below and bounded complexes, respectively.

For a ring A we denote by Mod-A the category of right A-modules.

We refer to [4], [8] and [14] for basic results in the theory of derived categories.

Definition 1. *For a complex $X^\bullet$, we denote by $\mathrm{Z}^i(X^\bullet)$ and $\mathrm{H}^i(X^\bullet)$ the ith cycle and the ith cohomology of $X^\bullet$, respectively.*

Definition 2 ([8])**.** *A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\bullet[-i], -)$ vanishes on $\mathcal{A}$ for $i \gg 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ the épaisse subcategory of $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ consisting of complexes of finite projective dimension.*

Note that the canonical functor $\mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ gives rise to equivalences of triangulated categories

$$\mathcal{K}^{\mathrm{b}}(\mathcal{P}_\mathcal{A}) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}.$$

Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$.

Definition 3. *A complex $X^\bullet \in \mathcal{K}(\mathcal{A})$ is said to be* $\mathrm{Hom}_\mathcal{A}(-,\mathcal{C})$*-exact if* $\mathrm{Hom}_\mathcal{A}(X^\bullet, C)$ *is exact for all $C \in \mathcal{C}$.*

Definition 4. *An exact sequence $0 \to M \to C^0 \to C^1 \to \cdots \to C^n \to \cdots$ in $\mathcal{A}$ is said to be a $\mathcal{C}$-coresolution of $M \in \mathcal{A}$ if $C^i \in \mathcal{C}$ for all i and the exact sequence is* $\mathrm{Hom}_\mathcal{A}(-,\mathcal{C})$*-exact.*

Definition 5 ([1, 7])**.** *An object $M \in \mathcal{A}$ is said to be Gorenstein projective if M admits a $\mathcal{P}$-coresolution. We denote by $\mathcal{GP}_\mathcal{A}$ the full subcategory of $\mathcal{A}$ consisting of Gorenstein projective objects $M \in \mathcal{A}$.*

We refer to [6] for basic facts on Gorenstein projective dimension.

3. Gorenstein projective dimension

In this section, we study Gorenstein projective objects and introduce the notion of complexes of finite Gorenstein projective dimension.

Definition 6. *We denote by* $\widehat{\mathcal{GP}}_{\mathcal{A}}$ *the full subcategory of* $\mathcal{A}$ *consisting of objects* $X \in \mathcal{A}$ *with* $\mathrm{Ext}^i_{\mathcal{A}}(X, \mathcal{P}_{\mathcal{A}}) = 0$ *for* $i > 0$.

Definition 7. *Let* $\widehat{\mathcal{GP}}_0 := \widehat{\mathcal{GP}}_{\mathcal{A}}$. *For* $n \geq 1$ *we denote by* $\widehat{\mathcal{GP}}_n$ *the full subcategory of* $\widehat{\mathcal{GP}}_{\mathcal{A}}$ *consisting of objects* X *admitting right resolutions in* $\mathcal{A}$ $0 \to X \to P^1 \to \cdots \to P^n \to Y \to 0$ *with* $Y \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ *and* $P^i \in \mathcal{P}$ *for* $1 \leq i \leq n$.

Proposition 8. *We have* $\mathcal{GP}_{\mathcal{A}} = \bigcap_{n \geq 0} \widehat{\mathcal{GP}}_n$.

Theorem 9. *Let* $X^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{GP}_{\mathcal{A}})$ *with* $X^i = 0$ *unless* $0 \leq i \leq l$. *Then there exists a quasi-isomorphism* $X^\bullet \to P^\bullet$ *with* $P^\bullet \in \mathcal{K}^+(\mathcal{P}_{\mathcal{A}})$ *such that* $\mathrm{Z}^{l+1}(P^\bullet) \in \mathcal{GP}_{\mathcal{A}}$ *and* $\mathrm{H}^{-i}(\mathrm{Hom}^\bullet_{\mathcal{A}}(P^\bullet, \mathcal{P}_{\mathcal{A}})) = 0$ *for* $i > l$.

Proposition 10. *Let* $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$. *The followings are equivalent:*

1. $X^\bullet \cong Y^\bullet$ *in* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *for some* $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{GP}_{\mathcal{A}})$.
2. *There exists a distinguished triangle* $X^\bullet \to Y^\bullet \to Z[l] \to$ *in* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *with* $Y^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$, $Z \in \mathcal{GP}_{\mathcal{A}}$ *and* $l \in \mathbb{Z}$.
3. $X^\bullet \cong Z[l]$ *in* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ *with* $Z \in \mathcal{GP}_{\mathcal{A}}$ *and* $l \in \mathbb{Z}$.

Definition 11. *A complex* $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *is said to have finite Gorenstein projective dimension if* $X^\bullet$ *satisfies the equivalent condition in Proposition 10. We denotes by* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ *the full subcategory of* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *consisting of all complexes in* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *having finite Gorenstein projective dimension.*

Theorem 12 (cf. [2] and [9, Proposition 3.5])**.** *The followings hold:*

1. *The embedding* $\widehat{\mathcal{GP}}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})$ *induces a fully faithful functor*
$$\widehat{\mathcal{GP}}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$$
2. *The embedding* $\mathcal{GP}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ *induces an equivalence*
$$\mathcal{GP}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$$

At the end of this section, using the quotient category, we characterize Gorenstein projective objects.

Theorem 13. *Let* $X \in \widehat{\mathcal{GP}}_{\mathcal{A}}$. *Then* $X \in \mathcal{GP}_{\mathcal{A}}$ *if and only if for each* $i > 0$ *there exists* $Y_i \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ *such that* $X \cong Y_i[-i]$ *in* $\mathcal{D}^{\mathrm{b}}(\mathcal{A})/\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$.

4. Derived equivalences

In this section, we deal with derived equivalences of abelian categories with enough projectives. Let $\mathcal{B}$ be an abelian category with enough projectives. Throughout this section we assume that there exists an equivalence of triangulated categories $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ with an integer $a > 0$ such that

$$(*) \qquad \mathrm{Hom}_{D(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{D(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_{\mathcal{A}}$ and $Q \in \mathcal{P}_{\mathcal{B}}$ unless $-a \leq i \leq a$ and G stands for a quasi-inverse of F.

Lemma 14. *$\mathrm{H}^i(FP) = 0$ for all $P \in \mathcal{P}_{\mathcal{A}}$ unless $-a \leq i \leq a$.*

Remark 15. *$\mathrm{H}^i(GQ) = 0$ for all $Q \in \mathcal{P}_{\mathcal{B}}$ unless $-a \leq i \leq a$.*

Proposition 16. *The equivalence F induces an equivalence of triangulated categories between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$.*

Proof. See [13, Proposition 8.2]. □

Lemma 17. *For each $X \in \widehat{\mathcal{GP}}_{\mathcal{A}}$ there exists $X' \in \widehat{\mathcal{GP}}_{\mathcal{B}}$ such that $FX \cong X'[a]$ in $\mathcal{D}^{\mathrm{b}}(\mathcal{B})/\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$.*

Theorem 18. *Let $F : \mathcal{D}^{\mathrm{b}}(\mathcal{A}) \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})$ be an equivalence of triangulated categories. If there exists $a > 0$ such that*

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(FP, Q[i]) = 0 = \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(Q, FP[i])$$

for all $P \in \mathcal{P}_{\mathcal{A}}$ and $Q \in \mathcal{P}_{\mathcal{B}}$ unless $-a \leq i \leq a$ then F induces an equivalence of triangulated categories between $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$.

Corollary 19. *The equivalence F induces an equivalence between $\mathcal{GP}_{\mathcal{A}}/\mathcal{P}_{\mathcal{A}}$ and $\mathcal{GP}_{\mathcal{B}}/\mathcal{P}_{\mathcal{B}}$.*

Corollary 20. *The following hold.*

(1) *$\mathcal{GP}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ if and only if $\mathcal{GP}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$.*
(2) *$\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{GP}_{\mathcal{A}}$ if and only if $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{GP}_{\mathcal{B}}$.*
(3) *$\widehat{\mathcal{GP}}_{\mathcal{A}} = \mathcal{P}_{\mathcal{A}}$ if and only if $\widehat{\mathcal{GP}}_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}}$.*

Proposition 21. *Let $F' : \mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fGpd}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fGpd}}$ be an equivalence of triangulated categories. Then F' induces an equivalence of triangulated categories $\mathcal{D}^{\mathrm{b}}(\mathcal{A})_{\mathrm{fpd}} \to \mathcal{D}^{\mathrm{b}}(\mathcal{B})_{\mathrm{fpd}}$ if both $\mathcal{A}$ and $\mathcal{B}$ satisfy the condition Ab4.*

Corollary 22. *Let A, B be rings. Then A and B are derived equivalent, i.e., $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}B)$ are equivalent as triangulated categories if and only if $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}A)_{\mathrm{fGpd}}$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{Mod}\text{-}B)_{\mathrm{fGpd}}$ are equivalent as triangulated categories.*

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CONSTRUCTION OF TWO-SIDED TILTING COMPLEXES FOR BRAUER TREE ALGEBRAS

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ABSTRACT. In this note, we explain how to construct two-sided tilting complexes corresponding to one-sided tilting complexes for Brauer tree algebras.

1. Introduction

For finite dimensional symmetric algebras Γ and Λ over an algebraically closed field k, the following is known.

Theorem 1. [2, 3, 4] *Let Γ and Λ be symmetric k-algebras. Then the following are equivalent.*

(1) Γ *and* Λ *are derived equivalent.*
(2) *There exists a complex T of* $K^b(\Gamma\text{-proj})$ *which satisfies the following conditions.*
 (i) $\mathrm{Hom}_{K^b(\Gamma\text{-proj})}(T, T[n]) = 0 \ \ (0 \neq \forall n \in \mathbb{Z})$.
 (ii) $\mathrm{add}(T)$ *generates* $K^b(\Gamma\text{-proj})$ *as a triangulated category.*
 (iii) $\mathrm{End}_{K^b(\Gamma\text{-proj})}(T) \cong \Lambda$.
(3) *There exists a complex C of* $K^b(\Gamma \otimes_k \Lambda^{op}\text{-mod})$ *which satisfies the following conditions.*
 (i) *All terms of C are projective as Γ-modules and as Λ^{op}-modules.*
 (ii) $C^* \otimes_\Gamma C \cong \Lambda$ *in* $K^b(\Lambda \otimes_k \Lambda^{op})$.

Definition 2. A complex T over Γ is called a one-sided tilting complex if it satisfies the conditions (i) and (ii) in Theorem 1 (2). A complex C over $\Gamma \otimes_k \Lambda^{op}$ is called a two-sided tilting complex if it satisfies the conditions (i) and (ii) in Theorem 1 (3).

Let C be a two-sided tilting complex over $\Gamma \otimes_k \Lambda^{op}$. It is known that if we consider C as a one-sided complex over Γ, then it is a one-sided tilting complex with endomorphism ring Λ. However it is difficult in general to construct the two-sided tilting complex corresponding to a one-sided tilting complex.

Let k be an algebraically closed field. Let A be a Brauer tree algebra over k associated to a Brauer tree with e edges and multiplicity μ of the exceptional vertex. Let $B(e, \mu)$ be a Brauer tree algebra over k with respect to a "star" with e edges and exceptional vertex with multiplicity μ in the center (or equivalently is a self-injective Nakayama algebra over k with e simple modules and the nilpotency degree of the radical being $e\mu + 1$). In [2], Rickard showed that A is derived equivalent to the algebra $B(e, \mu)$ by constructing a one-sided tilting complex T over A with endomorphism algebra $B(e, \mu)$. Our aim in this note is to construct a two-sided tilting complex C over $A \otimes_k B(e, \mu)^{op}$ corresponding to the one-sided tilting complex T constructed by Rickard in [2].

The detailed version of this paper will be submitted for publication elsewhere.

2. CONSTRUCTION OF TWO-SIDED TILTING COMPLEXES

Throughout this note algebras are of finite dimensional. First we recall the definition of stable equivalences of Morita type and properties of them.

Definition 3. Let Γ and Λ be symmetric k-algebras. Then Γ and Λ are said to be stably equivalent of Morita type if there exists a $\Gamma \otimes_k \Lambda^{op}$-module M such that

(1) M is projective as a Γ-module and as a Λ-module,
(2) $M \otimes_\Lambda M^* \cong \Gamma \oplus P$ as $\Gamma \otimes_k \Gamma^{op}$-modules, where P is a finitely generated projective $\Gamma \otimes_k \Gamma^{op}$-module and where $M^* = \mathrm{Hom}_k(M, k)$.

Proposition 4. [1, 3] *Two derived equivalent symmetric algebras are stably equivalent of Morita type.*

We know the Brauer tree algebras A and $B(e, \mu)$ defined in Section 1 are derived equivalent. Hence they are stably equivalent of Morita type by Proposition 4 since Brauer tree algebras are symmetric. Therefore there exists an $A \otimes_k B(e, \mu)^{op}$-module M inducing a stable equivalence of Morita type between A and $B(e, \mu)$.

Second we fix the notation to construct the two-sided tilting complex as mentioned above. For an edge corresponding to a simple A-module T, we define a positive integer $d(T)$ as the distance from the exceptional vertex to the furthest vertex of the edge. On this definition we put $m := \max\{d(T) | T : \text{simple } A\text{-module}\}$. Moreover let S be a simple A-module such that $d(S) = m$.

We need the next lemma later.

Lemma 5. *There exists an $A \otimes_k B(e, \mu)^{op}$-module M inducing a stable equivalence of Morita type between A and $B(e, \mu)$ such that $M^* \otimes_A S$ is simple.*

We construct the two-sided tilting complex by deleting some direct summand from each term of the projective resolution of M described in the Lemma5. Hence we consider the minimal projective resolution of M.

Lemma 6. [5] *Let Γ and Λ be symmetric k-algebras, and let $\mathcal{M}$ be a $\Gamma \otimes_k \Lambda^{op}$-module which is projective as a Γ-module and as a Λ-module. Then the projective cover of $\mathcal{M}$ is given by*

$$\bigoplus_W P(\mathcal{M} \otimes_\Lambda W) \otimes_k P(W)^*$$

where W runs over a complete set of representatives of isomorphism classes of simple Λ-modules.

If M is projective as an A-module and as a $B(e, \mu)^{op}$-module then $\Omega^n M$ is projective as an A-module and as a $B(e, \mu)^{op}$-module too for any integer n. Hence by Lemma 6 we obtain the minimal projective resolution of M:

$$
\begin{aligned}
\cdots \to & \bigoplus_{0\le i\le e-1} P(\Omega^{n-1}M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\to & \cdots \\
\to & \bigoplus_{0\le i\le e-1} P(\Omega M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_1} & \bigoplus_{0\le i\le e-1} P(M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_0} & M
\end{aligned}
$$

where V is the simple B-module $M^* \otimes_A S$ (see Lemma 5).

Lemma 7. *For the above projective resolution of M and $1 \le l \le m-2$,*

$$\pi_l(\bigoplus_{d(\mathrm{top}(\Omega^l M \otimes_B \Omega^{2i}V)) \le m-l-1} P(\Omega^l M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^*)$$

is contained in

$$\bigoplus_{d(\mathrm{top}(\Omega^{l-1} M \otimes_B \Omega^{2i}V)) \le m-l} P(\Omega^{l-1} M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^*.$$

We can construct the two-sided complex $C = (C_n, d)$ by deleting a direct summand in each term of the projective resolution of M as follows:

$$
\begin{cases}
C_0 = M & \\
C_n = \displaystyle\bigoplus_{d(\mathrm{top}(\Omega^{n-1} M \otimes_B \Omega^{2i}V)) \le m-n} P(\Omega^{n-1}M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^* & (1 \le n \le m-1) \\
C_n = 0 & (\text{otherwise})
\end{cases}
$$

and letting d_l be the restriction of π_l to C_l. By Lemma 7 we have that d_l is well-defined for each l. This two-sided complex C is a two-sided tilting complex and if we restrict action of $A \otimes_k B(e,\mu)^{op}$ to A then it coinsides with the one-sided tilting complex T constructed by Rickard in [2].

Theorem 8. *Let A be a Brauer tree algebra with e edges and multiplicity μ and let $B(e,\mu)$ be a Brauer tree algebra for a star with e edges and exceptional vertex with multiplicity μ in the center of the star. Then there exists a two-sided tilting complex C over $A \otimes_k B(e,\mu)^{op}$ such that when restricted to A, this complex coinsides with the one-sided tilting complex T constructed by Rickard in* [2].

3. OUTLINE OF THE PROOF

In this section, let C be the two-sided tilting complex over $A \otimes_k B(e,\mu)^{op}$ constructed in Section 2 and let $T = \bigoplus_{1\le i\le e} T_i$ be the one-sided tilting complex constructed by Rickard in [2], where T_i is the indecomposable summand for each $1 \le i \le e$.

To show that the two-sided complex C is a tilting complex, we show the following lemma.

Lemma 9. *For the two-sided complex C, the following hold.*

- $\mathrm{Hom}_{D^b(B\otimes_k B^{op})}(C^* \otimes_A C, V_i \otimes_k V_j) \cong \delta_{ij}k$
- $\mathrm{Hom}_{D^b(B\otimes_k B^{op})}(C^* \otimes_A C, V_i \otimes_k V_j[-n]) = 0$ *for* $1 \leq n \leq m-1$

where $\{V_1, \cdots, V_e\}$ *are the complete set of representatives of the isomorphism classes of simple* $B(e,\mu)$*-modules.*

This lemma shows that $C^* \otimes_A C \cong B$ in $K^b(B \otimes_k B^{op})$. Therefore C is the two-sided tilting complex.

Next, to show that if we restrict C to A then it coincides with T in the derived category $D^b(A)$, we show the following condition.

Lemma 10. *For the two-sided tilting complex C and the simple B-module V_i, there exists an indecomposable summand T_i of T such that it satisfies the following conditions:*

- $\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B V_i) \cong k$.
- *For any simple B-module V_j which is not isomorphic to V_i,*
$$\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B V_j) = 0.$$
- *For any nonzero integer n and any simple B-module U,*
$$\mathrm{Hom}_{D^b(A)}(T_i, C \otimes_B U[n]) = 0.$$

This lemma shows that $C \otimes_B B \cong T$ in the derived category $D^b(A)$. In other words, if we restrict C to A, then it coincides with T in the derived category $D^b(A)$. Therefore C is the requiered two-sided tilting complex.

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COTORSION PAIRS ON TRIANGULATED AND EXACT CATEGORIES

YU LIU

Abstract. We study hearts of cotorsion pairs in triangulated and exact categories. We show that they are equivalent to functor categories over cohearts of the cotorsion pairs.

1. Introduction

The notion of cotorsion pair in triangulated and exact categories is a general framework to study important structures in representation theory. Recently the notion of hearts of cotorsion pairs was introduced in [8] and [6], and they are proved to be abelian categories, which were known for the heart of t-structure [2] and the quotient category by cluster tilting subcategory. We refer to [7] and [1] for more results on hearts of cotorsion pairs.

In this talk, we give an equivalence between hearts and the functor categories over cohearts. For the details of functor category, see [4, Definition 2.9].

For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on a triangulated category $\mathcal{T}$, we introduce the notion of *cohearts* of a cotorsion pair, denote by

$$\mathcal{C}=\mathcal{U}[-1]\cap {}^{\perp}\mathcal{U}.$$

This is a generalization of coheart of a co-t-structure, which plays an important role in [5]. We have the following theorem in triangulated category.

Theorem 1. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on a triangulated category $\mathcal{T}$. If $\mathcal{U}[-1]\subseteq \mathcal{C}*\mathcal{U}$, then the heart of $(\mathcal{U},\mathcal{V})$ has enough projectives, and moreover it is equivalent to the functor category* $\mathrm{mod}\mathcal{C}$.

This generalizes [3, Theorem 3.4] which is for t-structure. One standard example of this theorem is the following: let A be a Noetherian ring with finite global dimension, then the standard t-structure of $\mathsf{D}^b(\mathrm{mod}A)$ has a heart $\mathrm{mod}A$ with co-heart $\mathrm{proj}A$, and we have an equivalence $\mathrm{mod}A\simeq \mathrm{mod}(\mathrm{proj}A)$ in this case.

For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on an exact category $\mathcal{E}$, we denote

$$\mathcal{C}=\mathcal{U}\cap {}^{\perp_1}\mathcal{U}$$

the *coheart* of $(\mathcal{U},\mathcal{V})$. We have the following theorem in exact category.

Theorem 2. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on an exact category $\mathcal{E}$ with enough projectives and injectives, if for any any object $U\in\mathcal{U}$, there exists an exact sequence $0\to U'\to C\to U\to 0$ where $U'\in\mathcal{U}$ and $C\in\mathcal{C}$, then the heart of $(\mathcal{U},\mathcal{V})$ has enough projectives, and moreover it is equivalent to the functor category* $\mathrm{mod}(\mathcal{C}/\mathcal{P})$, *where $\mathcal{P}$ is the subcategory of projetive objects on $\mathcal{E}$.*

The detailed version of this paper will be submitted for publication elsewhere.

2. Hearts on triangulated categories

Let $\mathcal{T}$ be a triangulated category.

Definition 3. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{T}$ which are closed under direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.

(b) For any object $T \in \mathcal{T}$, there exist two short exact sequences

$$T[-1] \to V_T \to U_T \to T, \quad T \to V^T \to U^T \to T[1]$$

satisfying $U_T, U^T \in \mathcal{U}$ and $V_T, V^T \in \mathcal{V}$.

For a cotorsion pairs $(\mathcal{U}, \mathcal{V})$, let $\mathcal{W} := \mathcal{U} \cap \mathcal{V}$. We denote the quotient of $\mathcal{T}$ by $\mathcal{W}$ as $\underline{\mathcal{T}} := \mathcal{T}/\mathcal{W}$. Let

$$\mathcal{T}^+ := \{T \in \mathcal{T} \mid U_T \in \mathcal{W}\}, \quad \mathcal{T}^- := \{T \in \mathcal{T} \mid V^T \in \mathcal{W}\}.$$

Let

$$\mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-$$

we call the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$. Under these settings, Abe, Nakaoka [1] introduced the homological functor $H : \mathcal{T} \to \underline{\mathcal{H}}$ associated with $(\mathcal{U}, \mathcal{V})$. We often use the following property of H: $H(\mathcal{U}) = 0 = H(\mathcal{V})$.

Let's start with an important property for H.

Proposition 4. *The functor $H : \mathcal{C} \to H(\mathcal{C})$ is an equivalence.*

Hence it is enough to show that $\underline{\mathcal{H}} \simeq \mathrm{mod}(H(\mathcal{C}))$.

Then we give the following theorem.

Theorem 5. *If $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$.*

Now we have the main result of this section.

Theorem 6. *If $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$, then $\underline{\mathcal{H}} \simeq \mathrm{mod}(H(\mathcal{C}))$.*

Note that the condition $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$ is satisfied in many cases. The following proposition is given as an example.

Proposition 7. *If $\mathcal{U}$ is covariantly finite and $\mathcal{T}$ is Krull-Schmidt, then $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$.*

3. Hearts on exact categories

Let $\mathcal{E}$ be a exact category with enough projectives $\mathcal{P}$ and enough injectives $\mathcal{I}$.

Definition 8. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{E}$ which are closed under direct summands. We call $(\mathcal{U}, \mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{U}, \mathcal{V}) = 0$.

(b) For any object $B \in \mathcal{E}$, there exits two short exact sequences

$$0 \to V_B \to U_B \to B \to 0, \quad 0 \to B \to V^B \to U^B \to 0$$

satisfying $U_B, U^B \in \mathcal{U}$ and $V_B, V^B \in \mathcal{V}$.

For a cotorsion pairs $(\mathcal{U}, \mathcal{V})$, we denote the quotient of $\mathcal{E}$ by $\mathcal{U} \cap \mathcal{V}$ as $\underline{\mathcal{E}} := \mathcal{E}/\mathcal{U} \cap \mathcal{V}$. Let

$$\mathcal{E}^+ := \{B \in \mathcal{E} \mid U_B \in \mathcal{W}\}, \quad \mathcal{E}^- := \{B \in \mathcal{E} \mid V^B \in \mathcal{W}\}.$$

Let

$$\mathcal{H} := \mathcal{E}^+ \cap \mathcal{E}^-$$

we denote the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U}, \mathcal{V})$. Let $H : \mathcal{E} \to \underline{\mathcal{H}}$ be the half exact functor associated with $(\mathcal{U}, \mathcal{V})$ [7]. We often use the following property of H: $H(\mathcal{U}) = 0 = H(\mathcal{V})$. Since $\mathcal{P} \subseteq \mathcal{U}$ and $\mathcal{I} \subseteq \mathcal{V}$, we have $H(\mathcal{P}) = 0 = H(\mathcal{I})$.

Let $\Omega\mathcal{C} = \{X \in \mathcal{E} \mid \text{X admits } 0 \to X \to P \to C \to 0 \text{ where } P \in \mathcal{P} \text{ and } C \in \underline{\mathcal{C}}\}$. Let $\pi : \Omega\mathcal{C} \to \Omega\mathcal{C}/\mathcal{P}$ be the quotient functor, since $H(\mathcal{P}) = 0$, we have a functor $\overline{H} : H(\Omega\mathcal{C}) \to \Omega\mathcal{C}/\mathcal{P}$ such that $\overline{H}\pi = H$.

As in the last section, we have the following proposition.

Proposition 9. *$\overline{H} : \Omega\mathcal{C}/\mathcal{P} \to H(\Omega\mathcal{C})$ is an equivalence.*

Since $\mathcal{C}/\mathcal{P} \simeq \Omega\mathcal{C}/\mathcal{P}$, it is enough to show that the heart of $(\mathcal{U}, \mathcal{V})$ to $\mathrm{mod}(H(\Omega\mathcal{C}))$.

Now we are ready to give the main theorem of this section.

Theorem 10. *If for any any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$.*

Theorem 11. *If for any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$, then $\underline{\mathcal{H}} \simeq \mathrm{mod}(H(\Omega\mathcal{C}))$.*

The following proposition shows that the assumption of Theorem 10 and 11 is satisfied in many cases.

Proposition 12. *If $\mathcal{U}$ is covariantly finite and contains $\mathcal{I}$, $\mathcal{E}$ is Krull-Schmidt , then for any object $U \in \mathcal{U}$, there exists an exact sequence $0 \to U' \to C \to U \to 0$ where $U' \in \mathcal{U}$ and $C \in \mathcal{C}$.*

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THE STRUCTURE OF PREENVELOPES WITH RESPECT TO MAXIMAL COHEN-MACAULAY MODULES

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Abstract. This article studies the structure of special preenvelopes with respect to maximal Cohen-Macaulay modules. We investigate the structure of them in terms of their kernels and cokernels. Moreover, using this result, we also study the structure of special proper coresolutions with respect to maximal Cohen-Macaulay modules over a Henselian Cohen-Macaulay local ring.

This article is based on [3]. Throughout this article, we assume that R is a d-dimensional Cohen-Macaulay local ring with canonical module ω. All R-modules are assumed to be finitely generated. Denote by $\mathsf{mod}R$ the category of finitely generated R-modules and by MCM the full subcategory of $\mathsf{mod}R$ consisting of maximal Cohen-Macaulay R-modules.

Auslander and Buchweitz showed the following result which plays an important role in the representation theory of commutative rings.

Theorem 1. [1] *For any R-module M, there exists a short exact sequence*

$$0 \to Y \to X \xrightarrow{\pi} M \to 0$$

such that $X \in \mathsf{MCM}$ and $\mathsf{id}_R Y < \infty$.

The morphism π is called a *maximal Cohen-Macalulay approximation* of M.

In this article, we mainly study a special MCM-preenvelope which is a categorical dual notion of a maximal Cohen-Macaulay approximation.

Definition 2. Let $\mu : M \to X$ be an R-homomorphism with $X \in \mathsf{MCM}$.

(1) μ is called an MCM-*preenvelope* of M if

$$\mathsf{Hom}_R(\mu, X') : \mathsf{Hom}_R(X, X') \to \mathsf{Hom}_R(M, X')$$

is an epimorphism for any $X' \in \mathsf{MCM}$.

(2) μ is called a *special* MCM-*preenvelope* of M if μ is an MCM-preenvelope and satisfies $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.

(3) μ is called an MCM-*envelope* of M if μ is an MCM-preenvelope and every $\phi \in \mathsf{End}_R(X)$ that satisfies $\phi\mu = \mu$ is an automorphism.

The notions of MCM-*precover*, *special* MCM-*precover*, and MCM-*cover* are defined dually.

Remark 3. (1) By definition, a maximal Cohen-Macaulay approximation is nothing but a special MCM-precover.

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(2) Owing to Wakamatsu's lemma, an MCM-envelope is a special MCM-preenvelope, and by definition, a special MCM-preenvelope is an MCM-preenvelope.
(3)
- (Auslander-Buchweitz [1])
  Every R-module has a special MCM-precover.
- (Yoshino [4])
  If R is Henselian (e.g. complete), then every R-module has an MCM-cover.
- (Holm [2])
  Every R-module has a special MCM-preenvelope, and if R is Henselian, every R-module has an MCM-envelope.

Since every MCM-precover is an epimorphism, for any R-homomorphism $\pi : X \to M$ with X maximal Cohen-Macaulay, the following are equivalent.
(1) π is a special MCM-precover.
(2) $\mathsf{Coker}\,\pi = 0$ and $\mathsf{Ext}^1_R(\mathsf{MCM}, \mathsf{Ker}\,\pi) = 0$.
(3) $\mathsf{Coker}\,\pi = 0$ and $\mathsf{id}(\mathsf{Ker}\,\pi) < \infty$.
Therefore, MCM-preenvelopes are characterized by using their kernels and cokernels. We consider the following question.

Question 4. *When is a given morphism $\mu : M \to X$ with $X \in \mathsf{MCM}$ a special MCM-preenvelope?*

The following result is our main theorem in this article and which gives an answer to this question.

Theorem 5. *Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*
(1) *μ is a special MCM-preenvelope of M.*
(2) *$\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$ and $\mathsf{Ext}^1_R(\mathsf{Coker}\,\mu, \mathsf{MCM}) = 0$.*
(3) *$\mathsf{codim}(\mathsf{Ker}\,\mu) > 0$, and there exists an exact sequence*

$$0 \to S \to \mathsf{Coker}\,\mu \to T \to U \to 0$$

such that
- *$\mathsf{codim}\,S > 1$, $\mathsf{codim}\,U > 2$,*
- *T satisfies Serre's condition (S_2),*
- *$\mathsf{id}_R T^\dagger < \infty$ and $T^\dagger$ satisfies Serre's condition (S_3).*

The condition (3) in Theorem 5 is rather complicated, but which characterizes a special MCM-preenvelope by some numerical conditions, and has an advantage that it does not contain vanishing condition of Ext-module, which is hard to check.

Next, we give some examples of special MCM-preenvelopes.

Example 6. (1) Let M be an R-module with $\mathsf{codim}\,M > 0$. Then $\mu : M \to 0$ is a special MCM-preenvelope.
(2) Let $\underline{x} = x_1, x_2, \ldots, x_n$ be an R-regular sequence with $n \geq 3$. Consider an exact sequence

$$0 \to M \xrightarrow{\mu} R^{\oplus n} \xrightarrow{(x_1,\ldots,x_n)} R \to R/(\underline{x}) \to 0.$$

Then μ is a special MCM-preenvelope.

(3) Let K and C be R-modules with $\operatorname{\mathsf{codim}} K > 0, \operatorname{\mathsf{codim}} C > 1$ and $\sigma \in \operatorname{\mathsf{Ext}}^2_R(C, K)$. σ defines an exact sequence

$$0 \to K \to M \xrightarrow{\mu} F \to C \to 0$$

with F a free R-module. Then μ is a special MCM-preenvelope of M.

Using the condition (3) in Theorem 5, we have a result about a special proper MCM-coresolution.

Definition 7. Let M be an R-module, and

$$(*) \qquad 0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots$$

be an R-complex with $X^i \in \mathsf{MCM}$ for each i. Put $\mu^0 := \delta^0$ and let $\mu^i : \operatorname{\mathsf{Coker}} \delta^{i-1} \to X^i$ be the induced morphism from δ^i for $i > 0$.

- If each μ^i is a special MCM-preenvelope (resp. an MCM-envelope), then we call (*) a special proper MCM-coresolution (resp. a minimal proper MCM-coresolution) of M.
- For a minimal proper MCM-coresolution (*), $\operatorname{\mathsf{Coker}} \mu^{i-1}$ is called an i-th minimal MCM-cosyzygy of M, and it is denoted by ${\mathsf{Cosyz}_{\mathsf{MCM}}}^i M$.

Remark 8. Suppose R is Henselian. For a special proper MCM-coresolution (*),

- $\operatorname{\mathsf{Coker}} \mu^i$ are unique up to free summands, and
- $\operatorname{\mathsf{Ker}} \mu^i$ are unique up to isomorphism.

Theorem 9. *Suppose R is Henselian. Let M be an R-module and*

$$0 \to M \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \xrightarrow{\delta^2} \cdots$$

a special proper MCM*-coresolution of M. Put $\mu^0 := \delta^0$ and let $\mu^i : \operatorname{\mathsf{Coker}} \delta^{i-1} \to X^i$ be the induced homomorphisms. Then for each $i \geq 0$, one has*

(1) $\operatorname{\mathsf{codim}}(\operatorname{\mathsf{Ker}} \mu^i) > i$,
(2) *there exists an exact sequence*

$$0 \to S^i \to \operatorname{\mathsf{Coker}} \mu^i \to T^i \to U^i \to 0$$

such that

- $\operatorname{\mathsf{codim}} S^i > i + 1$, $\operatorname{\mathsf{codim}} U^i > i + 2$,
- T^i *satisfies* (S_2),
- $\mathsf{id}_R(T^i)^\dagger < \infty$ *and* $(T^i)^\dagger$ *satisfies* (S_{i+3}).

From the above remark, we can show this theorem by constructing such a special proper MCM-coresoltion.

Letting $i = d - 2, d - 1$ in Theorem 9, we have the following corollary.

Corollary 10. *Suppose R is Henselian. For any R-module M,*

- ${\mathsf{Cosyz}_{\mathsf{MCM}}}^d M = 0$ *and*
- ${\mathsf{Cosyz}_{\mathsf{MCM}}}^{d-1} M$ *has finite length.*

In particular, for any R-module M, the minimal proper MCM*-coresolution of M has length at most* $\min\{0, d - 2\}$.

Remark 11. This corollary refines a theorem due to Holm [2, Theorem C]: For any R-module M, the minimal proper MCM-coresolution of M has length at most $\min\{0, d-1\}$.

For the last of this article, we give another characterization of special MCM-preenvelopes in terms of the existence of certain complexes.

Auslander and Buchweitz also state the following result

Theorem 12. [1] *Let $\pi : X \to M$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*

(1) *π is a special MCM-precover of M.*
(2) *There exists an R-complex*

$$C = (0 \to C_d \xrightarrow{\delta_{d-1}} C_{d-1} \xrightarrow{\delta_{d-2}} \cdots \to C_1 \xrightarrow{\delta_0} C_0 \xrightarrow{\delta_{-1}} C_{-1} \to 0)$$

such that
- *C_i is a finite direct sum of ω for $1 \le i \le d$,*
- *$\delta_{-1} = \pi$,*
- *C is exact.*

The following theorem is dual of this theorem.

Theorem 13. *Let $\mu : M \to X$ be an R-homomorphism such that $X \in \mathsf{MCM}$. Then the following conditions are equivalent;*

(1) *μ is a special MCM-preenvelope of M.*
(2) *There exists an R-complex*

$$C = (0 \to C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots \xrightarrow{\delta^{d-2}} C^{d-1} \to 0)$$

such that
- *C^i is free for $1 \le i \le d-1$,*
- *$\delta^{-1} = \mu$,*
- *$\operatorname{codim} \mathsf{H}^i(C) > i+1$ for any i.*

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HIGHER PRODUCTS ON YONEDA EXT ALGEBRAS..

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Abstract. We show that Massey products on a relative split Ext algebra can be computed in terms of relative split exact sequences. As an application we provide a proof and generalization of the result due to Gugenheim-May and Keller which states that in a suitable situation the Ext algebra $E = \mathrm{Ext}_A(S, S)$ of the direct sum S of all simple modules is generated by the degree 1-part E^1 as an algebra with "higher products". In our proof we see that this proposition is a trivial consequence of the elementary fact that every finite length module has composition series.

1. Introduction

We would like to recall a basic fact of Homological algebra. The n-th extension group $\mathrm{Ext}^n_A(N, M)$ the n-th derived functor of Hom functor has a description

$$\mathrm{Ext}^n_A(N, M) = \{0 \to M \to X_1 \to X_2 \to \cdots \to X_n \to N \to 0\} /(\text{equivalence})$$

and that under this description, the multiplication on the Ext algebra $\mathrm{Ext}_A(M, M) = \bigoplus_{n \geq 0} \mathrm{Ext}^n_A(M, M)$ corresponds to splicing exact sequences which represents corresponding elements.

Since the Ext algebra $\mathrm{Ext}^n_A(M, M)$ is the cohomology algebra of the endomorphism dg-algebra $\mathbf{R}\mathrm{Hom}_A(P, P)$ for a projective (or injective) resolution P of M, it has more structure than merely a graded associative multiplication. These are A_∞-structure and (higher, matric) Massey products, which are not 2-ary operations but multi-ary operations and are collectively called higher products. Such structure was found in topology and has been studied in many area. Recently, higher products have been becoming to play important role in representation theory (see e.g., [3]).

Now we meet a simple question that under the above description of Ext algebra by exact sequences, what operations for exact sequences correspond to higher products. It is not so obvious at the first sight. For example, presence of triple product tells us that there exists a way to construct an exact sequence $0 \to M \to X \to Y \to M \to 0$ from three short exact sequences $0 \to M \to U \to M \to 0, 0 \to M \to V \to M \to 0, 0 \to M \to W \to M \to 0$.

In this note, an answer is given for Massey products. Namely we show that Massey products on a Ext algebra can be computed in terms of exact sequences. As an application we provide a proof and generalization of the result due to Gugenheim-May and Keller which states that in a suitable situation the Ext algebra $E = \mathrm{Ext}_A(S, S)$ of the direct sum S of all simple modules is generated by the degree 1-part E^1 as an algebra with higher products.

In our proof we see that this proposition is a trivial consequence of the elementary fact that every finite length module has composition series.

The detailed version of this paper will be submitted for publication elsewhere.

2. Massey product

We recall the definition of Massey products (for detail see e.g. [5]). Let $\mathsf{C} = (\bigoplus_{d\in\mathbb{Z}} \mathsf{C}, \partial)$ be a dg-algebra and $\alpha_{i,i+1}$ be a homogeneous element of the cohomology algebra $\mathrm{H}(\mathsf{C})$ of degree $d_{i,i+1}$. We assume that $\alpha_{i,i+1}\alpha_{i+1,i+2} = 0$ for $i = 0, \ldots, n-2$. We define the integers d_{ij} for $0 \leq i+1 < j \leq n$ by the formula $d_{ij} = (\sum_{k=i+1}^{j-1} d_{ik}d_{kj}) - 1$ by using induction on $j - i$.

In this situation we say that the Massey product $\langle \alpha_{01}, \alpha_{12}, \ldots, \alpha_{n-1,n}\rangle$ is defined if there exist $\rho_{ij} \in \mathsf{C}^{d_{ij}}$ for $0 \leq i < j \leq n$ with $(i,j) \neq (0,n)$ such that (1) $\rho_{i,i+1}$ is a cocycle such that $[\rho_{i,i+1}] = \alpha_{i,i+1}$ and that (2) the following equations are satisfied:

$$\sum_{k=i+1}^{j-1} (-1)^{d_{ik}+1}\rho_{ik}\rho_{kj} = \partial\rho_{ij}.$$

In the case where the Massey product $\langle \alpha_{01}, \alpha_{12}, \ldots, \alpha_{n-1,n}\rangle$ is defined, the set of the Massey product $\langle \alpha_{01}, \alpha_{12}, \ldots, \alpha_{n-1,n}\rangle$ of $\alpha_{01}, \ldots, \alpha_{n-1,n}$ is defined to be a subset of $\mathrm{H}(\mathsf{C})$ which consists of the cohomology class $[\rho_{0n}]$ of the cocycles ρ_{0n} such that there exists a collection $(\rho_{ij})_{0\leq i<j\leq n,(i,j)\neq(0,n)}$ satisfying the above conditions such that

$$\rho_{0n} = \sum_{k=1}^{n-1} (-1)^{d_{0k}+1}\rho_{0k}\rho_{kn}.$$

3. Computation of Massey products of Yoneda Ext algebra in terms of exact sequences

For simplicity we deal with algebras A over a field k.

To compute Massey products on $\mathrm{Ext}_A(M,N)$, we use the model

$$\mathsf{C}(M,N) = \mathrm{Hom}_{A-A}(\mathsf{B}(A), \mathrm{Hom}_k(M,N))$$

where $\mathsf{B}(A)$ is the Bar resolution of A. We recall two things: the n-th cohomology group $\mathrm{H}^i(\mathsf{C}(M,N))$ is naturally isomorphic to the i-th extension group $\mathrm{Ext}^i_A(M,N)$. The coalgebra structure on $\mathsf{B}(A)$ induces the product

$$\mathsf{C}(M,N) \times \mathsf{C}(L,M) \to \mathsf{C}(L,N)$$

which is a morphism of complexes that become the Yoneda product after taking the cohomology group.

Let $M_0, M_1, \ldots, M_n$ be A-modules. A finite exhaustive filter $F : 0 = F_{-1} \subsetneq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = M$ of an A-module M is said to be of type $(M_0, M_1, \ldots, M_n)$ if we have isomorphisms $F_i/F_{i-1} \cong M_i$. Thus a filtered module M of type $(M_0, M_1, \ldots, M_n)$ is isomorphic to the direct sum $\bigoplus_{i=0}^n M_i$ as k-modules.

Theorem 1. *Let ξ be the following exact sequence*

$$0 \to M_0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{a-1}} X_a \xrightarrow{f_a} N \to 0.$$

Assume that the module X_1 has a filter of type $(M_0, M_1, \ldots, M_n)$ for some A-module $M_1, \ldots, M_n$. Then let X_1' be the kernel of the projection $\pi : X_1 \to M_n$ and $X_2' :=$

$X_2/f_1(X_1')$. *Let η be the exact sequence*

$$0 \to M_n \xrightarrow{g_1} X_2' \xrightarrow{g_2} X_3 \xrightarrow{f_3} \cdots \xrightarrow{f_{a-1}} X_a \xrightarrow{f_a} N \to 0$$

where g_1 and g_2 are induced morphisms. Let $\lambda_{i,i+1} \in \mathrm{Z}^1(\mathsf{C}(M_{i+1}, M_i))$ for $i = 0, 1, \ldots, n-1$ be the cocycle represent the extension induced from the filtration of X_1. Then the Massey product $\langle [\lambda_{01}], [\lambda_{12}], \ldots, [\lambda_{n-1n}], [\eta] \rangle$ is defined and contains $(-1)^{n+1}[\xi]$.

4. Applications: Generating condition of Ext algebra as algebra with higher products

Thanks to Theorem 1, the following theorems are nothing but a consequences the elementary fact that every finite length module has a composition series.

Theorem 2 ([1, Corollary 5.17],[2, 2.2.1.(b)])**.** *Let A be a locally finite non-negatively graded algebra over a field and S be a direct sum of all simple modules. Then the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using Massey products.*

Proof. Once we recall the fact that any element $\alpha \in \mathrm{Ext}^n_A(S, S)$ is represented by an exact sequence ξ such that each X_i is of finite length.

$$\xi : 0 \to S \to X_1 \to X_2 \to \cdots \to X_n \to S \to 0$$

Then by using Theorem 1, we can show that ξ is obtained as a consequence of iteration of Massey products from $\mathrm{Ext}^1_A(S, S)$. $\square$

Remark 3.
1. The proof of [1] required that the dg-algebra $\mathsf{C}(S, S)$ is augmented and the degree 0-part $\mathsf{C}^0(S, S)$ is semi-simple. The authors showed that the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using Matric Massey products.
2. In [2], any proof is not written. The author stated that Then the extension algebra $\mathrm{Ext}_A(S, S)$ is generated by $\mathrm{Ext}^0_A(S, S)$ and $\mathrm{Ext}^1_A(S, S)$ using A_∞ products.
3. By [4], for the conditions for a graded algebra E below we have the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3):
   1. E is generated by using Massey products by E^0 and E^1.
   2. E is generated by using Matric Massey products by E^0 and E^1.
   3. E is generated by using A_∞-products by E^0 and E^1.

   I don't know how is the converse.

Corollary 4. *Let A be a locally finite non-negatively graded algebra such that the augmentation algebra A^0 is semi-simple. Then the followings are equivalent:*
1. *A is Koszul.*
2. *the Ext algebra $E = \mathrm{Ext}_A(A_0, A_0)$ is generated by E^0 and E^1 as an ordinary algebra.*
3. *the higher A_∞-product on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*
4. *the Matric Massey products on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*
5. *the Massey products on the Ext algebra $\mathrm{Ext}_A(A_0, A_0)$ vanish.*

In the similar way we can prove the following theorem. We recall that a ring R is called a *Noetherian algebra* if the center $Z(R)$ is a (commutative) Noetherian ring and R is a finite $Z(R)$-module.

Theorem 5. *Let R be a Noetherian algebra with the center $Z = Z(R)$ and $\mathfrak{p}$ a maximal ideal of Z. We set $\kappa(\mathfrak{p}) := R \otimes_Z Z_{\mathfrak{p}}/\mathfrak{p}Z_{\mathfrak{p}}$. Then the Ext algebra $E = \mathrm{Ext}_R(\kappa(\mathfrak{p}), \kappa(\mathfrak{p}))$ is generated by E^0 and E^1 using Massey products.*

This can be proved by using the fact that every exact sequence which has $\kappa(\mathfrak{p})$ the most left term and the right most term is equivalent to a exact sequence with finite length middle terms X_i.

$$0 \to \kappa(\mathfrak{p}) \to X_1 \to X_2 \to \cdots \to X_n \to \kappa(\mathfrak{p}) \to 0$$

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HIGHER APR TILTING PRESERVE n-REPRESENTATION INFINITENESS

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Abstract. We show that m-APR tilting preserves n-representation infiniteness for $1 \leq m \leq n$. Moreover, we show that these tilting modules lift to tilting modules for the corresponding higher preprojective algebras, which is $(n+1)$-CY algebras. We also study the interplay of the two kinds of tilting modules.

1. Introduction

In this note, we show that m-APR tilting modules preserve n-representation infiniteness for m with $1 \leq m \leq n$. By this fact, we obtain a large family of n-representation infinite algebras. Our next result is that these modules lift to tilting modules over the corresponding $(n+1)$-preprojective algebras. Moreover, we show that the $(n+1)$-preprojective algebra of an m-APR tilted algebra is isomorphic to the endomorphism algebra of the corresponding tilting module induced by the m-APR tilting module. This also implies that we obtain a family of $(n+1)$-CY algebras, which are derived equivalent to each other.

$$\begin{tikzcd}
\left\{\begin{array}{c} n\text{-representation} \\ \text{infinite algebras} \end{array}\right\} \arrow[r, "\substack{\text{higher APR tilting} \\ \text{(Theorem 7)}}"] \arrow[d, "\substack{(n+1)\text{-preprojective} \\ \text{algebras}}"'] & \left\{\begin{array}{c} n\text{-representation} \\ \text{infinite algebras} \end{array}\right\} \arrow[d, "\substack{(n+1)\text{-preprojective} \\ \text{algebras}}"] \\
\left\{\begin{array}{c} \text{bimodule } (n+1)\text{-Calabi-Yau} \\ \text{algebras of G. P. 1} \end{array}\right\} \arrow[r, "\substack{\text{corresponding tilting} \\ \text{(Theorem 8)}}"'] & \left\{\begin{array}{c} \text{bimodule } (n+1)\text{-Calabi-Yau} \\ \text{algebras of G. P. 1} \end{array}\right\}
\end{tikzcd}$$

Notations. Let K be an algebraically closed field. We denote by $D := \operatorname{Hom}_K(-, K)$ the K-dual. An algebra means a K-algebra which is indecomposable as a ring. For an algebra Λ, we denote by $\operatorname{Mod}\Lambda$ the category of right Λ-modules and by $\operatorname{mod}\Lambda$ the category of finitely generated Λ-modules. If Λ is $\mathbb{Z}$-graded, we denote by $\operatorname{Mod}^{\mathbb{Z}}\Lambda$ the category of $\mathbb{Z}$-graded Λ-modules and by $\operatorname{mod}^{\mathbb{Z}}\Lambda$ the category of finitely generated $\mathbb{Z}$-graded Λ-modules.

2. Preliminaries

2.1. **n-representation infinite algebras.** Let Λ be a finite dimensional algebra of global dimension at most n. We let $\mathcal{D}^{\mathrm{b}}(\Lambda) := \mathcal{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)$ and denote the Nakayama functor by

$$\nu := - \otimes^{\mathbb{L}}_{\Lambda} D\Lambda \simeq D\,\mathbb{R}\operatorname{Hom}_{\Lambda}(-, \Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda).$$

The detailed version of this paper will be submitted for publication elsewhere.

Then ν gives a Serre functor, i.e. there exists a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(X,Y) \simeq D\,\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Y,\nu X)$$

for any $X,Y \in \mathcal{D}^{\mathrm{b}}(\Lambda)$. A quasi-inverse of ν is given by

$$\nu^{-} := \mathbb{R}\mathrm{Hom}_{\Lambda}(D\Lambda,-) \simeq - \otimes^{\mathbb{L}}_{\Lambda} \mathbb{R}\mathrm{Hom}_{\Lambda}(D\Lambda,\Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \longrightarrow \mathcal{D}^{\mathrm{b}}(\Lambda).$$

We let

$$\nu_n := \nu \circ [-n] \text{ and } \nu_n^{-} := \nu^{-} \circ [n].$$

Then we recall the definition of n-representation infinite algebras as follows.

Definition 1. [4] A finite dimensional algebra Λ is called *n-representation infinite* if it satisfies $\mathrm{gl.dim}\,\Lambda \leq n$ and $\nu_n^{-i}(X) \in \mathrm{mod}\Lambda$ for any $i \geq 0$.

2.2. **m-APR tilting modules.** Let Λ be an algebra. A Λ-module T is called *tilting* if it satisfies the following conditions.

(T1) There exists an exact sequence

$$0 \to P_m \to \cdots\cdots P_1 \to P_0 \to T \to 0$$

where each P_i is a finitely generated projective Λ-module.

(T2) $\mathrm{Ext}^i_{\Lambda}(T,T) = 0$ for any $i > 0$.

(T3) There exists an exact sequence

$$0 \to \Lambda \to T_0 \to T_1 \to \cdots\cdot \to T_m \to 0$$

where each T_i belongs to $\mathrm{add}\,T$.

In this case, there exists a triangle-equivalence between $\mathcal{D}^{\mathrm{b}}(\Lambda)$ and $\mathcal{D}^{\mathrm{b}}(\mathrm{End}(T))$.

The following tilting modules play a central role in this paper.

Definition 2. Let Λ be a finite dimensional algebra of global dimension at most n. We assume that there is a simple projective Λ-module S satisfying $\mathrm{Ext}^i_{\Lambda}(D\Lambda,S) = 0$ for any $1 \leq i < n$. Take a direct sum decomposition $\Lambda = S \oplus Q$ as a Λ-module. In [5, Proposition 3.2] (and its proof), it was shown that there exists a minimal projective resolution

$$0 \to S \xrightarrow{a_0} P_1 \xrightarrow{a_1} \cdots\cdots \xrightarrow{a_{n-1}} P_n \xrightarrow{a_n} \tau_n^{-}(S) \to 0$$

of $\tau_n^{-}(S)$ such that each P_i belongs to $\mathrm{add}\,Q$. Let $K_m := \mathrm{Im}\,a_m$ for $0 \leq m \leq n$. Note that $K_0 = S$ and $K_n = \tau_n^{-}(S)$. Then it was shown that $Q \oplus K_m$ is a tilting module with projective dimension m. Following [5], we call it the *m-APR* (=Auslander-Platzeck-Reiten) tilting module with respect to S.

If Λ is an n-representation infinite algebra with a simple projective module S, then the above condition $\mathrm{Ext}^i_{\Lambda}(D\Lambda,S) = 0$ $(1 \leq i < n)$ is automatically satisfied. Thus any simple projective module gives an m-APR tilting module for n-representation infinite algebras.

2.3. $(n+1)$**-preprojective algebras.** Next we recall the definition of $(n+1)$-preprojective algebras and their property. In the case of $n = 1$, the algebras coincide with the (classical) preprojective algebras.

Definition 3. [6] Let Λ be a finite dimensional algebra. The $(n+1)$*-preprojective algebra* $\widehat{\Lambda}$ for Λ is a tensor algebra

$$\widehat{\Lambda} := T_\Lambda(\operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda))$$

of $\Lambda^{\mathrm{op}} \otimes_K \Lambda$-module $\operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda)$. This algebra can be regarded as a positively graded algebra by

$$\widehat{\Lambda}_i = \operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda)^{\otimes^i_\Lambda} = \overbrace{\operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda) \otimes_\Lambda \cdots\cdots \otimes_\Lambda \operatorname{Ext}^n_\Lambda(D\Lambda, \Lambda)}^{i}.$$

We remark that the $(n+1)$-preprojective algebra is the 0-th homology of Keller's *derived* $(n+1)$*-preprojective DG algebra* [8].

Moreover, we recall the following definition, which is a graded analog of Ginzburg's Calabi-Yau algebras.

Definition 4. Let $A = \bigoplus_{i\geq 0} A_i$ be a positively graded algebra such that $\dim_K A_i < \infty$ for any $i \geq 0$. We denote by $A^e := A^{\mathrm{op}} \otimes_K A$. We call A *bimodule* n*-Calabi-Yau of Gorenstein parameter* 1 if it satisfies the following conditions.

(1) $A \in \mathcal{K}^{\mathrm{b}}(\operatorname{proj}^{\mathbb{Z}} A^e)$.
(2) $\mathbb{R}\mathrm{Hom}_{A^e}(A, A^e)[n](-1) \simeq A$ in $\mathcal{D}(\operatorname{Mod}^{\mathbb{Z}} A^e)$.

Then n-representation infinite algebras and bimodule CY algebras have a close relationship as follows (see [4, Theorem 4.35]).

Theorem 5. [1, 8, 9] *There is a one-to-one correspondence between isomorphism classes of n-representation infinite algebras Λ and isomorphism classes of graded bimodule $(n+1)$-CY algebras A of Gorenstein parameter 1. The correspondence is given by*

$$\Lambda \mapsto \widehat{\Lambda} \quad \textit{and} \quad A \mapsto A_0.$$

The following result implies that bimodule n-CY algebras provide n-CY triangulated categories.

Theorem 6. [7, Lemma 4.1][3, Proposition 3.2.4] *Let A be a bimodule n-CY algebra. Then there exists a functorial isomorphism*

$$\operatorname{Hom}_{\mathcal{D}(\operatorname{Mod} A)}(M, N) \simeq D\operatorname{Hom}_{\mathcal{D}(\operatorname{Mod} A)}(N, M[n])$$

for any $N \in \mathcal{D}(\operatorname{Mod} A)$ whose total homology is finite dimensional and any $M \in \mathcal{D}(\operatorname{Mod} A)$.

3. Our results

Let Λ be an n-representation infinite algebra. Assume that there exists a simple projective Λ-module S and take a direct sum decomposition $\Lambda = S \oplus Q$ as a Λ-module. As Definition 2, we have a minimal projective resolution

$$0 \to S \xrightarrow{a_0} P_1 \xrightarrow{a_1} \cdots\cdots \xrightarrow{a_{n-1}} P_n \xrightarrow{a_n} \tau_n^-(S) \to 0 \tag{3.1}$$

of $\tau_n^-(S)$ such that each P_i belongs to $\operatorname{add} Q$. Let $K_i := \operatorname{Im} a_i$ and we fix m with $0 \le m \le n$. Then we denote, respectively, the m-APR tilting Λ-module and the endomorphism algebra by

$$T := Q \oplus K_m \qquad \text{and} \qquad \Gamma := \operatorname{End}_\Lambda(T). \tag{3.2}$$

Our first result is the following one, which is a generalization of [4, Theorem 2.13].

Theorem 7. *Under the above setting, the algebra Γ is n-representation infinite.*

Moreover we show that m-APR tilting modules over n-representation infinite algebras lift to tilting modules over the corresponding $(n+1)$-preprojective algebras.

Let $\widehat{\Lambda} := \bigoplus_{i \ge 0} \widehat{\Lambda}_i$ and $\mathcal{D}(\widehat{\Lambda}) := \mathcal{D}(\operatorname{Mod} \widehat{\Lambda})$. For a $\mathbb{Z}$-graded $\widehat{\Lambda}$-module X, we write X_ℓ the degree ℓ-th part of X. For a $\mathbb{Z}$-graded finitely generated $\widehat{\Lambda}$-module X, the algebra $\operatorname{End}_{\widehat{\Lambda}}(X)$ can be regarded as a $\mathbb{Z}$-graded algebra by $\operatorname{End}_{\widehat{\Lambda}}(X)_i = \operatorname{Hom}_{\widehat{\Lambda}}(X, X(i))_0$, where (i) is a graded shift functor and $\operatorname{Hom}_{\widehat{\Lambda}}(X, X)_0 := \{f \in \operatorname{Hom}_{\widehat{\Lambda}}(X, X) \mid f(X_i) \subset X_i \text{ for any } i\}$.

Moreover, an algebra $\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda}$ can be regarded as a $\mathbb{Z}$-graded algebra by $(\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda})_i := \Lambda^{\mathrm{op}} \otimes_K (\widehat{\Lambda})_i$. Thus we regard $\widehat{\Lambda}$ as a $\mathbb{Z}$-graded $(\Lambda^{\mathrm{op}} \otimes_K \widehat{\Lambda})$-module and we have a functor

$$\widehat{(\ )} := - \otimes_\Lambda \widehat{\Lambda} : \operatorname{mod} \Lambda \longrightarrow \operatorname{mod}^{\mathbb{Z}} \widehat{\Lambda}.$$

Note that we have

$$\widehat{X}_i = \begin{cases} 0 & (i \le 0) \\ \tau_n^{-i}(X) & (i \ge 0) \end{cases}$$

for any $X \in \operatorname{mod} \Lambda$. Then we obtain the following results.

Theorem 8. *Under the above setting, the following assertions hold.*

(1) *$\widehat{T}$ is a tilting $\widehat{\Lambda}$-module of projective dimension m.*
(2) *$\operatorname{End}_{\widehat{\Lambda}}(\widehat{T})$ is isomorphic to the (n+1)-preprojective algebra $\widehat{\Gamma}$ of Γ. In particular, $\operatorname{End}_{\widehat{\Lambda}}(\widehat{T})$ is a graded bimodule (n+1)-CY algebra of Gorenstein parameter 1.*

For the case of $m = n$, m-APR tilting modules have a particularly nice property as stated below.

Corollary 9. *Assume that T is an n-APR tilting Λ-module. Then there exists an isomorphism $\widehat{\Lambda} \simeq \widehat{\Gamma}$ of algebras.*

Example 10.

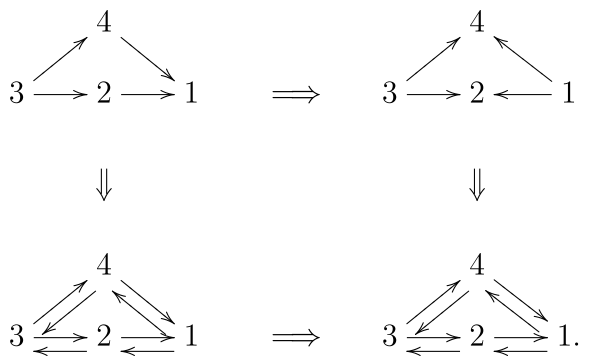
(1) First we give an example for the classical case, namely the case of $n = m = 1$. Let Q be the following quiver.

$$\begin{tikzcd} & 4 \arrow[dr] & \\ 3 \arrow[ur] \arrow[r] & 2 \arrow[r] & 1 \end{tikzcd}$$

We consider the path algebra $\Lambda := KQ$ of Q, which is 1-representation infinite, and the 1-APR tilting Λ-module T associated with vertex 1. Then $\Gamma := \operatorname{End}_\Lambda(T)$ is also a 1-representation infinite algebra, which is the path algebra of the quiver obtained from Q by reversing the arrows ending at the vertex 1.

It is known that the 2-preprojetive algebras $\widehat{\Lambda}$ and $\widehat{\Gamma}$ are given by the double quiver of the quiver of Λ and Γ with some relations respectively. Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T}) \simeq \widehat{\Lambda}$ by Theorem 8 and Proposition 9.

These results imply the compatibility of the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 2-preprojective algebras.



(2) Next we give an example for the case $m = 1 < 2 = n$. We note that the structure of 3-CY algebras has been extensively studied and it is known that they have a close relationship with quivers with potentials (QPs).

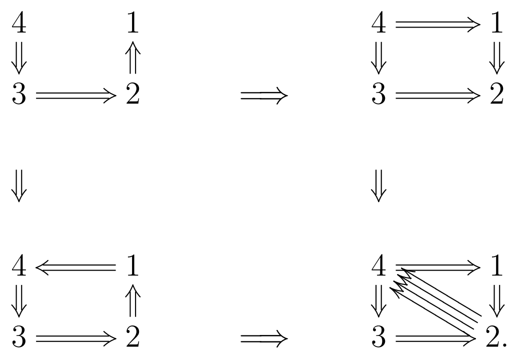
Let Q be a quiver

$$\begin{tikzcd} 4 \arrow[d, shift right, "y_3"'] \arrow[d, shift left, "x_3"] & 1 \arrow[l, shift right, "y_4"'] \arrow[l, shift left, "x_4"] \\ 3 \arrow[r, shift left, "x_2"] \arrow[r, shift right, "y_2"'] & 2, \arrow[u, shift left, "x_1"] \arrow[u, shift right, "y_1"'] \end{tikzcd}$$

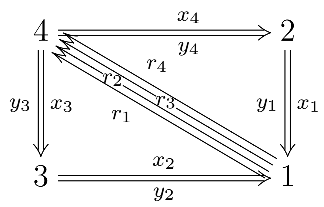
and $W := x_1x_2x_3x_4 - y_1y_2y_3y_4 + x_1y_2x_3y_4 - y_1x_2y_3x_4$ a *potential* on Q and $C := \{x_4, y_4\}$ a *cut*. Then the *truncated Jacobian algebra* Λ of (Q, W, C) is a 2-representation infinite algebra (see [1, section 6]), whose quiver is the left upper one in the picture below. We can consider the 1-APR tilting Λ-module T associated with vertex 1. By Theorem 7, $\Gamma := \mathrm{End}_\Lambda(T)$ is also a 2-representation infinite algebra. Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T})$.

In this example, we can understand the change of quivers with relations of tilts and the 3-preprojective algebras. Indeed, it is known that the quiver with relations of Γ can be calculated by applying mutation of graded QPs. On the other hand, the 3-preprojective algebra $\widehat{\Lambda}$ is given as the *Jacobian algebra* of (Q, W) (see [8]), and $\mathrm{End}_{\widehat{\Lambda}}(\widehat{T})$ is given as the Jacobian algebra of the QP obtained by mutating (Q, W).

Therefore, we have the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 3-preprojective algebras.



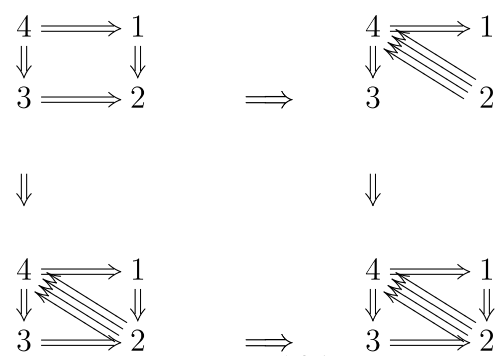
(3) Finally we give an example for the case $n = m = 2$. Let Q be a quiver



and $W := (x_1x_4 - x_2x_3)r_1 + (x_1y_4 - y_2x_3)r_2 + (y_1x_4 - x_2y_3)r_3 + (y_1y_4 - y_2y_3)r_4$ a potential on Q and $C := \{r_1, r_2, r_3, r_4\}$ a cut. Then the truncated Jacobian algebra Λ of (Q, W, C) is a 2-representation infinite algebra given in [4, Example 2.14], whose quiver is the left upper one in the picture below. We can consider the 2-APR tilting Λ-module T associated with vertex 2. By Theorem 7, $\Gamma := \mathrm{End}_\Lambda(T)$ is a 2-representation infinite algebra (this also follows from [4, Theorem 2.13]). Moreover T induces a tilting $\widehat{\Lambda}$-module $\widehat{T}$ with $\widehat{\Gamma} \simeq \mathrm{End}_{\widehat{\Lambda}}(\widehat{T}) \simeq \widehat{\Lambda}$.

In this example, the quiver of Γ can be calculated by the same argument of [5, Theorem 3.11], and the 3-preprojective alegbra $\widehat{\Lambda}$ is given as the Jacobian algebra of (Q, W).

Thus, we have the following diagram of quivers, where horizontal arrows indicate tilts of T and $\widehat{T}$, respectively, and vertical arrows indicate taking 3-preprojective algebras.



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2-DIMENSIONAL QUANTUM BEILINSON ALGEBRAS

IZURU MORI

ABSTRACT. A Fano algebra introduced by Minamoto is roughly speaking a finite dimensional algebra of finite global dimension which is derived equivalent to a (noncommutative) Fano variety [3]. Over such an algebra, a notion of regular module was introduced by Herschend, Iyama and Oppermann from the view point of representation theory of finite dimensional algebras [2]. In this article, we will recall the definitions of a Fano algebra and a regular module, and then explicitly calculate algebraic spaces parameterizing isomorphism classes of simple regular modules over typical examples of Fano algebras, namely, 2-dimensional quantum Beilinson algebras, using techniques of noncommutative algebraic geometry [5].

1. MOTIVATION

Throughout, let k be an algebraically closed field of characteristic 0. All algebras in this article are algebras over k. For a finite dimensional algebra R, we denote by $\operatorname{mod} R$ the category of finite dimensional right R-modules, and $D^b(\operatorname{mod} R)$ the bounded derived category of $\operatorname{mod} R$. If $\operatorname{gldim} R = d < \infty$, then we define an autoequivalence ν_d of $D^b(\operatorname{mod} R)$ by $\nu_d(X) := X \otimes_R^L DR[-d]$ where $DR = \operatorname{Hom}_k(R, k)$.

Definition 1. [3] A finite dimensional algebra R is called d-dimensional Fano if

(1) $\operatorname{gldim} R = d < \infty$, and
(2) $\nu_d^{-i}(R) \in \operatorname{mod} R$ for all $i \geq 0$.

If R is d-dimensional Fano as above, then we define the preprojective algebra of R by

$$\Pi R := T_R(\nu_d^{-1}(R)) = T_R(\operatorname{Ext}_R^d(DR, R))$$

as a graded algebra.

Theorem 2. [3] *A finite dimensional algebra is 1-dimensional Fano if and only if it is a hereditary algebra of infinite representation type.*

Remark 3. By the above theorem, Herschend, Iyama and Oppermann [2] call a d-dimensional Fano algebra R a d-representation infinite algebra. Moreover, they call R d-representation tame if ΠR is noetherian as an algebra.

For a hereditary algebra R of infinite representation type (that is, a 1-dimensional Fano algebra by the above theorem), classifying regular modules is essential in understanding $\operatorname{mod} R$. The notion of regular module was extended to a d-dimensional Fano algebra.

Definition 4. [2] Let R be a d-dimensional Fano algebra. A module $M \in \operatorname{mod} R$ is called d-regular if $\nu_d^i(M) \in \operatorname{mod} R$ for all $i \in \mathbb{Z}$.

The detailed version of this paper has been published in [5].
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The purpose of this ongoing project is to find an algebraic space $Reg(R)$ parameterizing isomorphism classes of simple d-regular modules over a d-dimensional Fano algebra R.

2. Quantum Beilinson Algebras

Let $A = \oplus_{i=0}^{\infty} A_i$ be a right noetherian graded algebra. We denote by $\operatorname{grmod} A$ the category of finitely generated graded right A-modules. For $M \in \operatorname{grmod} A$ and $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} \in \operatorname{grmod} A$ by $M_{\geq n} = \oplus_{i=n}^{\infty} M_i$, and the shift $M(n) \in \operatorname{grmod} A$ by $M(n)_i = M_{n+i}$. We say that A is connected graded if $A_0 = k$ and, in this case, $k = A/A_{\geq 1} \in \operatorname{grmod} A$.

For a right noetherian connected graded algebra A, we denote by $\operatorname{tors} A$ the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules over k, and $\operatorname{tails} A := \operatorname{grmod} A / \operatorname{tors} A$ the quotient category. Following [1], $\operatorname{Proj}_{\mathrm{nc}} A$ is an imaginary geometric object whose category of "coherent sheaves" is $\operatorname{tails} A$ since if A is commutative and generated in degree 1, then $\operatorname{tails} A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$.

Definition 5. A right noetherian connected graded algebra A is called d-dimensional AS-regular if

(1) $\operatorname{gldim} A = d < \infty$, and

(2) there exists $\ell \in \mathbb{N}^+$ such that $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If A is d-dimensional AS-regular as above, then we define the quantum Beilinson algebra of A by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Remark 6. Every AS-regular algebra as above is a skew Calabi-Yau algebra so that there exists a graded algebra automorphism $\mu \in \operatorname{Aut} A$, called the Nakayama automorphism, such that $\operatorname{Ext}^d_{A^e}(A, A^e) \cong {}_{\mu}A(\ell)$ as graded A-A bimodules where $A^e = A \otimes_k A^{op}$, and ${}_{\mu}A = A$ as a graded vector space with the new bimodule structure $a * x * b = \mu(a)xb$.

Theorem 7. [4] *If A is a d-dimensional AS-regular algebra, then*

(1) *∇A is a $(d-1)$-dimensional Fano algebra,*

(2) *$\operatorname{grmod} A \cong \operatorname{grmod} \Pi(\nabla A)$, and*

(3) *$D^b(\operatorname{tails} A) \cong D^b(\operatorname{mod} \nabla A)$.*

By the above theorem, we call R a d-dimensional quantum Beilinson algebra if there exists a $(d+1)$-dimensional AS-regular algebra A such that $R \cong \nabla A$.

Remark 8. If A is a d-dimensional AS-regular algebra, then $\operatorname{Proj}_{\mathrm{nc}} A$ can be viewed as a weighted quantum $\mathbb{P}^{d-1}$ since a commutative d-dimensional AS-regular algebra is exactly a weighted polynomial algebra in d variables. Since a (weighted) quantum projective space is one of the main objects of study in noncommutative algebraic geometry, the above theorem provides strong interactions between noncommutative algebraic geometry and representation theory of algebras.

The main observation in [5] claims that $Reg(\nabla A) = |\operatorname{Proj}_{\mathrm{nc}} A|$ the set of closed points of $\operatorname{Proj}_{\mathrm{nc}} A$, which is expected to have a structure of an algebraic stack. Instead of making this claim more precise, we will give explicit examples below.

3. Hereditary Cases

The results in this section are well-known in representation theory of algebras. We will recover these results using noncommutative algebraic geometry.

If $A = k[x, y]$ is a weighted polynomial algebra with $\deg x = a, \deg y = b \in \mathbb{N}^+$ such that $\gcd(a, b) = 1$, then A is a 2-dimensional AS-regular algebra with $\ell = a + b$, so ∇A is a hereditary algebra of infinite representation type (a 1-dimensional Fano algebra). In fact, $\nabla A = kQ$ is a path algebra where Q is a quiver of type $\widetilde{A_{\ell-1}}$.

Theorem 9. [5] *In the above setting,* $Reg(\nabla A) = \left[(\mathbb{A}^2 \setminus \{(0,0)\})/\sim\right]$ *the quotient stack where* $(x, y) \sim (\lambda^a x, \lambda^b y)$ *for* $0 \neq \lambda \in k$.

If $a = b = 1$, then $\left[(\mathbb{A}^2 \setminus \{(0,0)\})/\sim\right] = \mathbb{P}^1$ by the definition of $\mathbb{P}^1$. In general, $\left[(\mathbb{A}^2 \setminus \{(0,0)\})/\sim\right]$ is almost $\mathbb{P}^1$ but the point $(0,1) \in \mathbb{P}^1$ splits into a points, and the point $(1,0) \in \mathbb{P}^1$ splits into b points.

Recall that if R is a hereditary algebra of infinite representation type (a 1-dimensional Fano algebra), then two simple regular modules $M, N \in \operatorname{mod} R$ are in the same regular component if and only if they are in the same ν_1 orbit, so the regular components of R are parametrized by $Reg(R)/\langle\nu_1\rangle$. In the above setting, the split a points are in the same ν_1 orbit and the split b points are in the same ν_1 orbit, so we have the following result.

Theorem 10. [5] *In the above setting,* $Reg(\nabla A)/\langle\nu_1\rangle = \mathbb{P}^1$.

4. 2-dimensional Beilinson Algebras

Using the techniques in noncommutative algebraic geometry, we can show that 2-dimensional quantum Beilinson algebras can be constructed as follows. Let $g \in k[x, y, z]_3$ be a cubic polynomial, $E = \operatorname{Proj} k[x, y, z]/(g) \subset \mathbb{P}^2$ and $\sigma \in \operatorname{Aut} E$. Define an algebra $R(E, \sigma) = kQ/I$ where Q is the Beilinson quiver

$$\bullet \underset{z_1}{\overset{x_1}{\underset{y_1}{\rightrightarrows}}}\!\!\to \bullet \underset{z_2}{\overset{x_2}{\underset{y_2}{\rightrightarrows}}}\!\!\to \bullet$$

and

$$I = (\{f \in kQ_2 \mid f(p, \sigma(p)) = 0 \text{ for all } p \in E\}).$$

It can be shown that $R(E, \sigma)$ is generically a 2-dimensional quantum Beilinson algebra. Define

$$||\sigma|| := \inf\{i \in \mathbb{N}^+ \mid \text{ there exists } \tau \in \operatorname{Aut} \mathbb{P}^2 \text{ such that } \sigma^i = \tau\}.$$

Note that $||\sigma|| \leq |\sigma|$ the order of σ, and $||\sigma|| = 1$ if and only if $E = \mathbb{P}^2$.

Proposition 11. *Suppose that* $R(E, \sigma), R(E', \sigma')$ *are 2-dimensional quantum Beilinson algebras. If* $R(E, \sigma) \cong R(E', \sigma')$*, then* $E \cong E'$ *and* $||\sigma|| = ||\sigma'||$.

Since $R(E,\sigma) \cong R(E',\sigma')$ does not imply $|\sigma| = |\sigma'|$, $||\sigma||$ is more important than $|\sigma|$ to study $R(E,\sigma)$.

Theorem 12. [5] *Let $R(E,\sigma)$ be a 2-dimensional quantum Beilinson algebra. If $||\sigma|| = \infty$, then*

(1) *$RegR(E,\sigma) = E$, and*
(2) *$RegR(E,\sigma)/\langle \nu_2 \rangle = E/\langle \mu\sigma^3 \rangle$ where μ is the Nakayama automorphism.*

In the case of $||\sigma|| < \infty$, we only have a partial result.

Theorem 13. [5] *Let $R(E,\sigma)$ be a 2-dimensional quantum Beilinson algebra such that $E \subset \mathbb{P}^2$ is a triangle. Then $||\sigma|| < \infty$ if and only if $\Pi R(E,\sigma)$ is finite over its center (that is, $R(E,\sigma)$ is 2-representation tame), and, in this case,*

(1) *$RegR(E,\sigma) = E \sqcup (\mathbb{P}^2 \setminus E)$, and*
(2) *$RegR(E,\sigma)/\langle \nu_2 \rangle = E/\langle \mu\sigma^3 \rangle \sqcup (\mathbb{P}^2 \setminus E)$ where μ is the Nakayama automorphism.*

Example 14. Let $R = kQ/(y_1z_2 - \alpha z_1y_2, z_1x_2 - \beta x_1z_2, x_1y_2 - \gamma y_1x_2)$ where Q is the Beilinson quiver

$$\bullet \overset{x_1, y_1, z_1}{\rightrightarrows} \bullet \overset{x_2, y_2, z_2}{\rightrightarrows} \bullet .$$

If $\alpha\beta\gamma \neq 0, 1$, then $R = R(E,\sigma)$ is a 2-dimensional quantum Beilinson algebra where $E = \operatorname{Proj} k[x,y,z]/(xyz) = V(x) \cup V(y) \cup V(z) \subset \mathbb{P}^2$ is a triangle and $\sigma \in \operatorname{Aut} E$ is given by

$$\begin{aligned}
\sigma|_{V(x)}(0,b,c) &= (0,b,\alpha c) \\
\sigma|_{V(y)}(a,0,c) &= (\beta a,0,c) \\
\sigma|_{V(z)}(a,b,0) &= (a,\gamma b,0).
\end{aligned}$$

It is easy to see that

$$||\sigma|| = |\alpha\beta\gamma| \le \operatorname{lcm}(|\alpha|,|\beta|,|\gamma|) = |\sigma|,$$

so if $|\alpha\beta\gamma| = \infty$, then $Reg(R) = E$, and if $|\alpha\beta\gamma| < \infty$, then $Reg(R) = E \sqcup (\mathbb{P}^2 \setminus E)$.

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TILTING OBJECTS FOR NONCOMMUTATIVE QUOTIENT SINGULARITIES

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ABSTRACT. Tilting objects play a key role in the study of triangulated categories. Iyama and Takahashi proved that stable categories of graded maximal Cohen-Macaulay modules over Gorenstein isolated quotient singularities have tilting objects. As a consequence, it follows that these categories are triangle equivalent to derived categories of finite dimensional algebras. In this paper, using noncommutative algebraic geometry, we give a noncommutative generalization of Iyama and Takahashi's theorem with a more conceptual proof.

1. INTRODUCTION

In the study of triangulated categories, tilting objects play a key role. They often enable us to realize abstract triangulated categories as concrete derived categories of modules over algebras. One of the remarkable results on the existence of tilting objects has been obtained by Iyama and Takahashi.

Theorem 1. [2, Theorem 2.7, Corollary 2.10] *Let $S = k[x_1, \dots, x_d]$ be a polynomial algebra over an algebraically closed field k of characteristic 0 such that $\deg x_i = 1$ and $d \geq 2$. Let G be a finite subgroup of $\mathrm{SL}(d, k)$ acting linearly on S, and S^G the fixed subalgebra of S. Assume that S^G is an isolated singularity. Then the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$ of graded maximal Cohen-Macaulay modules has a tilting object. As a consequence, there exists a finite dimensional algebra Γ of finite global dimension such that*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\,\Gamma).$$

The stable categories of graded maximal Cohen-Macaulay modules are crucial objects studied in representation theory of algebras (see [1], [2] etc.) and also attract attention from the viewpoint of Kontsevich's homological mirror symmetry conjecture (see [3], [4] etc.). The aim of this paper is to generalize Theorem 1 to the noncommutative case using noncommutative algebraic geometry.

2. NONCOMMUTATIVE GORENSTEIN ISOLATED QUOTIENT SINGULARITIES

In this section, we will explain how to consider a noncommutative version of a "Gorenstein isolated quotient singularity". Throughout this paper, we fix an algebraically closed field k. Unless otherwise stated, a graded algebra means an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ over k. We denote by $\mathsf{GrMod}\,A$ the category of graded right A-modules, and by $\mathsf{grmod}\,A$ the full subcategory consisting of finitely generated modules. Morphisms in $\mathsf{GrMod}\,A$ are right A-module homomorphisms of degree zero. Graded left A-modules are identified with

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graded A^o-modules where A^o is the opposite graded algebra of A. For $M \in \mathsf{GrMod}\, A$ and $n \in \mathbb{Z}$, we define $M_{\geq n} = \bigoplus_{i \geq n} M_i \in \mathsf{GrMod}\, A$, and $M(n) \in \mathsf{GrMod}\, A$ by $M(n) = M$ as an ungraded right A-module with the new grading $M(n)_i = M_{n+i}$. The rule $M \mapsto M(n)$ is a k-linear autoequivalence for $\mathsf{GrMod}\, A$ and $\mathsf{grmod}\, A$, called the shift functor. For $M, N \in \mathsf{GrMod}\, A$, we write the graded vector space

$$\underline{\mathrm{Ext}}^i_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^i_{\mathsf{GrMod}\, A}(M, N(n)).$$

If $A_0 = k$, then we say that A is connected graded. Let A be a noetherian connected graded algebra. Then we view $k = A/A_{\geq 1} \in \mathsf{GrMod}\, A$ as a graded A-module.

Definition 2. A noetherian connected graded algebra A is called an AS-Gorenstein (resp. AS-regular) algebra of dimension d and of Gorenstein parameter ℓ if

- $\operatorname{injdim}_A A = \operatorname{injdim}_{A^o} A = d < \infty$ (resp. $\operatorname{gldim} A = d < \infty$), and
- $\underline{\mathrm{Ext}}^i_A(k, A) \cong \underline{\mathrm{Ext}}^i_{A^o}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

Definition 3. Let A be a connected graded algebra. A linear resolution of $M \in \mathsf{GrMod}\, A$ is a minimal free resolution of the form

$$\cdots \to \bigoplus A(-i) \to \cdots \to \bigoplus A(-1) \to \bigoplus A \to M \to 0.$$

We say that A is Koszul if $k \in \mathsf{GrMod}\, A$ has a linear resolution.

It is well-known that if A is Koszul, then A is quadratic, and its dual graded algebra $A^!$ is also Koszul, which is called the Koszul dual of A.

Let S be an AS-regular Koszul algebra of dimension d. Then the Gorenstein parameter is $\ell = d$. It is known that S is commutative if and only if S is isomorphic to $k[x_1, \dots, x_d]$ with $\deg x_i = 1$, so an AS-regular Koszul algebra is a noncommutative version of $k[x_1, \dots, x_d]$ generated in degree 1.

Next we give some conventions on group actions on algebras used in this paper. Let A be a noetherian connected graded algebra. We denote by $\operatorname{GrAut} A$ the group of graded k-algebra automorphisms of A. Let $G \leq \operatorname{GrAut} A$ be a finite subgroup. Then the fixed subalgebra A^G and the skew group algebra $A * G$ are graded by $(A^G)_i = A^G \cap A_i$ and $(A * G)_i = A_i \otimes_k kG$ for $i \in \mathbb{N}$. We tacitly assume that $\operatorname{char} k$ does not divide $|G|$. Note that this condition is equivalent to the condition that kG is semi-simple. Two idempotent elements

$$e := \frac{1}{|G|} \sum_{g \in G} g, \quad \text{and} \quad e' := 1 - e$$

of kG play crucial roles in this study. Since $kG \subset A * G$, we often view e, e' as idempotent elements of $A * G$. It is well-known that the map $\varphi : A^G \to e(A * G)e$ defined by $\varphi(c) = e(c * 1)e$ is an isomorphism of graded algebras. Thus for any $M \in \mathsf{GrMod}\, A * G$, the right A^G-module structure on Me is given by identifying A^G with $e(A * G)e$ via φ.

In [5], the following characterization of isolated quotient singularities was given.

Proposition 4 ([5, Corollary 3.11])**.** *Let $S = k[x_1, \dots, x_d]$ be a polynomial algebra generated in degree 1. If $\operatorname{char} k = 0$ and $G \leq \mathrm{SL}(d, k)$ is a finite subgroup, then the following are equivalent:*

(1) *S^G is an isolated singularity,*
(2) *$S * G/(e)$ is finite dimensional over k.*

Hence, combining the arguments of this section, if

- S is an AS-regular Koszul algebra of diminension d,
- $G \leq \mathrm{GrAut}\, S$ is a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$,
- S^G is AS-Gorenstein, and
- $S * G/(e)$ is finite dimensional over k,

then S^G can be considered as a noncommutative Gorenstein isolated quotient singularity.

3. Main Results

Let A be an AS-Gorenstein algebra. Then $M \in \mathsf{grmod}\, A$ is called graded maximal Cohen-Macaulay if $\underline{\mathrm{Ext}}^i_A(M, A) = 0$ for all $i > 0$. We denote by $\mathsf{CM}^{\mathbb{Z}}(A)$ the full subcategory of $\mathsf{grmod}\, A$ consisting of graded maximal Cohen-Macaulay modules. Then $\mathsf{CM}^{\mathbb{Z}}(A)$ is a Frobenius category. The stable category of $\mathsf{CM}^{\mathbb{Z}}(A)$ is denoted by $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$. Note that $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ is a triangulated category.

The following is the main result of this paper, saying that there exists a finite dimensional algebra Γ such that $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \Gamma)$ when A is a "noncommutative Gorenstein isolated quotient singularity".

Theorem 5 ([6])**.** *Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \mathrm{GrAut}\, S$ a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$, and let $e = \frac{1}{|G|} \sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e$. Assume that S^G is AS-Gorenstein and $S * G/(e)$ is finite dimensional over k. If we define the graded right $S*G$-module U by $U = \bigoplus_{i=1}^d \Omega^i_{S*G} kG(i)$, then*

$$e'Ue$$

is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$. As a consequence we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\ \mathrm{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue)).$$

We remark that if S is commutative, then this theorem recovers Theorem 1.

For the rest of this section, we will explain how to calculate $\mathrm{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue)$. Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \mathrm{GrAut}\, S$ a finite subgroup such that $\mathrm{char}\, k$ does not divide $|G|$, and let $e = \frac{1}{|G|} \sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e$. Then we consider the Koszul dual algebra

$$S^! := \bigoplus_{i \in \mathbb{N}} \underline{\mathrm{Ext}}^i_S(S_0, S_0).$$

Since S is an AS-regular algebra, it is known that $S^!$ is a graded self-injective algebra (see [10]). Moreover, the opposite group G^o acts on $S^!$ as explained in [9]. We can define the ungraded finite dimensional algebra

$$\nabla(S^!) := \begin{pmatrix} S^!_0 & S^!_1 & \cdots & S^!_{d-1} \\ 0 & S^!_0 & \cdots & S^!_{d-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S^!_0 \end{pmatrix}$$

called the Beilinson algebra of $S^!$. Then we can also define the action of G^o on $\nabla(S^!)$. Thus we have the skew group algebra

$$\nabla(S^!) * G^o.$$

It is easy to check that $\nabla(S^!) * G^o$ has the idempotent

$$\tilde{e'} = \begin{pmatrix} e' & 0 & \cdots & 0 \\ 0 & e' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e' \end{pmatrix} \in \begin{pmatrix} kG & * & \cdots & * \\ 0 & kG & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & kG \end{pmatrix} = \nabla(S^! * G^o) \cong \nabla(S^!) * G^o.$$

Theorem 6 ([6]). *As in the setting of Theorem 5, we have*

$$\operatorname{End}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)}(e'Ue) \cong \tilde{e'}(\nabla(S^!) * G^o)\tilde{e'}$$

as algebras.

Thanks to this theorem, we can calculate the endomorphism algebra of the tilting object found in Theorem 5. In the next section, we present an example.

4. An Example

The aim of this section is to provide an explicit example of Theorem 5 and Theorem 6. In this section, we assume that k is an algebraically closed field of characteristic 0.

Example 7 ([6]). Let S be $k\langle x_1, x_2, x_3, x_4\rangle$ having six defining relations

$$x_1^2 + x_2^2,\ \ x_1x_3 + x_3x_1,\ \ x_1x_4 + x_4x_1,\ \ x_2x_3 + x_3x_2,\ \ x_2x_4 + x_4x_2,\ \ x_3x_4 + x_4x_3,$$

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then S is a noetherian AS-regular Koszul algebra over k of dimension 4. Let G be a cyclic group generated by $g = \operatorname{diag}(1, -1, -1, -1)$. Then g defines a graded algebra automorphism of S, so G naturally acts on S. Clearly $|G| = 2$. One can check that S^G is AS-Gorenstein of dimension 4 (although $\det g \neq 1$). Moreover, by using a quiver presentation of $S * G/(e)$, we can check that $S * G/(e)$ is finite dimensional over k.

The Koszul dual $S^!$ is $k\langle x_1, x_2, x_3, x_4\rangle$ having ten defining relations

$$x_2^2 - x_1^2,\ \ x_3x_1 - x_1x_3,\ \ x_4x_1 - x_1x_4,\ \ x_2x_1,\ \ x_1x_2,$$
$$x_3x_2 - x_2x_3,\ \ x_4x_2 - x_2x_4,\ \ x_4x_3 - x_3x_4,\ \ x_3^2,\ \ x_4^2,$$

with $\deg x_1 = \deg x_2 = \deg x_3 = \deg x_4 = 1$. Then a quiver presentation of the Beilinson algebra $\nabla(S^!)$ is given as follows.

$$0 \overset{x_1}{\underset{x_4}{\overset{x_2}{\underset{x_3}{\rightrightarrows\rightrightarrows}}}} 1 \overset{x_1}{\underset{x_4}{\overset{x_2}{\underset{x_3}{\rightrightarrows\rightrightarrows}}}} 2 \overset{x_1}{\underset{x_4}{\overset{x_2}{\underset{x_3}{\rightrightarrows\rightrightarrows}}}} 3 \qquad \begin{array}{l} x_1^2 = x_2^2,\ x_3x_1 = x_1x_3,\ x_4x_1 = x_1x_4, \\ x_3x_2 = x_2x_3,\ x_4x_2 = x_2x_4,\ x_4x_3 = x_3x_4, \\ x_2x_1 = x_1x_2 = x_3^2 = x_4^2 = 0. \end{array}$$

By [8, Section 2.3], it follows that a quiver presentation of the skew group algebra $\nabla(S^!) * G^o$ is

$$
\begin{tikzcd}
(0,0) \arrow[r, "x_1"] \arrow[dr, shift left=2, "x_2"] \arrow[dr, "x_3"] \arrow[dr, shift right=2, "x_4"'] & (1,0) \arrow[r, "x_1"] \arrow[dr, shift left=2, "x_2"] \arrow[dr, "x_3"] \arrow[dr, shift right=2, "x_4"'] & (2,0) \arrow[r, "x_1"] \arrow[dr, shift left=2, "x_2"] \arrow[dr, "x_3"] \arrow[dr, shift right=2, "x_4"'] & (3,0) \\
(0,1) \arrow[r, "x_1"] \arrow[ur, shift left=2, "x_4"] \arrow[ur, "x_3"] \arrow[ur, shift right=2, "x_2"'] & (1,1) \arrow[r, "x_1"] \arrow[ur, shift left=2, "x_4"] \arrow[ur, "x_3"] \arrow[ur, shift right=2, "x_2"'] & (2,1) \arrow[r, "x_1"] \arrow[ur, shift left=2, "x_4"] \arrow[ur, "x_3"] \arrow[ur, shift right=2, "x_2"'] & (3,1)
\end{tikzcd}
\qquad
\begin{aligned}
&x_1^2 = x_2^2,\ x_3x_1 = x_1x_3,\\
&x_1x_4 = x_4x_1,\ x_3x_2 = x_2x_3,\\
&x_2x_4 = x_4x_2,\ x_4x_3 = x_3x_4,\\
&x_2x_1 = x_1x_2 = x_3^2 = x_4^2 = 0.
\end{aligned}
$$

Since $\tilde{e'} = (0,1) + (1,1) + (2,1) + (3,1)$ in the present setting, a quiver presentation of $\tilde{e'}(\nabla(S^!) * G^o)\tilde{e'}$ is obtained by

$$
\begin{tikzcd}
(0,1) \arrow[r, "x_1"] \arrow[rr, bend left=40, "x_2x_3"] \arrow[rr, bend left=30, "x_2x_4"] \arrow[rr, bend left=20, "x_3x_4"] & (1,1) \arrow[r, "x_1"] \arrow[rr, bend right=20, "x_2x_3"'] \arrow[rr, bend right=30, "x_2x_4"'] \arrow[rr, bend right=40, "x_3x_4"'] & (2,1) \arrow[r, "x_1"] & (3,1)
\end{tikzcd}
\qquad (4.1) \qquad
\begin{aligned}
&x_2x_3x_1 = x_1x_2x_3 = 0\\
&x_2x_4x_1 = x_1x_2x_4 = 0\\
&x_3x_4x_1 = x_1x_3x_4\\
&x_1^3 = 0
\end{aligned}
$$

Hence, if we denote by (Q, R) the quiver with relations in (4.1), then we have a triangle equivalence

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, kQ/(R))$$

by Theorem 5 and Theorem 6.

5. Key Results for the Proof of Theorem 5

Our proof of Theorem 5 is different from Iyama and Takahashi's proof of Theorem 1. In this section, we summarize key results for our proof. We use techniques of noncommutative algebraic geometry.

Let A be a noetherian graded algebra. We denote by $\mathsf{tors}\, A$ the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules. The noncommutative projective scheme associated to A is defined by the quotient category

$$\mathsf{tails}\, A := \mathsf{grmod}\, A/\mathsf{tors}\, A.$$

If A is a commutative graded algebra finitely generated in degree 1 over k, then $\mathsf{tails}\, A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$ by results of Serre, justifying the terminology. We denote by $\pi : \mathsf{grmod}\, A \to \mathsf{tails}\, A$ the quotient functor.

Before proving Theorem 5, we prove the following result about tilting objects in the derived categories $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$.

Theorem 8 ([6])**.** *Let S be an AS-regular Koszul algebra of dimension $d \geq 2$, $G \leq \operatorname{GrAut} S$ a finite subgroup such that $\operatorname{char} k$ does not divide $|G|$, and let $e = \frac{1}{|G|}\sum_{g \in G} g \in kG \subset S * G$ and $e' = 1 - e$. Assume that $S * G/(e)$ is finite dimensional over k. If we consider the graded right $S * G$-module $U = \bigoplus_{i=1}^{d} \Omega^i_{S*G} kG(i)$, then*

$$\pi U e$$

is a tilting object in $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$.

We now give an outline of the proof of Theorem 8. First we show that $(S * G)^!$ is an $\mathbb{N}$-graded self-injective Koszul algebra. (See [6] for our definition of Koszul for (non-connected) $\mathbb{N}$-graded algebras.) Using the BGG correspondence and the isolated singularity property of S^G, we have

$$\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G) \cong \underline{\mathsf{grmod}}(S * G)^!$$

as triangulated categories. Under this equivalence, we can show that $\pi U e$ corresponds to the tilting object in $\underline{\mathsf{grmod}}(S * G)^!$ which was obtained by Yamaura [11, Theorem 3.3 (2)]. Thus Theorem 8 follows.

Furthermore, in the setting of Theorem 8, if S^G is AS-Gorenstein, then there exists an embedding

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) \hookrightarrow \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G)$$

by Orlov's theorem [7]. We can verify that $e'Ue$ is sent to $\pi e'Ue$.

Combining these results, we have

$$\begin{array}{ccccc} \underline{\mathsf{CM}}^{\mathbb{Z}}(S^G) & \hookrightarrow & \mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, S^G) & \cong & \underline{\mathsf{grmod}}(S * G)^! \\ e'Ue & \mapsto & \pi e'Ue \quad \oplus \quad \pi U e & \mapsto & Y \end{array}$$

where Y is the Yamaura tilting object. Using this, we can give a conceptual proof that $e'Ue$ is a tilting object in $\underline{\mathsf{CM}}^{\mathbb{Z}}(S^G)$ in terms of triangulated categories.

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ON THE HOCHSCHILD COHOMOLOGY RING MODULO NILPOTENCE OF THE QUIVER ALGEBRA WITH QUANTUM-LIKE RELATIONS

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Abstract. This paper is based on my talk given at the 48th Symposium on Ring Theory and Representation Theory held at Nagoya University, Japan, 7–10 September 2015. In this paper, we consider some example of finite dimensional quiver algebra over a field k with quantum-like relations. We determine the projective bimodule resolution of the algebras using the total complex. And we have the ring structure of the Hochschild cohomology ring modulo nilpotence.

Introduction

Let A be an indecomposable finite dimensional algebra over a field k and $\operatorname{char} k = 0$. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n\geq 0}\mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$. In [6], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the ring structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [5], Snashal gave the question whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra. With respect to sufficient condition, Green, Snashall and Solberg have shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for self-injective algebras of finite representation type [1] and for monomial algebras [2].

Let Γ be the quiver with 4 vertices and 6 arrows as follows:

$$\begin{tikzcd} e_{(1,2)} \arrow[r, shift right, "a_{(1,2)}"'] & e_{(1,1)} \arrow[l, shift right, "a_{(1,1)}"'] \arrow[r, shift left, "a_{(2,1)}"] & e_{(2,2)} \arrow[l, shift left, "a_{(2,2)}"] \arrow[r, shift left, "a_{(3,1)}"] & e_{(3,2)} \arrow[l, shift left, "a_{(3,2)}"] \end{tikzcd}$$

and I the ideal of $k\Gamma$ generated by

$$X_1^{n_1}, X_2^{n_2}, X_3^{n_3}, X_1X_2 - X_2X_1, X_2X_3 - X_3X_2,$$
$$a_{(1,2)}a_{(2,1)}X_2^{l}a_{(3,1)}, a_{(3,2)}a_{(2,2)}X_2^{l'}a_{(1,1)} \text{ for } 0 \leq l, l' \leq n_2 - 1.$$

The detailed version of this paper will be submitted for publication elsewhere.

where $X_i = (a_{i,1} + a_{i,2})^2$ and n_i are integers with $n_i \geq 2$ for $1 \leq i \leq 3$. Paths in Γ are written from left to right. In this paper, we consider the quiver algebra $A = k\Gamma/I$.

In this paper,we determine the projective bimodule resolution of this algebra A and the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

In [3] and [4], we have the minimal projective bimodule resolution and the Hochschild cohomology ring modulo nilpotence of the quiver algebra defined by the two cycles and quantum-like relation. Then, the projective resolution of this algebra was given by the total complex depending on the projective bimodule resolutions of two Nakayama algebras. Similarly, the projective bimodule resolution of A is given by the total complex depending on the projective bimodule resolutions of the quiver algebra defined by two cycles and the Nakayama algebra, Using this resolution, we have the ring structure of the Hochschild cohomology ring of A modulo nilpotence.

The content of the paper is organized as follows. In Section 1, we give the complexes of the projective A-bimodule to form the total complex. In Section 2, we determine the projective bimodule resolution of A and the Hochschild cohomology ring of A modulo nilpotence.

1. The complexes of the projective A-bimodule

In this section, we we give the complexes to form the projective bimodule resolution of A. We set $e_{(1,1)} = e_{(2,1)}, e_{(2,2)} = e_{(3,1)}$. Let $\varepsilon_{(i,j,0),\{(s_1,s_2),(t_1,t_2)\}} = \varepsilon_{(i,j,0),\{(s_1,s_2),(t_1,t_2)\},(l_1,l_2)} = e_{(s_1,s_2)} \otimes e_{(t_1,t_2)}$. We define projective left A^e-modules, equivalently A-bimodules:

$P_0 = A\varepsilon_{(0,0),\{(1,1),(1,1)\}}A \oplus A\varepsilon_{(0,0),\{(1,2),(1,2)\}}A \oplus A\varepsilon_{(0,0),\{(2,2),(2,2)\}}A \oplus A\varepsilon_{(0,0),\{(3,2),(3,2)\}}A,$

$$Q_{(i,0,0)} = \begin{cases} \displaystyle\coprod_{k=1}^{2}(A\varepsilon_{(i,0,0),\{(k,1),(k,2)\}}A \oplus A\varepsilon_{(i,0,0),\{(k,2),(k,1)\}}A) \oplus \coprod_{\substack{l_1+l_2=i \\ l_1,l_2>0}} A\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)}A & \\ & \text{if } i \text{ is odd,} \\ \displaystyle\coprod_{k=1}^{2} A\varepsilon_{(i,0,0),\{(k,2),(k,2)\}}A \oplus \coprod_{\substack{l_1+l_2=i \\ l_1,l_2\geq 0}} A\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)}A & \text{if } i \text{ is even.} \end{cases}$$

$$Q_{(0,j,0)} = \begin{cases} A\varepsilon_{(0,j,0),\{(3,1),(3,2)\}}A \oplus A\varepsilon_{(0,j,0),\{(3,2),(3,1)\}}A & \text{if } j \text{ is odd,} \\ A\varepsilon_{(0,j,0),\{(3,1),(3,1)\}}A \oplus A\varepsilon_{(0,j,0),\{(3,2),(3,2)\}}A & \text{if } j \text{ is even.} \end{cases}$$

$$Q_{(i,j,0)} = \coprod_{\substack{l_1+l_2=i \\ l_1,l_2>0}} (A\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)}A) \oplus A\varepsilon_{(i,j,0),\{(3,1),(3,1)\}}A$$

$$\oplus \begin{cases} A\varepsilon_{(i,j,0),\{(1,1),(3,1)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,1)\}}A & \text{if } i,j \text{ are even,} \\ A\varepsilon_{(i,j,0),\{(1,1),(3,2)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,2),(1,1)\}}A & \text{if } i \text{ is even and } j \text{ is odd,} \\ A\varepsilon_{(i,j,0),\{(1,2),(3,1)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,1),(1,2)\}}A & \text{if } i \text{ is odd and } j \text{ is even,} \\ A\varepsilon_{(i,j,0),\{(1,2),(3,2)\}}A \oplus A\varepsilon_{(i,j,0),\{(3,2),(1,2)\}}A & \text{if } i,j \text{ are odd.} \end{cases}$$

$$
\begin{aligned}
&Q_{(i,1,k)} = \\
&\begin{cases}
\begin{cases}
A\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)}A \oplus A\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)}A \\
\oplus \displaystyle\coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2,\, l_2 \geq 1}} \coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1,l'_2:\text{even}} A\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)}A \\
\oplus \displaystyle\coprod_{l'_1,l'_2:\text{odd}} A\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)}A \Big) \oplus \begin{cases} A\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)}A & \text{if } k=1, \\ 0 & \text{others}, \end{cases}
\end{cases} \\
\hfill \text{if } k \text{ is odd}, \\
\begin{cases}
A\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(1,i-1)}A \oplus A\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(1,i-1)}A \\
\oplus \displaystyle\coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2,\, l_2 \geq 0}} \coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1:\text{odd},l'_2:\text{even}} A\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)}A \\
\oplus \displaystyle\coprod_{l'_1:\text{even},l'_2:\text{odd}} A\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)}A \Big)
\end{cases} \\
\hfill \text{if } k \text{ is even},
\end{cases} \\
&Q_{(i,j,k)} = \\
&\begin{cases}
\begin{cases}
\displaystyle\coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2,\, l_2 \geq 1}} A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2}A \\
\displaystyle\coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1,l'_2:\text{odd}} A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)}A \oplus \coprod_{l'_1,l'_2:\text{even}} A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)}A \Big) \\
\oplus \begin{cases}
A\varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)}A & \text{if } i \text{ is odd}, j \text{ is even}, \\
A\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)}A & \text{if } i \text{ is even}, j \text{ is odd}, \\
A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2}A & \\
\ \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2}A & \text{if } i,j \text{ are even}, \\
0 & \text{if } i,j \text{ are odd},
\end{cases}
\end{cases} \\
\hfill \text{if } k \text{ is odd}, \\
\begin{cases}
\displaystyle\coprod_{\substack{l_1+l_2=i \\ l_1 \geq 2,\, l_2 \geq 1}} A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(l_1,l_2)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(l_1,l_2)}A \\
\displaystyle\coprod_{l'_1+l'_2=k+1} \Big(\coprod_{l'_1:\text{even},l'_2:\text{odd}} A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(1,i-1),(l'_1,l'_2)}A \oplus \coprod_{l'_1:\text{odd},l'_2:\text{even}} A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(1,i-1),(l'_1,l'_2)}A \Big) \\
\oplus \begin{cases}
A\varepsilon_{(i,j,k),\{(1,2),(3,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,2)\},(i,0)}A & \text{if } i \text{ is odd}, j \text{ is even}, \\
A\varepsilon_{(i,j,k),\{(3,2),(1,1)\},(i,0)}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(3,2)\},(i,0)}A & \text{if } i \text{ is even}, j \text{ is odd}, \\
A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),2}A & \\
\ \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),1}A \oplus A\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),2}A & \text{if } i,j \text{ are even}, \\
0 & \text{if } i,j \text{ are odd},
\end{cases}
\end{cases} \\
\hfill \text{if } k \text{ is even},
\end{cases}
\end{aligned}
$$

Let $\pi\colon P_0 \to A$ be the multiplication map. Then, we have the following complexes.

Proposition 1. *For $1 \leq k \leq 3$, we set*

$$
\begin{aligned}
E_{(i,j,0),(k,1)} &= \varepsilon_{(i,j,0),\{(k,1),(k,2)\}} a_{(k,2)} + a_{(k,1)} \varepsilon_{(i,j,0),\{(k,2),(k,1)\}},\\
E_{(i,j,0),(k,2)} &= \varepsilon_{(i,j,0),\{(k,2),(k,1)\}} a_{(k,1)} + a_{(k,2)} \varepsilon_{(i,j,0),\{(k,1),(k,2)\}}
\end{aligned}
$$

(1) *We have the following complex depending on the projective bimodule resolution of the quiver algebra defined by the two cycles and quantum-like relation.*

$$
P_0 \overset{\partial_{(1,0,0),1}}{\longleftarrow} Q_{(1,0,0)} \overset{\partial_{(2,0,0),1}}{\longleftarrow} \cdots \overset{\partial_{(n,0,0),1}}{\longleftarrow} Q_{(n,0,0)} \leftarrow \cdots .
$$

where, for $i \geq 0$, the left A^e-homomorphisms $\partial_{(i+1,0,0),1}\colon Q_{(i+1,0,0)} \to Q_{(i,0,0)}$ are defined as follows.

$\partial_{(i+1,0,0)}$:

$$
\begin{cases}
\begin{cases}
\varepsilon_{(i+1,0,0),\{(k,1),(k,2)\}} \mapsto \varepsilon_{(i,0,0),\{(k,1),(k,1)\},(i,0)} a_{(k,1)} - a_{(k,1)} \varepsilon_{(i,0,0),\{(k,2),(k,2)\}} & \text{for } 1 \leq k \leq 2,\\
\varepsilon_{(i+1,0,0),\{(k,2),(k,1)\}} \mapsto \varepsilon_{(i,0,0),\{(k,2),(k,2)\}} a_{(k,2)} - a_{(k,2)} \varepsilon_{(i,0,0),\{(k,1),(k,1)\},(i,0)} & \text{for } 1 \leq k \leq 2,\\
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(l_1,l_2)} \mapsto & \\
\begin{cases}
-\sum_{l=0}^{n_1-1} X_1^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1-l} + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2 & \\
\quad - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \text{if } l_1 \text{ is even and } l_2 \text{ is odd},\\
\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1 - X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \\
\quad + \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} & \text{if } l_1 \text{ is odd and } l_2 \text{ is even},
\end{cases}
\end{cases} & \text{if } i \text{ is odd},\\
\begin{cases}
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(i+1,0)} \mapsto \sum_{l=0}^{n_1-1} X_1^l E_{(i,0,0),(1,1)} X_1^{n_1-1-l}, & \\
\varepsilon_{(i+1,0,0),\{(2,1),(2,1)\},(0,i+1)} \mapsto \sum_{l=0}^{n_2-1} X_2^l E_{(i,0,0),(2,1)} X_2^{n_2-1-l}, & \\
\varepsilon_{(i+1,0,0),\{(k,2),(k,2)\}} \mapsto \sum_{l=0}^{n_k-1} X_k^l E_{(i,0,0),(k,2)} X_k^{n_k-1-l} \text{ for } 1 \leq k \leq 2, & \\
\varepsilon_{(i+1,0,0),\{(1,1),(1,1)\},(l_1,l_2)} \mapsto & \\
\begin{cases}
-E_{(1,0,0),(2,1)} X_1 + X_1 E_{(1,0,0),(2,1)} + E_{(1,0,0),(1,1)} X_2 - X_2 E_{(1,0,0),(1,1)} & \text{if } l_1 = 1, l_2 = 1,\\
-\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i-1,1)} X_1 + X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i-1,1)} + E_{(i,0,0),(1,1)} X_2 - X_2 E_{(i,0,0),(1,1)} & \\
 & \text{if } l_1 = i, l_2 = 1,\\
-E_{(i,0,0),(2,1)} X_1 + X_1 E_{(i,0,0),(2,1)} + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1,i-1)} X_2 - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1,i-1)} & \\
 & \text{if } l_1 = 1, l_2 = i,\\
-\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1 + X_1 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} & \\
\quad + \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} & \text{if } l_1, l_2 \text{ are odd},\\
\sum_{l=0}^{n_1-1} X_1^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1-l} + \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} & \\
 & \text{if } l_1, l_2 \text{ are even},
\end{cases}
\end{cases} & \text{if } i \text{ is even},
\end{cases}
$$

(2) *We have the following complexe depending on the projective bimodule resolution of Nakayama algebra.*

$$P_0 \xleftarrow{\partial_{(0,1,0),2}} Q_{(0,1,0)} \xleftarrow{\partial_{(0,2,0),2}} \cdots \xleftarrow{\partial_{(0,n,0),2}} Q_{(0,n,0)} \leftarrow \cdots .$$

where, for $j \geq 0$, the left A^e-homomorphisms $\partial_{(0,j+1,0),2}$:$Q_{(0,j+1,0)} \to Q_{(0,j,0)}$ are defined as follows.

$$\partial_{(0,j+1,0)}:$$
$$\begin{cases} \begin{cases} \varepsilon_{(0,j+1,0),\{(3,1),(3,2)\}} \mapsto \varepsilon_{(0,j,0),\{(3,1),(3,1)\}}a_{(3,1)} - a_{(3,1)}\varepsilon_{(0,j,0),\{(3,2),(3,2)\}}, \\ \varepsilon_{(0,j+1,0),\{(3,2),(3,1)\}} \mapsto \varepsilon_{(0,j,0),\{(3,2),(3,2)\}}a_{(3,2)} - a_{(3,2)}\varepsilon_{(0,j,0),\{(3,1),(3,1)\}}, \end{cases} & \textit{if } j \textit{ is even,} \\ \begin{cases} \varepsilon_{(0,j+1,0),\{(3,1),(3,1)\}} \mapsto \sum_{l=0}^{n_3-1} X_3^l \varepsilon_{(0,j,0),\{(3,1),(3,1)\}} X_3^{n_3-1-l}, \\ \varepsilon_{(0,j+1,0),\{(3,2),(3,2)\}} \mapsto \sum_{l=0}^{n_3-1} X_3^l \varepsilon_{(0,j,0),\{(3,2),(3,2)\}} X_3^{n_3-1-l} \end{cases} & \textit{if } j \textit{ is odd.} \end{cases}$$

(3) *We have the following complex depending on the relations $X_1^{n_1}$, $X_2^{n_2}$ and $X_1X_2 - X_2X_1$.*

$$Q_{(0,j,0)} \xleftarrow{\partial_{(1,j,0),1}} Q_{(1,j,0)} \xleftarrow{\partial_{(2,j,0),1}} \cdots \xleftarrow{\partial_{(n,j,0),1}} Q_{(n,j,0)} \leftarrow \cdots .$$

where, for $i \geq 0$, the left A^e-homomorphisms $\partial_{(i,j,0),1}$:$Q_{(i,j,0)} \to Q_{(i-1,j,0)}$ are defined as follows.

$$\varepsilon_{(i,j,0),\{(1,2),(3,2)\}} \mapsto \begin{cases} a_{(1,2)}a_{(2,1)}\varepsilon_{(0,j,0),\{(3,1),(3,2)\}} & \textit{if } i=1 \textit{ and } j \textit{ is odd,} \\ a_{(1,2)}\varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} & \textit{if } i \textit{ is odd}(\neq 1) \textit{ and } j \textit{ is odd,} \end{cases}$$
$$\varepsilon_{(i,j,0),\{(1,1),(3,2)\}} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} \ \textit{if } i \textit{ is even and } j \textit{ is odd,}$$
$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\}} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,0),\{(1,2),(3,1)\}} \ \textit{if } i,j \textit{ are even,}$$
$$\varepsilon_{(i,j,0),\{(3,2),(1,2)\}} \mapsto \begin{cases} \varepsilon_{(0,j,0),\{(3,2),(3,1)\}}a_{(2,2)}a_{(1,1)} & \textit{if } i=1 \textit{ and } j \textit{ is odd,} \\ \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}}a_{(1,1)} & \textit{if } i \textit{ is odd}(\neq 1) \textit{ and } j \textit{ is odd,} \end{cases}$$
$$\varepsilon_{(i,j,0),\{(3,2),(1,1)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}}a_{(1,2)}X_1^{n_1-1} \textit{if } i \textit{ is even and } j \textit{ is odd,}$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,1),(1,2)\}}a_{(1,2)}X_1^{n_1-1} \textit{if } i,j \textit{ are even,}$$
$$\varepsilon_{(i,j,0),\{(1,2),(3,1)\}} \mapsto a_{(1,2)}\varepsilon_{(i-1,j,0),\{(1,1),(3,1)\}} \ \textit{if } i \textit{ is odd and } j \textit{ is even,}$$
$$\varepsilon_{(i,j,0),\{(3,1),(1,2)\}} \mapsto \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\}}a_{(1,1)}, \ \textit{if } i \textit{ is odd and } j \textit{ is even,}$$
$$\varepsilon_{(i,j,0),\{(2,2),(2,2)\}} \mapsto$$
$$\begin{cases} E_{(0,j,0),(3,1)}X_2 - X_2E_{(0,j,0),(3,1)} & \textit{if } i=1 \textit{ and } j \textit{ is odd,} \\ \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}}X_2 - X_2\varepsilon_{(0,j,0),\{(2,2),(2,2)\}} & \textit{if } i=1 \textit{ and } j \textit{ is even or } i \textit{ is odd}(i \neq 1), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} X_2^{n_2-1-l} & \textit{if } i \textit{ is even,} \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto$$

$$\begin{cases} \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} + \begin{cases} X_1 a_{(2,1)} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} & \text{if } l_1 = 1, \\ X_1 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1 \text{ is odd and } l_2 \text{ is even}, \\ X_1^{n_1-1} \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} \\ \quad - \begin{cases} \varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} a_{(3,2)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,2)\}} a_{(3,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1 \text{ is even and } l_2 \text{ is odd}, \\ \begin{cases} X_1 a_{(2,1)} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} & \text{if } l_1 = 1, \\ X_1 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} & \text{others}, \end{cases} \\ \quad - \begin{cases} a_{(1,1)} \varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} a_{(3,2)} X_2 - X_2 a_{(1,1)} \varepsilon_{(i-1,j,0),\{(1,2),(3,2)\}} a_{(3,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1, l_2 \text{ are odd}, \\ X_1^{n_1-1} \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1-1,l_2)} - \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(1,1),(3,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} \\ \qquad\qquad \text{if } l_1, l_2 \text{ are even}, \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto$$

$$\begin{cases} \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} + \begin{cases} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} a_{(2,2)} X_1 & \text{if } l_1 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1 & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1 \text{ is odd and } l_2 \text{ is even}, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1} \\ \quad - \begin{cases} a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}} X_2 - X_2 a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,1)\}} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1 \text{ is even and } l_2 \text{ is odd}, \\ \begin{cases} \varepsilon_{(i-1,j,0),\{(2,2),(2,2)\}} a_{(2,2)} X_1 & \text{if } l_1 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1 & \text{others}, \end{cases} \\ \quad - \begin{cases} a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}} a_{(1,2)} X_2 - X_2 a_{(3,1)} \varepsilon_{(i-1,j,0),\{(3,2),(1,2)\}} a_{(1,2)} & \text{if } l_2 = 1, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2 - X_2 \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} & \text{others}, \end{cases} \\ \qquad\qquad \text{if } l_1, l_2 \text{ are odd}, \\ \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1-1,l_2)} X_1^{n_1-1} - \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,0),\{(3,1),(1,1)\},(l_1,l_2-1)} X_2^{n_2-1-l} \\ \qquad\qquad \text{if } l_1, l_2 \text{ are even}, \end{cases}$$

(4) *We have the following complex depending on the relation* $X_3^{n_3}$.

$$Q_{(i,0,0)} \overset{\partial_{(i,1,0),2}}{\longleftarrow} Q_{(i,1,0)} \overset{\partial_{(i,2,0),2}}{\longleftarrow} \cdots \overset{\partial_{(i,n,0),2}}{\longleftarrow} Q_{(i,n,0)} \leftarrow \cdots .$$

where, for $j \geq 0$, the left A^e-homomorphisms $\partial_{(i,j,0),2}{:}Q_{(i,j,0)} \to Q_{(i,j-1,0)}$ are defined as follows.

$$\varepsilon_{(i,j,0),\{(1,2),(3,2)\}} \mapsto$$
$$\begin{cases} \varepsilon_{(1,0,0),\{(1,2),(1,1)\}}a_{(2,1)}a_{(3,1)} + a_{(1,2)}\varepsilon_{(1,0,0),\{(2,1),(2,2)\}}a_{(3,1)} & \text{if } i=j=1, \\ \varepsilon_{(i,0,0),\{(1,2),(1,1)\}}a_{(2,1)}a_{(3,1)} & \text{if } i \text{ is odd}(\neq 1) \text{ and } j=1, \\ \varepsilon_{(i,j-1,0),\{(1,2),(3,1)\},(l_1,l_2)}a_{(3,1)} & \text{if } i,j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,2),(1,2)\}} \mapsto$$
$$\begin{cases} a_{(3,2)}a_{(2,2)}\varepsilon_{(1,0,0),\{(1,1),(1,2)\}} + a_{(3,2)}\varepsilon_{(1,0,0),\{(2,2),(2,1)\}}a_{(1,1)} & \text{if } i=j=1, \\ a_{(3,2)}a_{(2,2)}\varepsilon_{(i,0,0),\{(1,1),(1,2)\}} & \text{if } i \text{ is odd}(\neq 1) \text{ and } j=1, \\ a_{(3,2)}\varepsilon_{(i,j-1,0),\{(3,1),(1,2)\},(l_1,l_2)} & \text{if } i,j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,2)\}} \mapsto \begin{cases} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i,0)}a_{(2,1)}a_{(3,1)} & \text{if } i \text{ is even and } j=1, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)}a_{(3,1)} & \text{if } i \text{ is even and } j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,2),(1,1)\}} \mapsto \begin{cases} a_{(3,2)}a_{(2,2)}\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(i,0)} & \text{if } i \text{ is even and } j=1, \\ a_{(3,2)}\varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{if } i \text{ is even and } j \text{ is odd}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,0),\{(1,2),(3,1)\}} \mapsto \varepsilon_{(i,j-1,0),\{(1,2),(3,2)\},(l_1,l_2)}a_{(3,2)}X_3^{n_3-1} \quad \text{if } i \text{ is odd and } j \text{ is even},$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,2)\}} \mapsto X_3^{n_3-1}a_{(3,1)}\varepsilon_{(i,j-1,0),\{(3,2),(1,2)\},(l_1,l_2)} \quad \text{if } i \text{ is odd and } j \text{ is even},$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\}} \mapsto \varepsilon_{(i,j-1,0),\{(1,1),(3,2)\},(l_1,l_2)}a_{(3,2)}X_3^{n_3-1} \quad \text{if } i,j \text{ are even},$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\}} \mapsto X_3^{n_3-1}a_{(3,1)}\varepsilon_{(i,j-1,0),\{(3,2),(1,1)\},(l_1,l_2)} \quad \text{if } i,j \text{ are even},$$

$$\varepsilon_{(i,j,0),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto$$
$$\begin{cases} \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1.i-1)}a_{(2,1)}X_3 + X_1\varepsilon_{(i,0,0),\{(2,1),(2,2)\}}X_3 & \text{if } i \text{ is odd}(\neq 1), j=1 \text{ and } l_1=1, \\ \varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)}a_{(2,1)}X_3 & \text{if } j=1 \text{ and ``} l_1 \neq 1 \text{ or } i \text{ is even''}, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)}X_3 & \text{if } j \text{ is odd}(\neq 1), \\ a_{(1,1)}\varepsilon_{(i,j-1,0),\{(1,2),(3,2)\},(1,i-1)}a_{(3,2)}X_3^{n_3-1} & \text{if } i \text{ is odd}(\neq 1), j=2 \text{ and } l_1=1, \\ \varepsilon_{(i,j-1,0),\{(1,1),(3,1)\},(l_1,l_2)}X_3^{n_3-1} & \text{others}, \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto$$
$$\begin{cases} X_3a_{(2,2)}\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(1.i-1)} + X_3\varepsilon_{(i,0,0),\{(2,2),(2,1)\}}X_1 & \text{if } i \text{ is odd}(\neq 1), j=1 \text{ and } l_1=1, \\ X_3a_{(2,2)}\varepsilon_{(i,0,0),\{(1,1),(1,1)\},(l_1,l_2)} & \text{if } j=1 \text{ and ``} l_1 \neq 1 \text{ or } i \text{ is even''}, \\ X_3\varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{if } j \text{ is odd}(\neq 1), \\ X_3^{n_3-1}a_{(3,1)}\varepsilon_{(i,j-1,0),\{(3,2),(1,2)\},(1,i-1)}a_{(1,2)} & \text{if } i \text{ is odd}(\neq 1), j=2 \text{ and } l_1=1, \\ X_3^{n_3-1}\varepsilon_{(i,j-1,0),\{(3,1),(1,1)\},(l_1,l_2)} & \text{others}, \end{cases}$$

$$\varepsilon_{(i,j,0),\{(3,1),(3,1)\}} \mapsto \begin{cases} \sum_{l=0}^{n_3-1} X_3^l\varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}}X_3^{n_3-1-l} & \text{if } j \text{ is even}, \\ E_{(i,0,0),(2,2)}X_3 - X_3E_{(i,0,0),(2,2)} & \text{if } i \text{ is odd and } j=1, \\ \varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}}X_3 - X_3\varepsilon_{(i,j-1,0),\{(3,1),(3,1)\}} & \text{others}. \end{cases}$$

(5) *We have the following complexes depending on the relations $a_{1,2}a_{(2,1)}X_2^l a_{(3,1)}$ and $a_{3,2}a_{(2,2)}X_2^{l'}a_{(1,1)}$.*

$$Q_{(i,j,0)} \overset{\partial_{(i,j,1),3}}{\longleftarrow} Q_{(i,j,1)} \overset{\partial_{(i,j,2),3}}{\longleftarrow} \cdots \overset{\partial_{(i,j,n),3}}{\longleftarrow} Q_{(i,j,n)} \leftarrow \cdots .$$

for $i \geq 2$, $j \geq 2$ where, for $k \geq 0$, the left A^e-homomorphisms $\partial_{(i,1,k),3}: Q_{(i,1,k)} \to Q_{(i,1,k-1)}$ and $\partial_{(i,j,k),3}: Q_{(i,j,k)} \to Q_{(i,j,k-1)}$ are defined as follows.

$$\begin{cases}
\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(1,i-1)}a_{(2,2)}X_1 + X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(1,i-1)},\\
\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(1,i-1)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(1,i-1)},\\
\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,2)}X_1 + X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2-1)},\\
\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,1,k-1),\{(3,1),(1,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2-1)},\\
\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto a_{(3,1)}\varepsilon_{(i,1,0),\{(3,2),(1,1)\},(i,0)}a_{(2,1)}X_3 - X_3a_{(2,2)}\varepsilon_{(i,1,0),\{(1,1),(3,2)\},(i,0)}a_{(3,2)},
\end{cases}$$

if $j = 1$ and k is odd,

$$\begin{cases}
\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(1,i-1)}a_{(2,1)}X_3 - X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(1,i-1)},\\
\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(1,i-1)} \mapsto \varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(1,i-1)}a_{(2,2)}X_1 - X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(1,i-1)},\\
\varepsilon_{(i,1,k),\{(1,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,1)}X_3 - X_1a_{(2,1)}\varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2-1)},\\
\varepsilon_{(i,1,k),\{(3,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,1,k-1),\{(3,1),(3,1)\},(l_1,l_2),(l'_1-1,l'_2)}a_{(2,2)}X_1 + X_3a_{(2,2)}\varepsilon_{(i,1,k-1),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2-1)},
\end{cases}$$

if $j = 1$ and k is even,

$$\begin{cases}
\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(1,i-1),(l'_1-1,l'_2)}a_{(2,2)}X_1 - X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(1,i-1),(l'_1,l'_2-1)},\\
\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)} \mapsto\\
\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(1,i-1),(l'_1-1,l'_2)}a_{(2,1)}X_3 + X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(1,i-1),(l'_1,l'_2-1)},\\
\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(l_1,l_2)}a_{(2,2)}X_1,\\
\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(l_1,l_2)},\\
\varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(1,2),(3,1)\},(i,0)}a_{(2,2)}X_1,\\
\varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,2)\},(i,0)},\\
\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(i,0),1}a_{(2,2)}X_1,\\
\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2} \mapsto X_1a_{(2,1)}\varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(i,0),1},\\
\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(3,1),(1,1)\},(i,0),2}a_{(2,1)}X_3,\\
\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,1)\},(i,0),2},\\
\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(3,2),(1,1)\},(i,0)}a_{(2,1)}X_3,\\
\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto X_3a_{(2,2)}\varepsilon_{(i,j,k-1),\{(1,1),(3,2)\},(i,0)},
\end{cases}$$

if $j \neq 1$ and k is odd,

$$
\begin{cases}
\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(1,i-1),(l'_1,l'_2)} \mapsto \\
\varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(1,i-1),(l'_1-1,l'_2)} a_{(2,1)} X_3 + X_1 a_{(2,1)} \varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2-1)}, \\
\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(1,i-1),(l'_1,l'_2)} \mapsto \\
\varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(1,i-1),(l'_1-1,l'_2)} a_{(2,2)} X_1 - X_3 a_{(2,2)} \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2-1)}, \\
\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(l_1,l_2)} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(l_1,l_2),1} a_{(2,1)} X_3, \\
\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(l_1,l_2)} \mapsto X_3 a_{(2,2)} \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(l_1,l_2),2}, \\
\varepsilon_{(i,j,k),\{(1,2),(3,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(1,2),(1,1)\},(i,0)} a_{(2,1)} X_3, \\
\varepsilon_{(i,j,k),\{(3,1),(1,2)\},(i,0)} \mapsto X_3 a_{(2,2)} \varepsilon_{(i,j,k-1),\{(1,1),(1,2)\},(i,0)}, \\
\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),1} \mapsto \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(i,0),1} a_{(2,1)} X_3, \\
\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),1} \mapsto X_3 a_{(2,2)} \varepsilon_{(i,j,k-1),\{(1,1),(1,1)\},(i,0),2}, \\
\varepsilon_{(i,j,k),\{(3,1),(1,1)\},(i,0),2} \mapsto \varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(i,0),1} a_{(2,2)} X_1, \\
\varepsilon_{(i,j,k),\{(1,1),(3,1)\},(i,0),2} \mapsto X_1 a_{(2,1)} \varepsilon_{(i,j,k-1),\{(3,1),(3,1)\},(i,0),2}, \\
\varepsilon_{(i,j,k),\{(3,2),(1,1)\},(i,0)} \mapsto \varepsilon_{(i,j,k-1),\{(3,2),(3,1)\},(i,0)} a_{(2,2)} X_1, \\
\varepsilon_{(i,j,k),\{(1,1),(3,2)\},(i,0),2} \mapsto X_1 a_{(2,1)} \varepsilon_{(i,j,k-1),\{(3,1),(3,2)\},(i,0)},
\end{cases}
$$

if $j \neq 1$ *and* k *is even.*

(6) *The following complex depend on the complexes of* (3):

$$
0 \leftarrow Q_{(1,j,k)} \xleftarrow{\partial_{(2,j,k),1}} Q_{(2,j,k)} \xleftarrow{\partial_{(3,j,k),1}} \cdots \xleftarrow{\partial_{(n,j,k),1}} Q_{(n,j,k)} \leftarrow \cdots .
$$

The left A^e*-homomorphism* $\partial_{(i,j,k),1}{:}Q_{(i,j,k)} \to Q_{(i-1,j,k)}$ *is defined as follows. We consider the case that* k *is odd. In the case* k *is even, we have the similar result.*

$$
\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(1,i-1)} \mapsto
$$

$$
\begin{cases}
0 & \textit{if } i = 2, \\
-\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)} X_2 + X_2 \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)} & \textit{if } i \textit{ is even}(\neq 2), \\
\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(1,i-2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd},
\end{cases}
$$

$$
\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(1,i-1)} \mapsto
$$

$$
\begin{cases}
0 & \textit{if } i = 2, \\
-\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)} X_2 + X_2 \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)} & \textit{if } i \textit{ is even}(\neq 2), \\
\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(1,i-2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd},
\end{cases}
$$

$$
\varepsilon_{(i,1,k),\{(1,1),(1,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto
$$

$$
\begin{cases}
-\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\
-\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1,l_2-1),(l'_1,l'_2)} & \textit{if } l_2 \textit{ is odd},
\end{cases}
$$

$$
+ \begin{cases}
X_1^{n_1-1} \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(k+1,0)} & \textit{if } l_1 \textit{ is even and } l'_1 = k+1, \\
X_1 \varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(k+1,0)} & \textit{if } l_1 \textit{ is odd and } l'_1 = k+1, \\
\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(0,k+1)} X_1^{n_1-1} & \textit{if } l_1 \textit{ is even and } l'_1 = 0, \\
\varepsilon_{(i-1,1,k),\{(1,1),(1,1)\},(l_1-1,l_2),(0,k+1)} X_1 & \textit{if } l_1 \textit{ is odd and } l'_1 = 0, \\
0 & \textit{others},
\end{cases}
$$

$$\varepsilon_{(i,1,k),\{(3,1),(3,1)\},(l_1,l_2),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,1,k),\{(3,1),(3,1)\},(l_1,l_2-1),(l'_1,l'_2)} & \textit{if } l_2 \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto 0 \textit{ if } k = 1.$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(1,i-1),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} 0 & \textit{if } i = 2, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} & \textit{if } i \textit{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(1,i-2),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(1,i-1),(l'_1,l'_2)} \mapsto$$
$$\begin{cases} 0 & \textit{if } i = 2, \\ -\varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} & \textit{if } i \textit{ is even}(\neq 2), \\ \sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(3,1),(3,1)\},(1,i-2),(l'_1,l'_2)} X_2^{n_2-1-l} & \textit{if } i \textit{ is odd}, \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),1} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),1} & \textit{if } l_2 \textit{ is odd}(\neq 1), \end{cases}$$
$$+ \begin{cases} X_1^{n_1-1} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),1} & \textit{if } l_1 \textit{ is even}, \\ X_1 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),1} & \textit{if } l_1 \textit{ is odd}(\neq 1), \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(l_1,l_2),2} \mapsto$$
$$\begin{cases} -\sum_{l=0}^{n_2-1} X_2^l \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} X_2^{n_2-1-l} & \textit{if } l_2 \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1,l_2-1),2} & \textit{if } l_2 \textit{ is odd}(\neq 1), \end{cases}$$
$$+ \begin{cases} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),2} X_1^{n_1-1} & \textit{if } l_1 \textit{ is even}, \\ \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(l_1-1,l_2),2} X_1 & \textit{if } l_1 \textit{ is odd}(\neq 1), \end{cases}$$
$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i-1,1),1} \mapsto$$
$$\begin{cases} -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1} \\ \qquad \textit{if } i \textit{ is odd and } j \textit{ is even}, \\ -\varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)} a_{(1,2)} X_2 + X_2 \varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)} a_{(1,2)} \\ \qquad \textit{if } i, j \textit{ are even}, \end{cases}$$
$$+ \begin{cases} X_1^{n_1-1} \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),1} & \textit{if } i \textit{ is odd}, \\ X_1 \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),1} & \textit{if } i \textit{ is even}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i-1,1),2} \mapsto$$

$$\begin{cases} -\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2}X_2 + X_2\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2} \\ \qquad\qquad\qquad\qquad\qquad \text{if } i \text{ is odd and } j \text{ is even}, \\ -a_{(1,1)}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)}X_2 + X_2 a_{1,1}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)} \\ \qquad\qquad\qquad\qquad\qquad \text{if } i, j \text{ are even}, \end{cases}$$

$$+ \begin{cases} X_1^{n_1-1}\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),2} & \text{if } i \text{ is odd}, \\ X_1\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-2,1),2} & \text{if } i \text{ is even}(\neq 1), \end{cases}$$

$$\varepsilon_{(i,j,k),\{(1,2),(1,1)\},(i,0)} \mapsto a_{(1,2)}\varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),1} \text{ if } i \text{ is odd and } j \text{ is even},$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,2)\},(i,0)} \mapsto \varepsilon_{(i-1,j,k),\{(1,1),(1,1)\},(i-1,0),2}a_{(1,1)} \text{ if } i \text{ is odd and } j \text{ is even},$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),1} \mapsto X_1^{n_1-1}a_{(1,1)}\varepsilon_{(i-1,j,k),\{(1,2),(1,1)\},(i-1,0)} \text{ if } i, j \text{ are even},$$

$$\varepsilon_{(i,j,k),\{(1,1),(1,1)\},(i,0),2} \mapsto \varepsilon_{(i-1,j,k),\{(1,1),(1,2)\},(i-1,0)}a_{(1,2)}X_1^{n_1-1} \text{ if } i, j \text{ are even},$$

$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto 0 \text{ if } i, j \text{ are even},,$$

$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto 0 \text{ if } i, j \text{ are even},$$

$$\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto 0 \text{ if } i \text{ is even and } j \text{ is odd},$$

$$\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto 0 \text{ if } i \text{ is even and } j \text{ is odd}.$$

(7) *The following complexes depend on the complexes of* (4):

$$0 \leftarrow Q_{(i,1,k)} \xleftarrow{\partial_{(i,2,k),2}} Q_{(i,2,k)} \xleftarrow{\partial_{(i,3,k),2}} \cdots \xleftarrow{\partial_{(i,n,k),2}} Q_{(i,n,k)} \leftarrow \cdots.$$

The left A^e-homomorphism $\partial_{(i,j,k),2}$:$Q_{(i,j,k)} \to Q_{(i,j-1,k)}$ is defined as follows. We consider the case that k is odd. In the case that k is even, we have the similar result.

$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),1} \mapsto X_3^{n_3-1}a_{(3,1)}\varepsilon_{(i,j-1,k),\{(3,2),(3,1)\},(i,0)} \text{ if } i, j \text{ are even},,$$

$$\varepsilon_{(i,j,k),\{(3,1),(3,1)\},(i,0),2} \mapsto \varepsilon_{(i,j-1,k),\{(3,1),(3,2)\},(i,0)}a_{(3,2)}X_3^{n_3-1} \text{ if } i, j \text{ are even},$$

$$\varepsilon_{(i,j,k),\{(3,2),(3,1)\},(i,0)} \mapsto a_{(3,2)}\varepsilon_{(i,j-1,k),\{(3,1),(3,1)\},(i,0),1} \text{ if } i \text{ is even and } j \text{ is odd},$$

$$\varepsilon_{(i,j,k),\{(3,1),(3,2)\},(i,0)} \mapsto \varepsilon_{(i,j-1,k),\{(3,1),(3,1)\},(i,0),2}a_{(3,1)} \text{ if } i \text{ is even and } j \text{ is odd},$$

$$\varepsilon_{(i,1,1),\{(3,1),(3,1)\},(i,0)} \mapsto 0 \text{ if } j = k = 1.$$

Then, the projective bimodule resolution of A is total complex of these complexes.

2. Main results

We have the projective bimodule resolution of A as the total complex of the complexes in Proposition 1. And, using this resolution, we determine the ring structure of the Hochschild cohomology ring of A modulo nilpotence.

Theorem 2. *We define the projective A-bimodules P_n for $n \geq 1$:*

$$P_n = \coprod_{i+j+k=n} Q_{(i,j,k)},$$

and the A^e-homomorphism d_n for $n \geq 1$:

$$\sum_{i+j=n} (\partial_{(i,j,0),1} + (-1)^{i+1}\partial_{(i,j,0),2}) + \sum_{i+j+k=n, k\geq 1} ((-1)^{j+k}\partial_{(i,j,k),1} + (-1)^{i+k+1}\partial_{(i,j,k),2} + \partial_{(i,j,k),3})$$

Then the following complex is the projective bimodule resolution of A:

$$0 \leftarrow A \xleftarrow{\pi} P_0 \xleftarrow{d_1} P_1 \xleftarrow{d_2} \cdots \xleftarrow{d_n} P_n \leftarrow \cdots .$$

Then the basis elements that are not nilpotent in Hochschild cohomology ring of A are 1_A and $e_{(2,1)} + e_{(2,2)} \in \mathrm{HH}^{2n}(A)$ for $n \geq 1$. The other elements are nilpotent elements. Therefore, we have the following result.

Theorem 3. *The Hochschild cohomology ring of A modulo nilpotence is isomorphic to the polynomial ring:*

$$\mathrm{H}^*(A)/\mathcal{N} \cong k[x]$$

where $x^n = e_{(2,1)} + e_{(2,2)} \in \mathrm{HH}^{2n}(A)$ for $n \geq 1$.

Now, we conjecture that the projective bimodule resolution of the finite dimensional algebra with quantum-like relations and monomial relations is given by the total complex of the complexes depending on the relations.

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THICK SUBCATEGORIES OF DERIVED CATEGORIES OF ISOLATED SINGULARITIES

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Abstract. Let R be a commutative noetherian local ring with residue field k. Denote by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of finitely generated R-modules. This article gives a classification of the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing k when R has an isolated singularity. If R is moreover Cohen–Macaulay and has minimal multiplicity, all the standard thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ are classified.

Key Words: Cohen–Macaulay ring, Derived category, Isolated singularity, Specialization-closed subset, Thick subcategory.

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1. Introduction

A *thick subcategory* of a triangulated category is by definition a full triangulated subcategory closed under direct summands. The notion of a thick subcategory has been introduced by Verdier [17] by the name of *épaisse* subcategory to develop the theory of Verdier localizations.

Classifying thick subcategories of triangulated categories is one of the most important subjects shared by homotopy theory, ring theory, algebraic geometry and representation theory. Classifying thick subcategories is one of the most important problems shared by homotopy theory, ring theory, algebraic geometry and representation theory. It was first done by Devinatz, Hopkins and Smith [4, 8] in the 1980s; they classified the thick subcategories of the triangulated category of compact objects in the p-local stable homotopy category. Later on, as an analogue of the Devinatz–Hopkins–Smith theorem for commutative rings, Hopkins and Neeman [7, 10] classified the thick subcategories of the derived category of perfect complexes over a commutative noetherian ring, and it was extended to a quasi-compact quasi-separated scheme by Thomason [16]. As an analogue of the Hopkins–Neeman theorem for finite groups, Benson, Carlson and Rickard [1] classified the thick subcategories of the stable category of finite dimensional representations of a finite p-group. It was extended to a finite group scheme by Friedlander and Pevtsova [6] and further generalized to the derived category of a finite group by Benson, Iyengar and Krause [2].

The celebrated Hopkins–Neeman theorem classifies the thick subcategories of perfect complexes over a commutative noetherian ring. To apply this for the whole derived category, let R be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then the theorem states that there is a bijection between the thick subcategories of the bounded

The detailed version [15] of this article will be submitted for publication elsewhere.

derived category $\mathsf{D}^{\mathsf{b}}(R)$ of finitely generated R-modules and the specialization-closed subsets of $\operatorname{Spec} R$. This theorem especially says that any nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ contains k. Since this fact is itself clear, the essential part of the Hopkins–Neeman theorem asserts that for a regular local ring R taking the *supports* makes a bijection from the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing k to the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$. The first main result of this article is the following theorem, which guatantees that this consequence of the Hopkins–Neeman theorem remains valid for a much wider class of rings, that is, the class of (catenary equidimensional) isolated singularities.

Theorem 1. *Let $(R, \mathfrak{m}, k)$ be a catenary equidimensional local ring with an isolated singularity. The assignments $f : \mathcal{X} \mapsto \operatorname{Supp}_R \mathcal{X}$ and $g : S \mapsto \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ make mutually inverse bijections*

$$\left\{\begin{gathered}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R)\\ \textit{containing } k\end{gathered}\right\} \underset{g}{\overset{f}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{gathered}\textit{Specialization-closed subsets of } \operatorname{Spec} R\\ \textit{containing } \mathfrak{m}\end{gathered}\right\}.$$

The assumption that R is catenary and equidimensional is quite weak; local rings that appear in algebraic geometry usually satisfy this assumption. For example, one-dimensional local rings, Cohen–Macaulay local rings, complete local domains and their localizations at prime ideals are all catenary and equidimensional. Also, the assumption that R has an isolated singularity is a mild condition; it is a standard assumption in the representation theory of Cohen–Macaulay rings. For instance, this assumption is indispensable to get the Auslander–Reiten quiver of the maximal Cohen–Macaulay R-modules. Reduced local rings of dimension one and normal local rings of dimension two are Cohen–Macaulay rings with an isolated singularity. Moreover, it turns out that without the isolated singularity assumption, Theorem 1 is no longer true; see Remark 17.

Several related results to Theorem 1 have been obtained so far. The author [14] classifies the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ containing R and k when R is a Gorenstein local ring that is locally a hypersurface on the punctured spectrum. Stevenson [12] obtains a complete classification of the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ in the case where R is a complete intersection. Thus, our next goal is to classify the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ for a non-complete-intersection local rings. However, this problem itself turns out to be quite hard; indeed, there seems even to be no example of a non-complete-intersection ring R such that all the thick subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ are classified. So it would be a reasonable approach to consider classifying the thick subcategories satisfying a certain condition which all the thick subcategories satisfy over complete intersections. The standard condition is such a one: We say that a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ is *standard* if it contains a nonzero object of finite projective dimension. Dwyer, Greenlees and Iyengar [5] prove that if R is a complete intersection, then every nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ is standard. As an application of Theorem 1, we obtain the following classification theorem of standard thick subcategories.

Theorem 2. *Let R be a nonregular local ring with an isolated singularity. Suppose that R is either*

(1) *a hypersurface, or*
(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then there is a one-to-one correspondence

$$\left\{\begin{matrix}\textit{Standard thick}\\ \textit{subcategories}\\ \textit{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \underset{\Gamma}{\overset{\Lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{1\text{–}1}}}} \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\} \sqcup \left\{\begin{matrix}\textit{Nonempty}\\ \textit{specialization-closed}\\ \textit{subsets of } \operatorname{Spec} R\end{matrix}\right\}.$$

Here, the maps Λ and Γ are defined by:

$$\Lambda(\mathcal{X}) = \begin{cases} (\operatorname{Supp} \mathcal{X}, 1) & \textit{if } \mathcal{X} \subseteq \mathsf{D}_{\mathsf{perf}}(R), \\ (\operatorname{Supp} \mathcal{X}, 2) & \textit{if } \mathcal{X} \nsubseteq \mathsf{D}_{\mathsf{perf}}(R), \end{cases}$$

$$\Gamma((S,i)) = \begin{cases} (\operatorname{Supp}^{-1} S) \cap \mathsf{D}_{\mathsf{perf}}(R) & \textit{if } i = 1, \\ \operatorname{Supp}^{-1} S & \textit{if } i = 2. \end{cases}$$

In the next Section 2 we make several necessary definitions and fundamental properties. In Sections 3 and 4 we give some comments on the above two theorems.

2. Basic definitions

Let us begin with fixing our conventions.

Convention 3. Throughout (the rest of) this article, let R be a commutative noetherian ring. We assume that all modules are finitely generated, and that all subcategories are nonempty, full and closed under isomorphism.

We denote by $\mathsf{mod}\, R$ the category of (finitely generated) R-modules, by $\mathsf{C}^{\mathsf{b}}(R)$ the category of bounded complexes of (finitely generated) R-modules and by $\mathsf{D}^{\mathsf{b}}(R)$ the bounded derived category of $\mathsf{mod}\, R$. Note that $\mathsf{mod}\, R$ and $\mathsf{C}^{\mathsf{b}}(R)$ are abelian categories and $\mathsf{D}^{\mathsf{b}}(R)$ is a triangulated category.

We make the definitions of thick subcategories of $\mathsf{mod}\, R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$.

Definition 4. (1) A subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property for short exact sequences.

(2) A subcategory $\mathcal{X}$ of $\mathsf{C}^{\mathsf{b}}(R)$ is called *thick* if it is closed under direct summands and shifts and satisfies the 2-out-of-3 property for short exact sequences.

(3) A subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ is called *thick* if it is closed under direct summands and satisfies the 2-out-of-3 property for exact triangles.

Let $\mathcal{C}$ be one of the categories $\mathsf{mod}\, R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$. For each subcategory $\mathcal{M}$ of $\mathcal{C}$ we denote by $\mathsf{thick}_{\mathcal{C}}\, \mathcal{M}$ the smallest thick subcategory of $\mathcal{C}$ containing $\mathcal{M}$, and call it the *thick closure* of $\mathcal{M}$ in $\mathcal{C}$.

Remark 5. (1) Every Serre subcategory of $\mathsf{mod}\, R$ is thick.

(2) The category $\mathsf{thick}_{\mathsf{mod}\, R}\, R$ consists of the R-modules of finite projective dimension.

(3) A thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ is closed under shifts. In fact, each object M of $\mathsf{D}^{\mathsf{b}}(R)$ admits exact triangles $M \to 0 \to M[1] \rightsquigarrow$ and $M[-1] \to 0 \to M \rightsquigarrow$ in $\mathsf{D}^{\mathsf{b}}(R)$. Since $\mathcal{X}$ contains 0, this shows that if M is in $\mathcal{X}$, then so is $M[\pm 1]$, and by induction so is $M[n]$ for all $n \in \mathbb{Z}$.

(4) Let R be a local ring with residue field k. Then $\mathsf{thick}_{\mathsf{mod}\, R}\, k$ consists of the R-modules of finite length. In particular, a thick subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ contains k if and only if $\mathcal{X}$ contains all the R-modules of finite length.

Let us recall the relationships among the categories $\mathsf{mod}\, R$, $\mathsf{C}^{\mathsf{b}}(R)$ and $\mathsf{D}^{\mathsf{b}}(R)$. Among these three categories there are natural functors

$$\mathsf{mod}\, R \xrightarrow{\alpha} \mathsf{C}^{\mathsf{b}}(R) \xrightarrow{\beta} \mathsf{D}^{\mathsf{b}}(R).$$

Each object C in $\mathsf{C}^{\mathsf{b}}(R)$ is sent by β to the same complex C, while each morphism $g : X \to Y$ in $\mathsf{C}^{\mathsf{b}}(R)$ is sent by β to the roof $X \xleftarrow{1} X \xrightarrow{g} Y$ in $\mathsf{D}^{\mathsf{b}}(R)$. For an object $M \in \mathsf{mod}\, R$ (resp. $C \in \mathsf{C}^{\mathsf{b}}(R)$) we often use the same letter M (resp. C) to denote $\alpha(M)$ (resp. $\beta(C)$).

Next we recall the defintion of a specialization-closed subset.

Definition 6. A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if S contains $\mathrm{V}(\mathfrak{p})$ for all $\mathfrak{p} \in S$. Here, for an ideal I of R we denote by $\mathrm{V}(I)$ the set of prime ideals of R containing I.

Remark 7. (1) A specialization-closed subset of $\operatorname{Spec} R$ is nothing but a (possibly infinite) union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.
(2) Let R be a local ring with maximal ideal $\mathfrak{m}$. Then a specialization-closed subset of $\operatorname{Spec} R$ is nonempty if and only if it contains $\mathfrak{m}$.

Now we introduce the notion of supports for the module category $\mathsf{mod}\, R$.

Definition 8. (1) For each module $M \in \mathsf{mod}\, R$ we denote by $\operatorname{Supp}_R M$ the set of prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \ncong 0$ in $\mathsf{mod}\, R_{\mathfrak{p}}$, and call this the *support* of M in $\mathsf{mod}\, R$.
(2) For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ we set $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp}_R X$, and call this the *support* of $\mathcal{X}$.
(3) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}_{\mathsf{mod}\, R}^{-1} S$ the subcategory of $\mathsf{mod}\, R$ consisting of all modules whose supports are contained in S.

Remark 9. (1) For an exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{mod}\, R$ it holds that $\operatorname{Supp}_R M = \operatorname{Supp}_R L \cup \operatorname{Supp}_R N$.
(2) Let M be an R-module. Then $\operatorname{Supp}_R M$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology.
(3) Let S be a set of prime ideals of R. Then $\operatorname{Supp}_{\mathsf{mod}\, R}^{-1} S$ is a Serre subcategory of $\mathsf{mod}\, R$, and in particular, a thick subcategory of $\mathsf{mod}\, R$.

Next we introduce the notion of supports for the derived category $\mathsf{D}^{\mathsf{b}}(R)$.

Definition 10. (1) Let X be an object of $\mathsf{D}^{\mathsf{b}}(R)$. Then the following sets of prime ideals of R are the same.
- $\operatorname{Supp}_R \mathrm{H}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}(X)_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{mod}\, R_{\mathfrak{p}}\}$.
- $\bigcup_{i \in \mathbb{Z}} \operatorname{Supp}_R \mathrm{H}^i(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{H}^i(X)_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{mod}\, R_{\mathfrak{p}} \text{ for some } i \in \mathbb{Z}\}$.
- $\{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{D}^{\mathsf{b}}(R_{\mathfrak{p}})\}$.
- $\{\mathfrak{p} \in \operatorname{Spec} R \mid \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} X_{\mathfrak{p}} \ncong 0 \text{ in } \mathsf{D}^{-}(R_{\mathfrak{p}})\}$.

Here $\mathsf{D}^{-}(R_{\mathfrak{p}})$ stands for the derived category of bounded-above $R_{\mathfrak{p}}$-complexes. We denote these four sets by $\operatorname{Supp}_R X$ and call it the *support* of X in $\mathsf{D}^{\mathsf{b}}(R)$.
(2) For a subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ we set $\operatorname{Supp}_R \mathcal{X} = \bigcup_{X \in \mathcal{X}} \operatorname{Supp}_R X$ and call it the *support* of $\mathcal{X}$.

(3) For a subset S of $\operatorname{Spec} R$ we denote by $\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of objects whose supports are contained in S.

The supports for $\mathsf{D}^{\mathsf{b}}(R)$ have the same notation as those for $\operatorname{\mathsf{mod}} R$, but there would be no danger of confusion since the support of an object M of $\operatorname{\mathsf{mod}} R$ is equal to the support of the object $\beta\alpha(M)$ of $\mathsf{D}^{\mathsf{b}}(R)$.

Remark 11. (1) (a) One has $\operatorname{Supp}_R(X \oplus Y) = \operatorname{Supp}_R X \cup \operatorname{Supp}_R Y$ for $X, Y \in \mathsf{D}^{\mathsf{b}}(R)$.
(b) One has $\operatorname{Supp}_R(X[n]) = \operatorname{Supp}_R X$ for $X \in \mathsf{D}^{\mathsf{b}}(R)$ and $n \in \mathbb{Z}$.
(c) Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D}^{\mathsf{b}}(R)$. Then for any permutation A, B, C of X, Y, Z one has $\operatorname{Supp}_R A \subseteq \operatorname{Supp}_R B \cup \operatorname{Supp}_R C$.
(2) For $X \in \mathsf{D}^{\mathsf{b}}(R)$ the subset $\operatorname{Supp}_R X$ of $\operatorname{Spec} R$ is closed in the Zariski topology.
(3) For a subset S of $\operatorname{Spec} R$ the subcategory $\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ of $\mathsf{D}^{\mathsf{b}}(R)$ is thick.
(4) Let $\mathcal{X}$ be a subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Then $\operatorname{Supp}_R \mathcal{X}$ is a specialization-closed subset of $\operatorname{Spec} R$. Furthermore, it holds that $\operatorname{Supp}_R \mathcal{X} = \operatorname{Supp}_R(\operatorname{\mathsf{thick}}_{\mathsf{D}^{\mathsf{b}}(R)} \mathcal{X})$, because $\operatorname{Supp}^{-1}(\operatorname{Supp} \mathcal{X})$ is a thick subcategory containing $\mathcal{X}$, whence contains $\operatorname{\mathsf{thick}} \mathcal{X}$.

Let us recall the defintion of (the derived category of) perfect complexes.

Definition 12. A *perfect* complex is by defintion a bounded complex of finitely generated projective modules. We denote by $\mathsf{D}_{\mathsf{perf}}(R)$ the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of perfect complexes. This is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$, and hence a triangulated category. For each subset S of $\operatorname{Spec} R$ we set $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} S = (\operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S) \cap \mathsf{D}_{\mathsf{perf}}(R)$.

Remark 13. (1) Every thick subcategory of $\mathsf{D}_{\mathsf{perf}}(R)$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$.
(2) One has $\operatorname{\mathsf{thick}}_{\mathsf{D}^{\mathsf{b}}(R)} R = \mathsf{D}_{\mathsf{perf}}(R)$.
(3) For a set S of prime ideals of R, the subcategory $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} S$ of $\mathsf{D}_{\mathsf{perf}}(R)$ is thick.

Finally, we recall the definitions of a hypersurface, a Cohen–Macaulay ring with minimal multiplicity and a disjoint union of sets.

Definition 14. (1) A local ring R is called a *hypersurface* if the completion of R is isomorphic to a quotient of a regular local ring by a nonzero element.
(2) Let R be a Cohen–Macaulay local ring. Then R satisfies the inequality

$$\mathrm{e}(R) \geq \operatorname{edim} R - \dim R + 1, \tag{2.1}$$

where $\mathrm{e}(R)$ and $\operatorname{edim} R$ denote the multiplicity of R and the embedding dimension of R, respectively. We say that R has *minimal multiplicity* (or *maximal embedding dimension*) if the equality of (2.1) holds.
(3) Let A_1, A_2 be sets whose intersection is possibly nonempty. The *disjoint union* of A_1 and A_2 is defined as

$$A_1 \sqcup A_2 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) = \{(x, 1), (y, 2) \mid x \in A_1, y \in A_2\}.$$

In the case where $A_1 \cap A_2$ is empty, the set $A_1 \sqcup A_2$ is identified with the union $A_1 \cup A_2$, namely, it is the usual disjoint union.

3. Comments on Theorem 1

The essentail part of the proof of Theorem 1 is played by the following result. This is shown by using contradiction, considering complexes of objects in $\mathsf{C}^{\mathsf{b}}(R)$ and applying the Hopkins–Neeman theorem [10, Theorem 1.5].

Proposition 15. *Let $(R, \mathfrak{m}, k)$ be a catenary equidimensional local ring with an isolated singularity. Let X be a non-acyclic bounded complex of R-modules. Then one has*

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, X\} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}.$$

Remark 16. The equality in Proposition 15 is no longer true if we remove k from the left-hand side; the equality

$$\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)} X = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{R/\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X\}$$

holds for $X = R$ if and only if $\mathsf{D}_{\mathsf{perf}}(R) = \mathsf{D}^{\mathsf{b}}(R)$, if and only if R is regular. This is one of the reasons why we consider thick subcategories containing k.

We should remark that unless R has only an isolated singularity, Theorem 1 does not necessarily hold. To be more precise, if R does not have an isolated singularity, then there may exist a thick subcategory $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ containing k such that $\mathcal{X} \neq \operatorname{Supp}^{-1} S$ for all nonempty specialization-closed subsets S of $\operatorname{Spec} R$.

Remark 17. Let $(R, \mathfrak{m}, k)$ be a local ring, and suppose that R does not have an isolated singularity. Set $\mathcal{X} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}\{k, R\}$. Then $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, but $\mathcal{X} \neq \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ for all subsets S of $\operatorname{Spec} R$.

Note that in the above remark $\mathcal{X} \neq \operatorname{Supp}^{-1}_{\mathsf{D}^{\mathsf{b}}(R)} S$ for all subsets S of $\operatorname{Spec} R$, not necessarily nonempty specialization-closed ones.

As a consequence of Theorem 1, we obtain the following one-to-one correspondence without prime ideals.

Corollary 18. *Let R be a catenary equidimensional local ring with an isolated singularity. Then one has a one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Thick subcategories of } \mathsf{D}^{\mathsf{b}}(R) \\ \textit{containing } k\end{array}\right\} \underset{\psi}{\overset{\phi}{\underset{1-1}{\rightleftarrows}}} \left\{\begin{array}{c}\textit{Nonzero thick subcategories} \\ \textit{of } \mathsf{D}_{\mathsf{perf}}(R)\end{array}\right\},$$

where ϕ, ψ are defined by $\phi(\mathcal{X}) = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$ and $\psi(\mathcal{Y}) = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$ for subcategories $\mathcal{X}$ of $\mathsf{D}^{\mathsf{b}}(R)$ and $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$.

Proof. Let S be a specialization-closed subset of $\operatorname{Spec} R$ containing $\mathfrak{m}$. Take a sequence $\boldsymbol{x}$ of elements of R which generates $\mathfrak{m}$. Then $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)} S$ contains the Koszul complex $\mathrm{K}(\boldsymbol{x}, R)$, and hence it is a nonzero thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. Conversely, for any nonzero thick subcategory $\mathcal{Y}$ of $\mathsf{D}_{\mathsf{perf}}(R)$, the support $\operatorname{Supp}_R \mathcal{Y}$ contains $\mathfrak{m}$. Thus, the Hopkins–Neeman theorem [10, Theorem 1.5] implies that Supp_R and $\operatorname{Supp}^{-1}_{\mathsf{D}_{\mathsf{perf}}(R)}$ make mutually inverse bijections between the nonzero thick subcategories of $\mathsf{D}_{\mathsf{perf}}(R)$ and the specialization-closed subsets of $\operatorname{Spec} R$ containing $\mathfrak{m}$.

Let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ containing k, and let $\mathcal{Y}$ be a nonzero thick subcategory of $\mathsf{D}_{\mathsf{perf}}(R)$. Combining our Theorem 1 with the above one-to-one correspondence, one has only to verify the equalities

(1) $\operatorname{Supp}_{\mathsf{D}_{\mathsf{perf}}(R)}^{-1}\operatorname{Supp}\mathcal{X} = \mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R)$,

(2) $\operatorname{Supp}_{\mathsf{D}^{\mathsf{b}}(R)}^{-1}\operatorname{Supp}\mathcal{Y} = \mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\})$.

We have $\mathcal{X} \cap \mathsf{D}_{\mathsf{perf}}(R) \subseteq \operatorname{Supp}_{\mathsf{D}_{\mathsf{perf}}(R)}^{-1}(\operatorname{Supp}\mathcal{X}) \subseteq \operatorname{Supp}_{\mathsf{D}^{\mathsf{b}}(R)}^{-1}(\operatorname{Supp}\mathcal{X}) = \mathcal{X}$, where the last equality follows from Theorem 1. This shows the equality (1). On the other hand, it holds that $\operatorname{Supp}\mathcal{Y} = \operatorname{Supp}(\mathcal{Y} \cup \{k\}) = \operatorname{Supp}(\mathsf{thick}_{\mathsf{D}^{\mathsf{b}}(R)}(\mathcal{Y} \cup \{k\}))$, where the second equality follows from the fact that $\mathcal{Y}$ is nonzero. Applying $\operatorname{Supp}_{\mathsf{D}^{\mathsf{b}}(R)}^{-1}$ and using Theorem 1, we obtain the equality (2). $\square$

4. Comments on Theorem 2

The *stable derived category* $\mathsf{D}_{\mathsf{sg}}(R)$ of R, which is also called the *singularity category* of R, is defined as the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(R)$ by $\mathsf{D}_{\mathsf{perf}}(R)$. This has been introduced by Buchweitz [3] in relation to maximal Cohen–Macaulay modules over Gorenstein rings, and explored by Orlov [11] in relation to the Homological Mirror Symmetry Conjecture.

The essential part of the proof of Theorem 2 is played by the following result on the stable derived category. The assertion immediately follows from [13, Main Theorem] in the case (1). As for the case (2), it is shown by taking a minimal reduction of the maximal ideal.

Proposition 19. *Let R be a local ring with an isolated singularity. Suppose that R is either*

(1) *a hypersurface, or*

(2) *a Cohen–Macaulay ring with minimal multiplicity and infinite residue field.*

Then $\mathsf{D}_{\mathsf{sg}}(R)$ has no nontrivial thick subcategory.

Let us give some examples of a ring satisfying Theorem 2(2).

Example 20. Let k be an infinite field, and let x, y be indeterminates over k. Then it is easy to observe that $k[[x,y]]/(x^2, xy, y^2)$, $k[[x,y,z]]/(x^2-yz, y^2-zx, z^2-xy)$ and $k[[x^3, x^2y, xy^2, y^3]]$ are non-Gorenstein rings satisfying the condition (2) in Theorem 2. In general, normal local domains of dimension two with rational singularities satisfy Theorem 2(2); see [9, Theorem 3.1].

Remark 21. (1) Theorem 2(1) can also be deduced from [12, Theorem 4.9].
(2) Theorem 2(2) especially says the following.

> *Let R be a Cohen–Macaulay local ring with an isolated singularity and infinite residue field, and assume that R has minimal multiplicity. Let $\mathcal{X}$ be a standard thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ which is not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Then $\mathcal{X}$ contains the residue field of R.*

This statement is no longer true without the assumption that R has minimal multiplicity.

Indeed, let $R = k[x,y]/(x^2, y^2)$ with k a field, and let $\mathcal{X}$ be the thick closure of R and $R/(x)$ in $\mathsf{D}^{\mathsf{b}}(R)$. Then R is an artinian complete intersection local ring, and $\mathcal{X}$ is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(R)$. As $\mathcal{X}$ contains R, it is standard. As $R/(x)$ has infinite projective dimension as an R-module, $\mathcal{X}$ is not contained in $\mathsf{D}_{\mathsf{perf}}(R)$. Note that both R and $R/(x)$ have complexity at most one. Since the subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ consisting of objects having complexity at most one is thick, every object in $\mathcal{X}$ have complexity at most one. Since k has complexity two, $\mathcal{X}$ does not contain k.

Consequently, the assumption in Theorem 2(2) that R has minimal multiplicity is indispensable.

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EXAPMLES OF ORE EXTENSIONS WHICH ARE MAXIMAL ORDERS WHOSE BASED RINGS ARE NOT MAXIMAL ORDERS

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Abstract. Let R be a prime Goldie ring and (σ, δ) be a skew derivation on R. It is well known that if R is a maximal order, then the Ore extension $R[x; \sigma, \delta]$ is a maximal order. It was a long standing open question that the convserse is true or not in case $\sigma \neq 1$ and $\delta \neq 0$. We give an example of non-maximal order R with a skew derivation (σ, δ) on R ($\sigma \neq 1, \delta \neq 0$) such that $R[x; \sigma, \delta]$ is a maximal order.

1. Introduction

Let σ be an automorphism of a ring R and let δ be a left σ-derivation of R. Then we say (σ, δ) is a skew derivation on R. The aim of this paper is to obtain an example such that the Ore extension $R[x; \sigma, \delta]$ is a maximal order but R is not a maximal order.

In case δ is trivial, the following example is known (see [1, Proposition 2.6]). Let D be a hereditary Noetherian prime ring (an HNP ring for short) satisfying the following:

(a) there is a cycle $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ ($n \geq 2$) such that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$ for some $a \in D$ and

(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ ($1 \leq i \leq n$) is invertible.

We define a skew derivation (σ, δ) on D by $\sigma(r) = ara^{-1}$ and $\delta(r) = 0$ for all $r \in D$. Then D is clearly not a maximal order and the Ore extension $D[x; \sigma, 0]$ is a maximal order. But in case σ and δ are both non-tirvial, we need to consider the Ore extension of a polynomial ring over D and we must specify v-ideals of it.

Therefore let $R = D[t]$ be the polynomial ring over D in an indeterminate t. Then (σ, δ) on D is extended to a skew derivation on R by $\sigma(t) = t$ and $\delta(t) = a$ (see [4, Lemma 1.2]) and it is proved that the Ore extension $R[x; \sigma, \delta]$ is maximal order but R is not a maximal order (Theorem 12).

Section 2 contains preliminary results which are used in Section 3. In Section 3, we describe the structure of prime invertible ideals of $R[x; \sigma, \delta]$ (Proposition 9) and Theorem 12 is proved by showing that any v-ideal is v-invertible.

We refer the readers to [12] and [13] for terminology not defined in the paper.

2. Preliminary results

Let S be a Noetherian prime ring with quotient ring Q and A be a fractional S-ideal. We use the following notation:

$$(S : A)_l = \{q \in Q \mid qA \subseteq S\}, \quad (S : A)_r = \{q \in Q \mid Aq \subseteq S\} \text{ and}$$
$$A_v = (S : (S : A)_l)_r \supseteq A \quad \text{and} \quad {}_vA = (S : (S : A)_r)_l \supseteq A.$$

The detailed version of this paper will be submitted for publication elsewhere.

A is called a *v-ideal* if ${}_vA = A = A_v$. A v-ideal A is said to be *v-invertible* (*invertible*) if ${}_v((S:A)_lA) = S = (A(S:A)_r)_v$ $((S:A)_lA = S = A(S:A)_r)$, respectively.

Note that if A is v-invertible, then it is easy to see that $O_r(A) = S = O_l(A)$ and $(S:A)_l = A^{-1} = (S:A)_r$, where $O_l(A) = \{q \in Q \mid qA \subseteq A\}$, a *left order* of A, $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, a *right order* of A and $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.

Concerning invertible ideals and v-invertible ideals of S, the next lemma holds.

Lemma 1. *A v-ideal is invertible if and only if it is v-invertible and projective (left and right projective).*

In the remainder of this section, *let D be a hereditary Noetherian prime ring (an HNP ring for short) with quotient ring $K = Q(D)$ and $R = D[t]$.* Let σ be an inner automorphism induced by a regular element a of D, that is, $\sigma(r) = ara^{-1}$ for all $r \in D$ and δ be a trivial left σ-derivation on D, that is, $\delta(r) = 0$ for all $r \in D$.

Put $R = D[t]$, the polynomial ring over D in an indeterminate t. σ and δ are extended to an automorphism σ of R and a left σ-derivation δ on R as follows ([4, Lemma 1.2]);

$$\sigma(t) = t \quad \text{and} \quad \delta(t) = a.$$

It is well-known that a skew derivation (σ, δ) is naturally extended to a skew derivation on K ([12, p. 132]). Also we note that $\sigma\delta = \delta\sigma$ holds.

We put

$$\begin{aligned} V_r(R) &= \{\mathfrak{a}: \text{ ideals } \mid \mathfrak{a} = \mathfrak{a}_v\} \supseteq V_{(m,r)}(R) = \{\mathfrak{a} \in V_r(R) \mid \mathfrak{a} \text{ is maximal in } V_r(R)\}, \\ V_l(R) &= \{\mathfrak{a}: \text{ ideals } \mid \mathfrak{a} = {}_v\mathfrak{a}\} \supseteq V_{(m,l)}(R) = \{\mathfrak{a} \in V_l(R) \mid \mathfrak{a} \text{ is maximal in } V_l(R)\} \text{ and} \\ \mathrm{Spec}_0(R) &= \{\mathfrak{b}: \text{ prime ideals } \mid \mathfrak{b} \cap D = (0) \text{ and } \mathfrak{b} \text{ is a v-ideal}\}. \end{aligned}$$

Note that for each fractional R-ideal $\mathfrak{a}$, $\mathfrak{a} = \mathfrak{a}_v$ if and only if $\mathfrak{a}$ is right projective by [2, Proposition 5.2] and that there is a one-to-one correspondence between $\mathrm{Spec}_0(R)$ and $\mathrm{Spec}(K[t])$ (see [12, Proposition 2.3.17]).

Using these facts, we can prove the following lemma.

Lemma 2. $V_{(m,r)}(R) = V_{(m,l)}(R)$ *and is equal to*

$$V_m(R) = \{\mathfrak{m}[t], \mathfrak{b} \mid \mathfrak{m} \textit{ runs over all maximal ideals of } D \textit{ and } \mathfrak{b} \in \mathit{Spec}_0(R)\}.$$

From Lemmas 1 and 2, we have the following.

Lemma 3. *If $\mathfrak{b} \in Spec_0(R)$, then $\mathfrak{b}$ is invertible.*

Now we can determine the maximal invertible ideals of R by Lemmas 2 and 3.

Proposition 4. $\{\mathfrak{p}[t] = \mathfrak{m}_1[t] \cap \cdots \cap \mathfrak{m}_k[t], \mathfrak{b} \mid \mathfrak{m}_1, \ldots, \mathfrak{m}_k$ *is a cycle of* $D, k \geq 1, \mathfrak{b} \in$ *$Spec_0(R)\}$ is the full set of maximal invertible ideals of R (ideals maximal amongst the invertible ideals).*

The following proposition follows from the proof of [3, Proposition 2.1 and Theorem 2.9].

Proposition 5. *The invertible ideals of R generate an Abelian group whose generators are maximal invertible ideals.*

In case D has enough invertible ideals, it is shown in [9] that $R = D[t]$ is a v-HC order with enough v-invertible ideals, which is a Krull type generalization of HNP rings. Recall the notion of v-HC orders: A Noetherian prime ring S is called a *v-HC order* if ${}_v(A(S:A)_l) = O_l(A)$ for any ideal A of S with $A = {}_vA$ and $((R:S)_rB)_v = O_r(B)$ for any ideal B of S with $B = B_v$. A v-HC order S is said to be *having enough v-invertible ideals* if any v-ideal of S contains a v-ideal which is v-invertible. A v-ideal C is called *eventually v-idempotent* if $(C^n)_v$ is v-idempotent for some $n \geq 1$, that is, $(({(C^n)_v}^2)_v = (C^n)_v$.

Ideal theory in HNP rings are generalized to one in v-HC orders with enough v-invertible ideals. The following two lemmas are very useful to investigate the structure of v-ideals of v-HC orders (for their proofs, see [8, Lemma 1.1] and [10, Lemma 1 and Proposition 3]).

Lemma 6. *Let S be a prime Goldie ring and A, B be fractional S-ideals.*

(1) $(AB)_v = (AB_v)_v$.
(2) $(A_vB)_v = (AB)_v$ *if B is v-invertible.*
(3) $(AB)_v = A_vB$ *if B is invertible.*

Lemma 7. *Let S be a v-HC order with enough v-invertible ideals and A be a fractional S-ideal.*

(1) ${}_vA = A_v$.
(2) $A_v = (BC)_v$ *for some v-invertible ideal B and eventually v-idempotent ideal C.*
(3) *Let C be an eventually v-idempotent ideal and let $M_1, \dots, M_k$ be the full set of maximal v-ideals containing C. Then $(C^k)_v = ((M_1 \cap \cdots \cap M_k)^k)_v$ and is v-idempotent.*

Remark. A v-ideal of S is eventually v-idempotent if and only if it is not contained in any v-invertible ideals (see the proofs of [3, Propositions 4.3 and 4.5]).

3. Examples

Throughout this section, *D is an HNP ring with quotient ring K satisfying the following*:

(a) there is a cycle $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ such that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$ for some $a \in D$.
(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ is invertible.

Examples of an HNP ring D satisfying the conditions (a) and (b) are found in [6] and [1]. The simplest example is $D = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the ring of integers and p is a prime number.

Unless otherwise stated, $R = D[t]$, σ is an automorphism of R and δ is a left σ-derivation as in Section 1, that is, $\sigma(r) = ara^{-1}$, $\delta(r) = 0$ for all $r \in D$, $\sigma(t) = t$ and $\delta(t) = a$.

Note that $\sigma(\mathfrak{m}_i) = \mathfrak{m}_{i+1}$ $(1 \leq i \leq n-1)$, $\sigma(\mathfrak{m}_n) = \mathfrak{m}_1$ and $\sigma(\mathfrak{n}) = \mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ with $\mathfrak{n} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$ by [5, Theorem 14] and [9, Corollary 2.3]. Furthermore, by Lemma 2 and Proposition 4,

$$V_m(R) = \{\mathfrak{m}_i[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{n} \neq \mathfrak{m}_i \text{ and } \mathfrak{b} \in \mathrm{Spec}_0(R)\}$$

and

$$I_m(R) = \{\mathfrak{p}[t],\ \mathfrak{n}[t],\ \mathfrak{b} \mid \mathfrak{p} = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n, \mathfrak{n} \neq \mathfrak{m}_i \text{ and } \mathfrak{b} \in \mathrm{Spec}_0(R)\}$$

is the set of all maximal invertible ideals of R.

Note that a maximal ideal of $K[t]$ is either $tK[t]$ or $\omega K[t]$ for some $\omega = k_l t^l + \cdots + k_0 \in Z(K[t])$ with $k_l \neq 0, k_0 \neq 0, l \geq 1$, where $Z(K[t])$ is the center of $K[t]$ (see [12, Theorem 2.3.10]) and so any $\mathfrak{b} \in \mathrm{Spec}_0(R)$ is either $\mathfrak{b} = tR$ or $\mathfrak{b} = \omega K[t] \cap R$, where $\omega \in Z(K[t])$ and $\omega K[t]$ is a maximal ideal ([12, Proposition 2.3.17]).

A fractional R-ideal $\mathfrak{a}$ is called *σ-invariant* if $\sigma(\mathfrak{a}) = \mathfrak{a}$ and is called *δ-stable* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$. A σ-invariant and δ-stable fractional R-ideal is said to be *(σ, δ)-stable.*

The following lemma is crutial to study ideals of R and is proved by using the results obtained in section 2.

Lemma 8. (1) *Any projective ideal of R is a product of an invertible ideal and an eventually v-idempotent ideal.*
(2) *Any eventually v-idempotent ideal is not σ-invariant.*
(3) *$\mathfrak{n}[t]$ and $\mathfrak{p}[t]$ are (σ, δ)-stable.*
(4) *Let $\omega = t$ or $\omega \in Z(K[t])$ and let $\mathfrak{b} = \omega K[t] \cap R$, which is a maximal invertible ideal of R. Then*
  (i) *$\mathfrak{b}^n$ is σ-invariant for any $n \geq 1$.*
  (ii) *$\mathfrak{b}^n$ is δ-stable if and only if $\omega^n K[t]$ is δ-stable if and only if $\delta(\omega^n) = 0$.*
  (iii) (a) *If char $K = 0$, then $\mathfrak{b}^n$ is not δ-stable for any n.*
    (b) *If char $K = p \neq 0$ and $\delta(\omega) \neq 0$, then $\mathfrak{b}^p$ is (σ, δ)-stable and $\mathfrak{b}^i$ is not (σ, δ)-stable $(1 \leq i < p)$.*
    (c) *If char $K = p \neq 0$ and $\delta(\omega) = 0$, then $\mathfrak{b}^n$ is (σ, δ)-stable for all $n \geq 1$.*

In the remainder of this section, *let $S = R[x; \sigma, \delta]$, an Ore extension in an indeterminate x and $T = Q[x; \sigma, \delta]$, where $Q = Q(R)$, the quotient ring of R.* We will prove that S is a maximal order. To prove maximality of S, it is enough to show that each v-ideal of S is v-invertible. For this purpose, we will describe all v-ideals of S.

Note that for an ideal $\mathfrak{a}$ of R, $\mathfrak{a}[x; \sigma, \delta]$ is an ideal of S if and only if $\mathfrak{a}$ is (σ, δ)-stable.

From Lemma 8, we have the following Proposition 9 and we can prove invertibility of a v-ideal A of S such as $A \cap R \neq (0)$ by using Proposition 9.

Proposition 9. *Under the same notations as in Lemma 8, let A be an ideal of S such that $A = A_v$ and is maximal in $\{B : \text{ ideal } \mid B = B_v\}$. If $A \cap R = \mathfrak{a} \neq (0)$, then A is equal to one of $P = \mathfrak{p}[t][x; \sigma, \delta]$, $N = \mathfrak{n}[t][x; \sigma, \delta]$, $B = \mathfrak{b}[x; \sigma, \delta]$ (in case $\mathfrak{b}$ is (σ, δ)-stable) or $C = \mathfrak{b}^p[x; \sigma, \delta]$ (in case $\mathfrak{b}$ is σ-invariant but not δ-stable) and each of these is a prime invertible ideal of S.*

Lemma 10. *Let A be an ideal of S such that $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then $\mathfrak{a}$ is a (σ, δ)-stable invertible ideal and $A = \mathfrak{a}[x; \sigma, \delta]$.*

Outline of Proof. Assume that $A \supset \mathfrak{a}[x; \sigma, \delta]$ and that it is maximal for this property. Then, by Proposition 9, there is a $P_0 = \mathfrak{p}_0[x; \sigma, \delta] \supset A$, where $\mathfrak{p}_0 = \mathfrak{p}_[t]$ or $\mathfrak{n}[t]$ or $\mathfrak{b}$ or $\mathfrak{c}$ and $S \supseteq AP_0^{-1} \supset A$. Then $AP_0^{-1} = \mathfrak{a}'[x; \sigma, \delta]$ for some (σ, δ)-stable v-ideal $\mathfrak{a}'$, and $A = ((AP_0^{-1})P_0)_v = (\mathfrak{a}'\mathfrak{p}_0)_v[x; \sigma, \delta]$, which is a contradiction. □

By Lemma 10, we can prove also v-invertibility of a v-ideal A such as $A \cap R = (0)$.

Lemma 11. *Let A be an ideal of S such that $A = A_v$ and $A \cap R = (0)$. Then A is v-invertible.*

Outline of Proof. $T = (S : A)_l AT$ holds and so $(S : A)_l A \cap R \neq (0)$. Then ${}_v((S : A)_l A)$ is invertible by the left version of Lemma 10. Suppose ${}_v((S : A)_l A) \subset S$. Then there is a maximal invertible ideal P_0 which is prime and $P_0 \supseteq {}_v((S : A)_l A)$. Then the localizaion S_{P_0} is a local Dedekind prime ring and

$$S_{P_0} = (S_{P_0} : AS_{P_0})_l AS_{P_0} \subseteq S_{P_0}(S : A)_l AS_{P_0} \subseteq S_{P_0} P_0 S_{P_0} = J(S_{P_0}),$$

the Jacobson radical of S_{P_0}, which is a contradiction. □

Now we obtain the main theorem of this paper by Lemmas 10 and 11.

Theorem 12. *$S = R[x; \sigma, \delta]$ is a maximal order and R is not a maximal order.*

Proof. Let A be any non-zero ideal of S. Since $S \subseteq O_l(A) \subseteq O_l(A_v)$, in order to prove $O_l(A) = S$, we may assume that $A = A_v$. By Lemmas 10 and 11, A is (v)-invertible . Hence $O_l(A) = S$ and similarly $O_r(A) = S$, that is, S is a maximal order. Of course R is not a maximal order. □

As an application of Theorem 12, we give the example related to unique factorization rings. A Noetherian prime ring R is called a *unique factorization ring* (a UFR for short) if each prime ideal P with $P = P_v$ (or $P = {}_vP$) is principal, that is, $P = bR = Rb$ for some $b \in R$. We note that R is a UFR if and only if R is a maximal order and each v-ideal is principal, and if R is a maximal order, then every prime v-ideal is a maximal v-ideal.

Then we have the following.

Proposition 13. *Suppose char $D = 0$ and any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ ($1 \leq i \leq n$) is principal. Then $S = R[x; \sigma, \delta]$ is a UFR but R is not a UFR.*

At the end, we state an open problem concerning Ore extensions.

Problem. Let R be a prime Goldie ring and consider the Ore extension $R[x; \sigma, \delta]$ of R, where (σ, δ) is a skew derivation on R. Then what is necessary and sufficient condition for $R[x; \sigma, \delta]$ to be a maximal order or unique factorization ring?

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QUIVERS, OPERATORS ON HILBERT SPACES AND OPERATOR ALGEBRAS

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ABSTRACT. One of the aims of the theory of representations of finite dimensional algebras is to describe how linear transformations can act simultaneously on a finite dimensional vector space. We consider bounded linear operators on a infinite-dimensional Hilbert space instead of linear transformations on a finite dimensional vector space. We describe similarities and differences between ring theory and theory of operator algebras.

1. Introduction

Operator algebraists import many ideas from ring theory without paying anything. Ring theorists import few ideas from theory of operator algebras because it is based on functional analysis. But we expect more fruitful interactions between these two theory. For example, quivers are related with operator algebras in the following (at least) three different stages:
(1)Cuntz-Krieger algebras [1]
(2)Principal graphs for subfactors [5], [4],[8]
(3)Hilbert representations of quivers [2], [3]

First We describe similarities and differences between ring theory and theory of operator algebras.

(1)Cuntz-Krieger algebras [1]

Strongly connected quivers generate an important class of purely infinite simple C^*-algebras, called Cuntz-Krieger algebras, and they are classified by their K-groups. The vertices are represented by orthogonal subspaces and the arrows are represented by partial isometries with the orthognal ranges.

(2)Principal graphs for subfactors [5], [4],[8]

The category of bimodules for a subfactor forms a principal graph (a certain bitartite graph) and a good invariant in subfactor theory. In particular, irreducible hyperfinite subfactors with Jones index less than four have Dynkin diagrams A,D and E. The vertices are constructed by irreducible bimodules and arrows are constructed by bimodule homomorphisms.

The paper is in a final form and no version of it will be submitted for publication elsewhere.

Ring Theory, Representation Theory	*Theory of Operator algebras*
Algebra	Functional analysis
Finite dimensional alg. are improtant	Finite dimensional alg. are trivial
Infinite dimensional alg. are interesting	Infinite dimensional alg. are essential
vector space	Hilbert space (need inner product)
ring	*-algebra
non-commutative	non-commutative
over a field K	over $\mathbb{C}$
no measure theory	need measure theory
category is an essential tool	category is a useful language
combinatrics	topotopological approximation
?	positivity
charateristic p	?
polynomial ring	algebra of continuous functions
path algebra	Cuntz-Krieger algebra

(3)Hilbert representations of quivers [2], [3]

A Hilbert representation of a quiver is to associate Hilbert spaces for the vertices and bounded operators for arrows. Jordan blocks correspond to strongly irreducible operators. The invaraint subspace problem is one of the famous unsolved problems in functional anaysis and rephrased by the existence of a simple Hilbert representation of a quiver.

We study operator algebras instead of finite dimensional algebras. We have two important classes of operator algebras, that is, C^*-algebras and von Neumann algebras. A C^*-algebra is a *-subalgebra of the algebra $B(H)$ of bounded operators on a Hilbert space H closed under operator-norm-topology. A von Neumann algebra is a *-subalgebra of $B(H)$ closed under weak-operator-topology. C^*-algebras are regarded as quantized (locally) compact Housdorff spaces. Von Neumann algebras are regarded as quantized measure spaces.

We can associated C^*-algebras for topological dynamical systems and von Neumann algebras for measurable dynamical systems. In the half of this note, we will show our study on C^*-algebras associated complex dynamical systems ([6]) and self-similar dynamical systems ([7]) and on Hilbert representations. These results are based on joint works with Tsuyoshi Kajiwara and Masatoshi Enomoto.

In order to "feel" the difference between purely algebraic setting and functional analytic setting, let us consider the following typical examples: Let L_1 be one-loop quiver, that is, L_1 is a quiver with one vertex $\{1\}$ and 1-loop $\{\alpha\}$ such that $s(\alpha) = r(\alpha) = 1$. Consider two infinite-dimensional spaces the polynomial ring $\mathbb{C}[z]$ and the Hardy space $H^2(\mathbb{T})$. Then $\mathbb{C}[z]$ is dense in $H^2(\mathbb{T})$ with respect to the Hilbert space norm.

Define a purely algebraic representation (V, T) of L_1 by $V_1 = \mathbb{C}[z]$ and the multiplication operator T_α by z. That is, $T_\alpha h(z) = zh(z)$ for a polynomial $h(z) = \sum_n a_n z^n$. Since

$End(V,T) \cong \mathbb{C}[z]$ have no idempotents, the purely algebraic representation (V,T) is indecomposable.

Next we define a Hilbert representation (H,S) by $H_1 = H^2(\mathbb{T})$ and $S_\alpha = T_z$ the Toeplitz operator with the symbol z. Then $S_\alpha = T_z$ is the multiplication operator by z on $H^2(\mathbb{T})$ and is identified with the unilateral shift. Then

$$\begin{aligned} End(H,S) &\cong \{A \in B(H^2(\mathbb{T})) \mid AT_z = T_z A\} \\ &= \{T_\phi \in B(H^2(\mathbb{T})) \mid \phi \in H^\infty(\mathbb{T})\} \end{aligned}$$

is the algebra of analytic Toeplitz operators and isomorphic to $H^\infty(\mathbb{T})$. By the F. and M. Riesz Theorem, if $f \in H^2(\mathbb{T})$ has the zero set of positive measure, then $f = 0$. Since $H^\infty(\mathbb{T}) = H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, $H^\infty(\mathbb{T})$ has no non-trivial idempotents. Thus there exists no non-trivial idempotents which commutes with T_z and Hilbert space (H,S) is indecomposable. In this sense, the analytical aspect of Hardy space is quite important in our setting.

Any subrepresentation of the purely algebraic representation (V,T) is given by the restriction to an ideal $J = p(z)\mathbb{C}[z]$ for some polynomial $p(z)$. Any subrepresentation of the Hilbert representation (H,S) is given by an invariant subspace of the shift operator T_z. Beurling theorem shows that any subrepresentation of (H,S) is given by the restriction to an invariant subspace $M = \varphi H^2(\mathbb{T})$ for some inner function φ. For example, if an ideal J is defined by

$$J = \{f(z) \in \mathbb{C}[z] \mid f(\lambda_1) = \cdots = f(\lambda_n) = 0\}$$

for some distinct numbers $\lambda_1, \dots \lambda_n \in \mathbb{C}$, then the corresponding polynomial $p(z)$ is given by $p(z) = (z-\lambda_1)\dots(z-\lambda_n)$. The case of Hardy space is much more analytic. We shall identify $H^2(\mathbb{T})$ with a subspace $H^2(\mathbb{D})$ of analytic functions on the open unit disc $\mathbb{D}$. If an invariant subspace M is defined by

$$M = \{f \in H^2(\mathbb{D}) \mid \ f(\lambda_1) = \cdots = f(\lambda_n) = 0\}$$

for some distinct numbers $\lambda_1, \dots \lambda_n \in \mathbb{D}$, then the corresponding inner function φ is given by a finite Blaschke product

$$\varphi(z) = \frac{(z-\lambda_1)}{1-\overline{\lambda_1}z} \cdots \frac{(z-\lambda_n)}{1-\overline{\lambda_n}z}.$$

Here we cannot use the notion of degree like polynomials and we must manage to treat orthogonality to find such an inner function φ.

2. path algebras and Cuntz-Krieger algebras

The elements of a path algebra of a quiver are finite linear sums of paths in the quiver. Similarly the elements of a Cuntz-Krieger algebra of a quiver are generated by partial isometry operators representating paths. The difference is that the ranges of generating partial isometries are orthogonal and we add the adjoint T^* of any operator T in the Cuntz-Krieger algebra. But usually the Cuntz-Krieger algebra is described using the 0-1 matrix A corresponding the quiver as follows: The Cuntz-Krieger algebra $\mathcal{O}_A$ is the

symbolic dynamical system	complex dynamical system
quiver (or 0-1 matrix A)	rational function R
irreducible matrix	restriction of R on the Julia set J_R
Cantor set	closed subset of Riemann sphere
one-sided Markov shift	branched covering map
Cuntz-Krieger algebra $\mathcal{O}_A$	Cuntz-Pimsner algebra $\mathcal{O}_R(J_R)$
maximal abelian subalgebra $C(X_A)$	maximal abelian subalgebra $C(J_R)$
étale groupoid	not étale groupoid in general
K-group is a good invariant	K-group is not a good invariant

universal algebra generated by partial isometries $S_1, S_2, \ldots, S_n$ with orthogonal ranges satisfying that

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^* \quad \text{and} \quad \sum_{j=1}^{n} S_j S_j^* = I$$

Theorem 1. (Cuntz-Krieger) Let A be an irreducible $0-1$ matrix which is not a permutation. Then the correspoding Cuntz-Krieger algebra $\mathcal{O}_A$ is simple, purely infinite, nuclear C^*-algebra. Furthermore the K-groups are the following:

$$K_0(\mathcal{O}_A) = \mathbb{Z}^n/(I - A^t)\mathbb{Z}^n \quad K_1(\mathcal{O}_A) = Ker\ (I - A^t) \subset \mathbb{Z}^n$$

It is known that the K-groups are complete invarint of a certain class of simple nuclear C^*-algebras containing Cuntz-Krieger algebras.

3. C^*-ALGEBRAS ASSOCIATED WITH COMPLEX DYNAMICAL SYSTEMS

We can regard Cuntz-Krieger algebras are C^*-algebraic version of path algebras for quivers. But we usually consider that Cuntz-Krieger algebras are associated with certain symbolic dynamical systems ,i.e. Markov shifts. Similarly many C^*-algebras are constructed from dynamical systems.

Let R be a rational function of the form $R(z) = \frac{P(z)}{Q(z)}$ with relatively prime polynomials P and Q. The degree of R is denoted by $N = \deg R := \max\{\deg P, \deg Q\}$.

We regard a rational function R as a N-fold branched covering map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

The sequence $(R^n)_n$ of iterations of R gives a complex analytic dynamical system on $\hat{\mathbb{C}}$. The Fatou set F_R of R is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R^n)_n$ is equicontinuous (or a normal family), and the Julia set J_R of R is the complement of the Fatou set in $\hat{\mathbb{C}}$. The Fatou set F_R is a stable part and the Julia set J_R is an unstable part. Since the Riemann sphere $\hat{\mathbb{C}}$ is decomposed into the union of the Julia set J_R and Fatou set F_R, we associated three C^*-algebras $\mathcal{O}_R = \mathcal{O}_R(\hat{\mathbb{C}})$, $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$ by considering R as dynamical systems on $\hat{\mathbb{C}}$, J_R and F_R respectively.

Theorem 2. (Kajiwara-Watatani) Let R be a rational function with $\deg R \geq 2$. Then the C^*-algebra $\mathcal{O}_R(J_R)$ associated with R on the Julia set J_R is simple and purely infinite.

Theorem 3. (Kajiwara-Watatani) Let R be a rational function with $\deg R \geq 2$. Then the following are equivalent:
(1)The core $\mathcal{O}_R(J_R)^\gamma$ is simple.
(2)The Julia set J_R contains no branched points i.e. $J_R \cap B_R = \emptyset$.

4. Hilbert representation of quivers

Let $\Gamma = (V, E, s, r)$ be a finite quiver. We say that (H, f) is a *Hilbert representation* of Γ if $H = (H_v)_{v \in V}$ is a family of Hilbert spaces and $f = (f_\alpha)_{\alpha \in E}$ is a family of bounded linear operators such that $f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)}$ for $\alpha \in E$.

Hilbert representation (H, f) of Γ is called *decomposable* if (H, f) is isomorphic to a direct sum of two non-zero Hilbert representations. A non-zero Hilbert representation (H, f) of Γ is said to be *indecomposable* if it is not decomposable, that is, if $(H, f) \cong (K, g) \oplus (K', g')$ then $(K, g) \cong 0$ or $(K', g') \cong 0$.

A Hilbert representation (H, f) of a quiver Γ is called *transitive* if $End(H, f) = \mathbb{C}I$. It is clear that if a Hilbert representation (H, f) is canonically simple, then (H, f) is transitive. If a Hilbert representation (H, f) of Γ is transitive , then (H, f) is indecomposable. In fact, since $End(H, f) = \mathbb{C}I$, any idempotent endomorphism T is 0 or I. In purely algebraic setting, a representation of a quiver is called a *brick* if its endomorphism ring is a division ring. But for a Hilbert representation (H, f) of a quiver, $End(H, f)$ is a Banach algebra and not isomorphic to its purely algebraic endomorphism ring in general, because we only consider bounded endomorphisms. By Gelfand-Mazur theorem, any Banach algebra over $\mathbb{C}$ which is a division ring must be isomorphic to $\mathbb{C}$. Therefore the reader may use "brick" instead of "transitive Hilbert representation" if he does not confuse the difference between purely algebraic endomorphism ring and $End(H, f)$.

A lattice $\mathcal{L}$ of subspaces of a Hilbert space H containing 0 and H is called a transitive lattice if

$$\{A \in B(H) \mid AM \subset M \text{ for any } M \in \mathcal{L}\} = \mathbb{C}I.$$

Let $\mathcal{L} = \{0, M_1, M_2, \ldots, M_n, H\}$ be a finite lattice. Consider a n subspace quiver $R_n = (V, E, s, r)$, that is, $V = \{1, 2, \ldots, n, n+1\}$ and $E = \{\alpha_k \mid k = 1, \ldots, n\}$ with $s(\alpha_k) = k$ and $r(\alpha_k) = n + 1$ for $k = 1, \ldots, n$. Then there exists a Hilbert representation (K, f) of R_n such that $K_k = M_k$, $K_{n+1} = H$ and $f_{\alpha_k} : M_k \to H$ is an inclusion for $k = 1, \ldots, n$. Then the lattice $\mathcal{L}$ is transitive if and only if the corresponding Hilbert representation (H, f) is transitive. This fact guarantees the terminology "transitive" in the above.

Theorem 4. (Enomoto-Watatani) Let Γ be a quiver whose underlying undirected graph is an extended Dynkin diagram. Then there exists an infinite-dimensional transitive Hilbert representation of Γ if and only if Γ is not an oriented cyclic quiver.

A non-zero Hilbert representation (H, f) of a quiver Γ is called *simple* if (H, f) has only trivial subrepresentations 0 and (H, f). A Hilbert representation (H, f) of Γ is called *canonically simple* if there exists a vertex $v_0 \in V$ such that $H_{v_0} = \mathbb{C}$, $H_v = 0$ for any other vertex $v \neq v_0$ and $f_\alpha = 0$ for any $\alpha \in E$. It is clear that if a Hilbert representation (H, f) of Γ is canonically simple, then (H, f) is simple. It is trivial that if a Hilbert representation (H, f) of Γ is simple, then (H, f) is indecomposable.

Operator theory	*Representation of quivers*
Hilbert space	vertex
bounded operator	edge
direct sum	direct sum
irreducible operator	irreducible representation
strongly irreducible operator T	indecomposable representation
commutant $\{T\}'$	endomorphism ring
up to similar	up to isomorphism
Fredholm index	defect
no non-trivial invariant subspace	simple
transitive operator	transitive representation

We can rephrase the invariant subspace problem in functional analysis in terms of simple representations of a one-loop quiver. Let L_1 be one-loop quiver, so that L_1 has one vertex 1 and one arrow $\alpha : 1 \to 1$. Any bounded operator A on a non-zero Hilbert space H gives a Hilbert representation (H^A, f^A) of L_1 such that $H_1^A = H$ and $f_\alpha^A = A$. Then the operator A has only trivial invariant subspaces if and only if the Hilbert representation (H^A, f^A) of L_1 is simple. If H is one-dimensional and A is a non-zero scalar operator, then the Hilbert representation (H^A, f^A) of L_1 is simple but is not canonically simple. If H is finite-dimensional with $\dim H \geq 2$, then the Hilbert representation (H^A, f^A) of L_1 is not simple, because any operator A on H has a non-trivial invariant subspace. If H is countably infinite-dimensional, then we do not know whether the Hilbert representation (H^A, f^A) of L_1 is not simple. In fact this is the invariant subspace problem, that is, the question whether any operator A on H has a non-trivial (closed) invariant subspace.

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ON WEAKLY SEPARABLE EXTENSIONS AND WEAKLY QUASI-SEPARABLE EXTENSIONS

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Abstract. Separable extensions of noncommutative rings have already been studied extensively. Recently, N. Hamaguchi and A. Nakajima introduced the notions of weakly separable extensions and weakly quasi-separable extensions. The purpose of this paper is to give some improvements and generalizations of Hamaguchi and Nakajima's results. We shall characterize weakly separable polynomials, and we shall show the difference between the separability and the weakly separability in skew polynomial rings.

1. Introduction

Throughout this paper, A/B will represent a ring extension with common identity 1. Let M be an A-A-bimodule, and x, y arbitrary elements in A. An additive map δ is said to be a *B-derivation* of A to M if $\delta(xy) = \delta(x)y + x\delta(y)$ and $\delta(\alpha) = 0$ for any $\alpha \in B$. Moreover, δ is called *central* if $\delta(x)y = y\delta(x)$, and δ is called *inner* if $\delta(x) = mx - xm$ for some fixed element $m \in M$. We say that a ring extension A/B is *separable* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $a \otimes b \mapsto ab$ splits. It is well known that A/B is separable if and only if for any A-A-bimodule M, every B-derivation of A to M is inner (cf. [1, Satz 4.2]). In [12], Y. Nakai introduced the notion of a quasi-separable extension of commutative rings by using the module differentials, and in the noncommutative case, it was characterized by H. Komatsu [8, Lemma 2.1] as follows : A/B is *quasi-separable* if and only if for any A-A-bimodule M, every central B-derivation of A to M is zero. Recently, N. Hamaguchi and A. Nakajima gave the following definitions as generalizations of separable extensions and quasi-separable extensions.

Definition 1. ([2, Definition 2.1]) (1) A/B is called *weakly separable* if every B-derivation of A to A is inner.

(2) A/B is called *weakly quasi-separable* if every central B-derivation of A to A is zero.

Obviously, a separable extension is weakly separable and a quasi-separable extension is weakly quasi-separable. Moreover, a separable extension is quasi-separable by [8, Theorem 2.4].

Let B be a ring, ρ an automorphism of B, D a ρ-derivation, that is, D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ for any $\alpha, \beta \in B$. $B[X; \rho, D]$ will mean the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We write $B[X; \rho] = B[X; \rho, 0]$ and $B[X; D] = B[X; 1, D]$. By $B[X; \rho, D]_{(0)}$ we denote the set of all monic polynomials g in $B[X; \rho, D]$ such that $gB[X; \rho, D] = B[X; \rho, D]g$. Let f be in $B[X; \rho, D]_{(0)}$. Then the residue ring $B[X; \rho, D]/fB[X; \rho, D]$ is a

The detailed version of this paper [15] will be published.

free ring extension of B. We say that f is a *separable* (resp. *weakly separable*, *weakly quasi-separable*) *polynomial* in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable, weakly quasi-separable) over B.

In section 2, we treat weakly separable polynomials over a commutative ring B. In [10, Theorem 2.3], T. Nagahara showed that a separable polynomial $f(X)$ in $B[X]$ has a close relationship with the invertibilities of its derivative $f'(X)$ and its discriminant $\delta(f(X))$. We shall characterize the weakly separability of $f(X)$ in terms of the properties of $f'(X)$ and $\delta(f(X))$.

In section 3, we study weakly separable polynomials and weakly quasi-separable polynomials in skew polynomial rings. When B is an integral domain, N. Hamaguchi and A. Nakajima gave necessary and sufficient conditions for weakly separable polynomials in $B[X;\rho]$ and $B[X;D]$ (cf. [2, Theorem 4.1.4 and Theorem 4.2.3]). We shall try to give sharpenings of their results for a noncommutative coefficient ring B. Moreover, we shall show the difference between the separability and the weakly separability in skew polynomial rings $B[X;\rho]$ and $B[X;D]$, respectively.

2. Weakly separable polynomials over a commutative ring

In this section, we shall study weakly separable polynomials over a commutative ring. It is well known that a (noncommutative) ring extension A/B is separable if and only if there exists $\sum_j x_j \otimes y_j \in (A\otimes_B A)^A$ such that $\sum_j x_j y_j = 1$, where $(A\otimes_B A)^A = \{\mu \in A\otimes_B A \,|\, x\mu = \mu x \text{ for any } x \in A\}$ (cf. [3, Definition 2]). First we shall state the following.

Lemma 2. ([15, Lemma 2.1]) *Let A/B be a commutative ring extension. If there exists $\sum_j x_j \otimes y_j \in (A\otimes_B A)^A$ such that $\sum_j x_j y_j$ is a non-zero-divisor in A, then A/B is weakly separable.*

Now, let B be a commutative ring. For a monic polynomial $f(X) \in B[X]$, $f'(X)$ and $\delta(f(X))$ will mean the derivative of $f(X)$ and the discriminant of $f(X)$, respectively. As was shown in [10, Theorem 2.3], a polynomial $f(X)$ in $B[X]$ is separable if and only if $\delta(f(X))$ is invertible in B, or equivalently, $f'(X)$ is invertible in $B[X]$ modulo $(f(X))$. In [2], N. Hamaguchi and A. Nakajima proved that $f(X) = X^m - Xa - b$ is weakly separable in $B[X]$ if and only if $\delta(f(X))$ is a non-zero-divisor in B, or equivalently, $f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$ (cf [2, Theorem 3.1 and Corollary 3.2]). For a monic polynomial $f(X)$, we have the following.

Theorem 3. ([15, Theorem 2.2]) *Let B be a commutative ring, and $f(X)$ a monic polynomial in $B[X]$. The following are equivalent.*
(1) *$f(X)$ is weakly separable in $B[X]$.*
(2) *$f'(X)$ is a non-zero-divisor in $B[X]$ modulo $(f(X))$.*
(3) *$\delta(f(X))$ is a non-zero-divisor in B.*

Remark 4. In Theorem 3, it is already known that (2) and (3) are equivalent by [10, Theorem 1.3].

3. Weakly separable polynomials in skew polynomial rings

In [2], N. Hamaguchi and A. Nakajima characterized weakly separable polynomials and weakly quasi-separable polynomials in skew polynomial rings $B[X;\rho]$ and $B[X;D]$ when B is an integral domain. In this section, we shall generalize their results for a noncommutative coefficient ring B.

We shall use the following conventions :

$Z =$ the center of B

$V_A(B) =$ the centralizer of B in A

$B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$

$B^D = \{\alpha \in B \mid D(\alpha) = 0\}$ and $Z^D = Z \cap B^D$

$D(B) = \{D(\alpha) \mid \alpha \in B\}$

3.1. **Automorphism type.** We consider a polynomial f in $B[X;\rho]_{(0)}$ of the form

$$f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0 = \sum_{j=0}^{m} X^j a_j \quad (a_m = 1,\ m \geq 2).$$

By [4, Lemma 1.3], f is in $B[X;\rho]_{(0)}$ if and only if

$$\begin{cases} \alpha a_j = a_j \rho^{m-j}(\alpha) \quad (\alpha \in B,\ 0 \leq j \leq m-1), \\ \rho(a_j) - a_j = a_{j+1}(\rho(a_{m-1}) - a_{m-1}) \ (0 \leq j \leq m-2), \\ a_0(\rho(a_{m-1}) - a_{m-1}) = 0. \end{cases}$$

We set $A = B[X;\rho]/fB[X;\rho]$, and $x = X + fB[X;\rho] \in A$. Now we let f be in $B[X;\rho]_{(0)} \cap B^\rho[X]$. Then there is an automorphism $\tilde\rho$ of A which is naturally induced by ρ, that is, $\tilde\rho$ is defined by $\tilde\rho(\sum_{j=0}^{m-1} x^j c_j) = \sum_{j=0}^{m-1} x^j \rho(c_j)$. We write $J_{\rho^k} = \{h \in A \mid \alpha h = h\rho^k(\alpha)\ (\alpha \in B)\}$ $(k \geq 1)$, $V = V_A(B)$, and $V^{\tilde\rho} = \{h \in V \mid \tilde\rho(h) = h\}$. Then we consider a $V^{\tilde\rho}$-$V^{\tilde\rho}$-homomorphism $\tau : J_\rho \longrightarrow J_{\rho^m}$ defined by

$$\begin{aligned} \tau(h) &= x^{m-1}\sum_{j=0}^{m-1}\tilde\rho^j(h) + x^{m-2}\sum_{j=0}^{m-2}\tilde\rho^j(h)a_{m-1} + \cdots + x\{\tilde\rho(h) + h\}a_2 + ha_1 \\ &= \sum_{k=0}^{m-1} x^k \sum_{j=0}^{k} \tilde\rho^j(h) a_{k+1}. \end{aligned}$$

First we shall state the following.

Lemma 5. ([15, Lemma 3.1]) *If δ is a B-derivation of A, then $\delta(x) \in J_\rho$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in J_\rho$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.*

Now we shall give the following which is a sharpening of [2, Theorem 4.1.4].

Theorem 6. ([15, Theorem 3.2]) *Let $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ be in $B[X;\rho]_{(0)} \cap B^\rho[X]$. Then f is weakly separable in $B[X;\rho]$ if and only if*

$$\{g \in J_\rho \mid \tau(g) = 0\} = \{x(\tilde\rho(h) - h) \mid h \in V\}.$$

In virtue of Theorem 6, we shall show the difference between the separability and the weakly separability in $B[X;\rho]$ as follows.

Theorem 7. ([15, Theorem 3.4]) *Let m be the order of ρ, $f = X^m + X^{m-1}a_{m-1} + \cdots + Xa_1 + a_0$ in $B[X;\rho]_{(0)} \cap B^\rho[X]$, $C(A)$ a center of A, and I_x an inner derivation of A by x (that is, $I_x(h) = hx - xh$ for any $h \in A$).*

(1) *f is weakly separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde\rho}$-$V^{\tilde\rho}$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \xrightarrow{\text{inj}} V \xrightarrow{I_x} J_\rho \xrightarrow{\tau} V^{\tilde\rho}.$$

(2) *f is separable in $B[X;\rho]$ if and only if the following sequence of $V^{\tilde\rho}$-$V^{\tilde\rho}$-homomorphisms is exact:*

$$0 \longrightarrow C(A) \xrightarrow{\text{inj}} V \xrightarrow{I_x} J_\rho \xrightarrow{\tau} V^{\tilde\rho} \longrightarrow 0.$$

Remark 8. In this section, we assumed that f is in $B[X;\rho]_{(0)} \cap B^\rho[X]$. However, in general case, a polynomial which is in $B[X;\rho]_{(0)}$ is not always in $B^\rho[X]$. Concerning this, we have already known by [4, Corollary 1.5] that if B is a semiprime ring then every polynomial in $B[X;\rho]_{(0)}$ is in $C(B^\rho)[X]$, where $C(B^\rho)$ is the center of B^ρ.

At the end of this section, we shall mention briefly on weakly quasi-separable polynomials in $B[X;\rho]$. When B is an integral domain, N. Hamaguchi and A. Nakajima proved that every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable (cf. [2, Theorem 4.1.1]). For an arbitrary ring B, we have the following.

Proposition 9. ([15, Proposition 3.5]) (1) *If $\rho \neq 1$ and $\{\rho(c) - c \,|\, c \in B\}$ contains a non-zero divisor, then every polynomial in $B[X;\rho]_{(0)}$ is weakly quasi-separable.*

(2) *Let $f = X^m - u$ be in $B[X;\rho]_{(0)}$. If m and u are non-zero-divisors in B, then f is weakly quasi-separable in $B[X;\rho]$.*

3.2. **Derivation type.** In this section, let B be of prime characteristic p, and we consider a p-polynomial f in $B[X;D]_{(0)}$ of the form

$$f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0 = \sum_{j=0}^{e} X^{p^j} b_{j+1} + b_0 \ (b_{e+1} = 1).$$

We set $A = B[X;D]/fB[X;D]$, and $x = X + fB[X;D] \in A$. By [4, Corollary 1.7], f is in $B[X;D]_{(0)}$ if and only if

$$\begin{cases} b_0 \in B^D, \quad b_{j+1} \in Z^D \ (0 \leq j \leq e-1), \\ \displaystyle\sum_{j=0}^{e} D^{p^j}(\alpha) b_{j+1} = b_0\alpha - \alpha b_0 \quad (\alpha \in B). \end{cases}$$

Since f is in $B^D[X]$, there is a derivation $\tilde D$ of A which is naturally induced by D, that is, $\tilde D$ is defined by $\tilde D(\sum_{j=0}^{p^e-1} x^j c_j) = \sum_{j=0}^{p^e-1} x^j D(c_j)$. We write $V = V_A(B)$, $\tilde D(V) = \{\tilde D(h) \,|\, h \in$

$V\}$, and $V^{\tilde{D}} = \{v \in V \mid \tilde{D}(v) = 0\}$. Then we consider a $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphism $\tau : V \longrightarrow V^{\tilde{D}}$ defined by

$$\begin{aligned}\tau(h) &= \tilde{D}^{p^e-1}(h) + \tilde{D}^{p^{e-1}-1}(h)b_e + \cdots + \tilde{D}^{p-1}(h)b_2 + hb_1 \\ &= \sum_{j=0}^{e} \tilde{D}^{p^j-1}(h)b_{j+1}.\end{aligned}$$

First we shall state the following.

Lemma 10. ([15, Lemma 3.7]) *If δ is a B-derivation of A, then $\delta(x) \in V$ and $\tau(\delta(x)) = 0$. Conversely, if $g \in V$ with $\tau(g) = 0$, then there exists a B-derivation δ of A such that $\delta(x) = g$.*

Now we shall give a sharpening of [2, Theorem 4.2.3]

Theorem 11. ([15, Theorem 3.8]) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$. Then f is weakly separable in $B[X;D]$ if and only if*

$$\{g \in V \mid \tau(g) = 0\} = \tilde{D}(V).$$

In virtue of Theorem 11, we shall show the difference between the separability and the weakly separability in $B[X;D]$ as follows.

Theorem 12. ([15, Theorem 3.10]) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$.*

(1) *f is weakly separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:*

$$0 \longrightarrow V^{\tilde{D}} \xrightarrow{\text{inj}} V \xrightarrow{\tilde{D}} V \xrightarrow{\tau} V^{\tilde{D}}.$$

(2) *f is separable in $B[X;D]$ if and only if the following sequence of $V^{\tilde{D}}$-$V^{\tilde{D}}$-homomorphisms is exact:*

$$0 \longrightarrow V^{\tilde{D}} \xrightarrow{\text{inj}} V \xrightarrow{\tilde{D}} V \xrightarrow{\tau} V^{\tilde{D}} \longrightarrow 0.$$

Remark 13. Also when B is arbitrary ring and f is a monic polynomial in $B[X;D]_{(0)}$, Theorem 11 is true. However, we can not prove yet Theorem 12 in the case.

Finally, we shall mention briefly on weakly quasi-separable polynomials in $B[X;D]$. As same as automorphism type, N. Hamaguchi and A. Nakajima proved that every polynomial in $B[X;D]_{(0)}$ is weakly quasi-parable when B is an integral domain (cf. [2, Theorem 4.2.1]). For an arbitrary ring B, we have the following.

Proposition 14. ([15, Proposition 3.11]) (1) *If $D(B)$ contains a non-zero-divisor, then every polynomial in $B[X;D]_{(0)}$ is weakly quasi-seprable.*

(2) *Let $f = X^{p^e} + X^{p^{e-1}}b_e + \cdots + X^p b_2 + Xb_1 + b_0$ be in $B[X;D]_{(0)}$. If b_1 is a non-zero-divisor in B, then f is wakly quasi-separable.*

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ALMOST GORENSTEIN REES ALGEBRAS

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Abstract. Let A be a Cohen-Macaulay local ring, and let $I \subset A$ be an $\mathfrak{m}$-primary ideal. Let $\mathcal{R} = R(I)$ be the Rees algebra of I and $\mathfrak{M}$ the unique graded maximal ideal of $\mathcal{R}$. We ask the following question: When is the Rees algebra $\mathcal{R}$ (resp. $\mathcal{R}_{\mathfrak{M}}$) an almost Gorenstein graded ring (resp. local ring)?

We give several answers to the questions as above in the case of parameter ideals, p_g-ideals, and socle ideals.

Key Words: Commutative Algebra, almost Gorenstein local/graded ring, Rees algebra

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1. Introduction

この講演を通して, A は可換ネーター局所環とし, $\mathfrak{m}$ をその極大イデアルとする. 有限生成 A-加群 M に対して, $\ell_A(M)$, $\mu_A(M)$, $\dim_A M$ はそれぞれ M の**長さ (the length)**, **極小生成系の個数 (the minimal number of generators)**, **次元 (dimension)** を表す. $\mathfrak{m}$-準素イデアル I に対して,

$$e_I(M) = \lim_{n\to\infty} \frac{(\dim M)!}{n^{\dim M}} \cdot \ell_A(M/I^{n+1}M)$$

とおき, M の I に関する**重複度 (multiplicity)** と言う.

可換環における基本的なクラスを思いだしておこう. $\mathrm{emb}(A) := \mu_A(\mathfrak{m})$ を A の埋入次元と言う. このとき, 一般に $\mu_A(\mathfrak{m}) \geq \dim A$ が成り立つが, 等号が成立するとき, A は**正則局所環**であるという.

パラメーターイデアル Q に対して, $e_Q^0(A) \leq \ell_A(A/Q)$ が成り立つが, 等号が成立するとき, A は **Cohen-Macaulay 環**であると言う. 同様に, Cohen-Macaulay A-加群も定義される.

K_A を A の**標準加群 (canonical module)** とするとき, A が Cohen-Macaulay 環で, A-加群として $A \cong K_A$ であるとき, A は **Gorenstein 環**であると言う. A が Cohen-Macaulay 局所環のとき, Gorenstein 環の準同型像であることと, 標準加群を持つこととは同値であることが知られている.

本講演のタイトルにもある almost Gorenstein 性の概念を局所環, 次数付き環それぞれの場合に対して定義する.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 1 (**Almost Gorenstein local ring** [5]). A を標準加群 K_A を持つ Cohen-Macaulay 局所環とする. 短完全列

$$0 \to A \xrightarrow{\varphi} K_A \to C \to 0 \quad (\text{ただし}, \mu_A(C) = e^0_{\mathfrak{m}}(C))$$

が存在するとき, A は **almost Gorenstein 局所環 (local ring)** であると言う.

上の定義において, $C = 0$ とした場合が Gorenstein 環である. 従って, Gorenstein 環は常に almost Gorenstein 環である.

また, 上の定義において $C \neq 0$ であるとき, C は $(d-1)$-次元の **Ulrich A-加群**, すなわち, $\mu_A(C) = e^0_{\mathfrak{m}}(C)$ をみたす Cohen-Macaulay A-加群である ([5, Lemma 3.1]). ここで, $d = \dim A$ である.

$R = \bigoplus_{n \geq 0} R_n$ を $R_0 = A$ 上の次数付き環とし, そのただ一つの斉次極大イデアルを $\mathfrak{M} = \mathfrak{m}R + R_+$ とする. $a = a(R) = -\min\{n \in \mathbb{Z} \mid [K_R]_n \neq 0\}$ を R の a-invariant と言う. また, 次数付き R-加群 M と $k \in \mathbb{Z}$ に対して, $M(k)$ は R-加群としては M と一致し, 次数を $[M(k)]_n = M_{k+n}$ と定めた次数付き R 加群を表す.

Definition 2 (**Almost Gorenstein graded ring** [5]). $R = \bigoplus_{n \geq 0} R_n$ を標準加群 K_R を持つ Cohen-Macaulay 次数付き環とする. もし, 短完全列

$$0 \to R \xrightarrow{\varphi} K_R(-a) \to C \to 0 \quad (\text{ただし}, \mu_{R_{\mathfrak{M}}}(C_{\mathfrak{M}}) = e^0_{\mathfrak{M}R_{\mathfrak{M}}}(C_{\mathfrak{M}}))$$

が存在するならば, A は **almost Gorenstein 次数付き環 (graded ring)** であると言う.

上の状況において, 次数付き環 R に対して, $(K_R)_{\mathfrak{M}} \cong K_{R_{\mathfrak{M}}}$ であることに注意すると, 次の主張が成り立つ.

Proposition 3. *R が almost Gorenstein 次数付き環であるならば, $R_{\mathfrak{M}}$ は almost Gorenstein 局所環である.*

同様の主張は, Cohen-Macaulay 性, Gorenstein 性についても言える. これらの性質については逆が正しいにも関わらず, **almost Gorenstein 性については逆は成り立たない**.

Almost Gorenstein 局所/次数付き環の例をあげておこう. $\dim A = 0$ (すなわち, A がアルチン環) のとき, A が almost Gorenstein 局所環であることと, A が Gorenstein 環であることとは同値である ([5, Lemma 3.1]).

$\dim A = 1$ の例として, 数値的半群 H に付随する数値的半群環

$$K[[H]] = \{t^h \mid h \in H\} \big(\subset K[[t]] \big)$$

を考えると, H が概対称性 (almost symmetric) であることと, $K[[H]]$ が almost Gorenstein 局所環であることとは同値である ([1], [3]). 例えば, $H = \langle 3, a, b \rangle$ $(3 < a < b, \gcd(3, a, b) = 1)$ を考えると, $a < b \leq 2a - 3$ であり, H が概対称性を持つことと, $b = 2a - 3$ であることとは同値である.

2 次元以下の Cohen-Macaulay 局所環が, Cohen-Macaulay 有限表現型を持つならば, almost Gorenstein 性を持つ (cf. [5, Section 12]). 高次元の場合でも今までに知られている Cohen-Macaulay 有限表現型を持つ Cohen-Macaulay 局所環はすべて almost Gorenstein であることが確認されているが, 概念的な証明は知られていない.

2 次元有理特異点を持つ局所環は almost Gorenstein である ([5, Section 11]). 有理特異点と almost Gorenstein 性との関係は興味深い.

一方, 高次元の almost Gorenstein 性の研究は発展途上であり, 東谷氏による斉次代数, 特に日比環に関する研究成果と, Stanley-Reisner 環に関する村井氏・松岡氏の共同研究の成果以外はあまり知られていないようである.

次の結果は, almost Gorenstein 局所環から almost Gorenstein 次数付き環を構成する方法として有用である.

Theorem 4 ([5, Theorem 11.1]). *A を $\mathrm{emb}(A) = e^0_{\mathfrak{m}}(A) + \dim A - 1$ をみたす almost Gorenstein 局所環とするとき,*

$$G = \mathrm{gr}_{\mathfrak{m}}(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

は almost Gorenstein 次数付き環である.

本講演における問題を明示するために, Rees 環の定義を思い出しておこう.

Definition 5. $(0) \neq I \subset A$ をネーター局所整域のイデアルとする. 次数付き環

$$\mathcal{R} = R(I) = \sum_{n \geq 0} I^n t^n \subset A[t]$$

をイデアル I の **Rees 代数**と言う.

$\mathcal{R} = R(I)$ に対して,

$$\dim \mathcal{R} = d + 1, \qquad a(\mathcal{R}) = -1$$

が成り立つ.

本講演における主問題をあげておこう.

Question. A を Cohen-Macaulay 局所整域とし, $I \subset A$ をイデアルとする. $\mathcal{R} = R(I)$ を I の Rees 代数, $\mathfrak{M} = \mathfrak{m}\mathcal{R} + \mathcal{R}_+$ を $\mathcal{R}$ の斉次極大イデアルとする. このとき,

(1) いつ $\mathcal{R}$ は almost Gorenstein 次数付き環か？

(2) いつ $\mathcal{R}_{\mathfrak{M}}$ は almost Gorenstein 局所環か？

2. Parameter ideals

この節を通して, $(A, \mathfrak{m})$ を d-次元の Cohen-Macaulay 局所環とする. $A/(a_1, \ldots, a_d)$ の長さが有限であるとき, A の元の列 $a_1, a_2, \ldots, a_d$ を A の **s.o.p.(パラメーター系)** と言う. また, $0 \leq r \leq d$ に対して, $a_1, a_2, \ldots, a_r$ が A の s.o.p. の一部であるとき, **s.s.o.p.** であると言う. 次の結果は良く知られている.

Lemma 6. *A の s.s.o.p. $a_1, a_2, \ldots, a_r$ に対して, $Q = (a_1, \ldots, a_r)A$ とおく. $r \geq 2$ のとき,*

(1) $\mathcal{R}(Q)$ は Cohen-Macaulay である.

(2) $\mathcal{R}(Q)$ が Gorenstein であるのは, A が Gorenstein で, $r = 2$ の場合に限る ([4]).

最初に, $r = 2$ の場合の結果を述べておこう. この場合は, 非常にシンプルな結果である.

Proposition 7 ([6]). *$Q = (a_1, a_2)$ を s.s.o.p. で生成されたイデアルとするとき, 次の条件は同値である :*

(1) A は Gorenstein である.

(2) $\mathcal{R}(Q)$ は *Gorenstein* である.
(3) $\mathcal{R}(Q)_{\mathfrak{M}}$ は *almost Gorenstein* 局所環である.
(4) $\mathcal{R}(Q)$ は *almost Gorenstein* 次数付き環である.

上の状況において, $\mathcal{R}(Q) \cong A[T_1, T_2]/(a_2T_1 - a_1T_2)$ が成り立つ.

次の定理は高次元の almost Gorenstein 局所環/次数付き環の例を提供する. A の Gorenstein 性をはずした場合の結果はまだ知られていない.

Theorem 8 ([6])**.** *A は Gorenstein 局所環と仮定する. $Q = (a_1, \ldots, a_r)$ を s.s.o.p. で生成されたイデアルとする. $r \geq 3$ のとき, 次の条件は同値である:*

(1) $\mathcal{R}(Q)_{\mathfrak{M}}$ は *almost Gorenstein* 局所環である.
(2) A は正則局所環である.

Theorem 9 ([6])**.** *A は Gorenstein 局所環と仮定する. $Q = (a_1, \ldots, a_r)$ を s.s.o.p. で生成されたイデアルとする. $r \geq 3$ のとき, 次は同値である:*

(1) $\mathcal{R}(Q)$ は *almost Gorenstein* 次数付き環である.
(2) A は正則局所環であり, $a_1, \ldots, a_r$ は正則パラメーター系の一部である.

Example 10. A を正則局所環とし, $d = \dim A \geq 3$ とする. $Q = (a_1, \ldots, a_d) \neq \mathfrak{m}$ をパラメーターイデアルとするとき,

(1) $\mathcal{R}(Q)_{\mathfrak{M}}$ は almost Gorenstein 局所環である.
(2) $\mathcal{R}(Q)$ は almost Gorenstein 次数付き環である.

特に, $A = K[x_1, x_2, x_3]$, $Q = (x_1, x_2, x_3^k)$ $(k \geq 2)$ のとき,

$$\mathcal{R}(Q) \cong K[x_1, x_2, x_3, y_1, y_2, y_3]/I_2\begin{pmatrix} x_1 & x_2 & x_3^k \\ y_1 & y_2 & y_3 \end{pmatrix}$$

は (局所化すると) almost Gorenstein 正規局所整域であるが, almost Gorenstein 次数付き環ではない.

定理の証明の鍵は, Eagon-Northcott 複体である. 実際, それにより Rees 代数の標準加群が表現されることが重要である. $Q = (a_1, \ldots, a_r)A$ を Gorenstein 局所環 A の中で s.s.o.p. で生成されたイデアルとする.

$\Psi : S = R[X_1, \ldots, X_r] \to \mathcal{R} := \mathcal{R}(Q)$ を環準同型とすると, $\ker \Psi = I_2(\mathbb{A})$ と書くことができる. ただし,

$$\mathbb{A} = \begin{pmatrix} X_1 & X_2 & \cdots & X_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

とする. このとき, $\mathbb{A}$ に付随する Eagon-Northcott 複体を

$$\mathcal{C}_\bullet : \ 0 \to C_r \to C_{r-1} \to \cdots \to C_0 = S$$

とすると, これは R の S 上の次数付き極小自由分解を与える. $S(-r)$-双対を取ると, $K_{\mathcal{R}}$ の次のような表現が得られる:

$$\bigoplus_{i=1}^{r-2} S(-(i+1))^{\oplus r} \to \bigoplus_{i=1}^{r-1} S(-i) \to K_{\mathcal{R}} \to 0 \quad (ex).$$

3. p_g-IDEALS

この節を通して, A は 2 次元の excellent 正規局所整域とする. 特異点解消 $f\colon X \to \mathrm{Spec}A$ を持つと仮定する. $p_g(A) = \ell_A(H^1(X, \mathcal{O}_X))$ を A の**幾何種数 (the geometric genus** と言う.

Lemma 11. 上記の A に対して, 任意の $\mathfrak{m}$-準素整閉イデアル I はある特異点解消 $X' \to \mathrm{Spec}A$ とその上のアンチネフサイクル Z を用いて, $I = I_Z := H^0(X', \mathcal{O}_{X'}(-Z))$, かつ $I\mathcal{O}_{X'} = \mathcal{O}_{X'}(-Z)$ と書くことができる.

Theorem 12 ([9])**.** 上記の A に対して, $\mathcal{O}_X(-Z)$ が固定成分を持たないと仮定するとき, 不等式

$$\ell_A(H^1(X, \mathcal{O}_X(-Z)) \leq p_g(A).$$

が成り立つ. 等号が成立するとき, $\mathcal{O}_X(-Z)$ は大域切断で生成される.

また, 上の定理において等号が成立する条件は特異点解消の取り方にはよらない. そこで, 次の概念が定義される

Definition 13 ([9])**.** 上記の A に対して, $I = I_Z$ が **p_g-イデアル (p_g-ideal)** であるとは, 上の等号が成立すること, すなわち, $\ell_A(H^1(X, \mathcal{O}_X(-Z)) = p_g(A)$ が成立することと定める.

Remark 14 ([9]). 任意の 2 次元 excellent 正規局所整域は p_g-イデアルを持つ.

Theorem 15 ([10])**.** *A を 2 次元の excellent 正規局所整域とする. $\mathfrak{m}$-準素イデアル $I \subset A$ に対して, 次は同値である:*

(1) I は p_g-イデアルである.
(2) あるパラメーターイデアル $Q \subset I$ に対して, $I^2 = QI$ が成立し, 各 $n \geq 1$ に対して I^n は整閉である.
(3) $\mathcal{R}(I)$ は *Cohen-Macaulay* 整閉整域である.

Proposition 16 ([9])**.** 定理の A に対して, I, J は p_g-イデアルとする. このとき, $a \in I$, $b \in J$ が存在して, $IJ = aJ + bI$ が成立する. 特に, *multi-Rees* 代数 $R(I, J) = A[It_1, Jt_2]$ も *Cohen-Macaulay* 整閉整域である.

次の結果が本節の主結果である.

Theorem 17 ([7])**.** A は 2 次元の *Gorenstein excellent* 正規局所整域とし, $I \subset A$ を p_g-イデアルとする. このとき, $\mathcal{R}(I)$ は *almost Gorenstein* 次数付き環である.

$p_g(A) = 0$ のとき, A は**有理特異点 (rational singularity)** であると言う. p_g-イデアルの概念は, 有理特異点の整閉イデアルの概念を拡張したものに他ならない.

Proposition 18 (cf. Lipman)**.** A が 2 次元有理特異点ならば, 任意の $\mathfrak{m}$-準素整閉イデアルは p_g-イデアルである.

次において, Gorenstein 性を外せるかどうかは未解決である.

Corollary 19. A が 2 次元 *Gorenstein* 有理特異点ならば, 任意の $\mathfrak{m}$-準素整閉イデアルに対して, $\mathcal{R}(I)$ は正規 *almost Gorenstein* 次数付き環である.

Example 20. A を 2 次元正則局所環とすると, 任意の $\mathfrak{m}$-準素整閉イデアル I に対して, $\mathcal{R}(I)$ は正規 almost Gorenstein 次数付き環である.

Example 21. $p \geq 1$ を整数とする.

(1) $A = k[[x,y,z]]/(x^2+y^3+z^{6p+1})$ とおくと, 各 $k = 1,2,\dots,3p$ に対して, $I_k = (x,y,z^k)$ は p_g-イデアルである.

(2) $A = k[[x,y,z]]/(x^2+y^4+z^{4p+1})$ とおくと, 各 $k = 2,\dots,2p$ に対して, $I_k = (x,y,z^k)$ は p_g-イデアルであるが, $I_1 = \mathfrak{m}$ は p_g-イデアルではない.

どちらの場合も $p_g(A) = p$ である.

定理の証明のスケッチを述べておこう. I を p_g-イデアルと仮定しよう. このとき, $J = Q\colon I$ も p_g-イデアルである ([11]). さらに, $\mathfrak{m}$ は整閉イデアルだから, $f \in \mathfrak{m}$, $g \in I$ と $h \in J$ を

$$IJ = gJ + Ih, \qquad \mathfrak{m}J = fJ + \mathfrak{m}h$$

をみたすように取れる ([13]). これより, $\mathfrak{M} \cdot J\mathcal{R} \subset (f, gt)J\mathcal{R} + \mathcal{R}h$ を導くことができる. 他方, $K_{\mathcal{R}}(1) = J\mathcal{R}$, 及び $a(\mathcal{R}) = -1$ なので, $\varphi(1) = h$ として定めるとき,

$$\mathcal{R} \xrightarrow{\varphi} J\mathcal{R} \to C \to 0 \text{ (ex)}$$

を得る. $\dim C_{\mathcal{M}} \leq 2 < \dim \mathcal{R}$ ゆえ, φ は単射であり ([5, Lemma 3.1]), 求める完全列を得る.

4. Socle ideals

この節を通じて, $(A, \mathfrak{m})$ を d 次元の正則局所環とする. Q を A のパラメーターイデアルとし, $I = Q\colon \mathfrak{m}$ とおく. このようなイデアルは**ソークルイデアル (socle ideal)** と呼ばれる.

Lemma 22 ([14])**.** *$I = Q\colon \mathfrak{m} \subset A$ をソークルイデアルとする. もし, $d \geq 3$ であるか, もしくは, $d = 2$ で $Q \subset \mathfrak{m}^2$ ならば, $I^2 = QI$ が成り立つ. 特に, $\mathcal{R}(I)$ は Cohen-Macaulay 整域である.*

ソークルイデアル I の Rees 代数はほとんど almost Gorenstein 局所環にならないことを例をあげて主張しておこう (※講演終了後に, 詳細な結果が得られた).

A を 2 次元の正則局所環で, $\mathfrak{m} = (x, y)$ とする. A のパラメーターイデアル $Q = (a, b)$ に対して, $I = Q\colon \mathfrak{m}$ とおく. $Q \subset \mathfrak{m}^2$ と仮定すると, Wang の定理より, $I^2 = QI$ を得る. また, $\mu(I) = 3$ であるから, $I = (a, b, c)$ と書くことができる. このとき, $xc, yc \in Q$ だから, 次の 2 つの等式

$$f_1 a + f_2 b + xc = 0, \qquad g_1 a + g_2 b + yc = 0.$$

を得る.

Theorem 23 ([8])**.** *$(f_1, f_2, g_1, g_2) \subset \mathfrak{m}^2$ (e.g. $Q \subset \mathfrak{m}^3$) ならば, $\mathcal{R}_{\mathfrak{M}}$ は almost Gorenstein 局所環ではない.*

高次元の場合は, 次数付き環に限定しているが, やはり almost Gorenstein にはなりにくいことを示すことができる.

Theorem 24 ([6])**.** *A は正則局所環で, $d = \dim A \geq 3$ とする. Q をパラメータイデアルで, $Q \neq \mathfrak{m}$ なるものとすると, ソークルイデアル $I = Q\colon \mathfrak{m}$ に対して, 次は同値である:*

(1) *$\mathcal{R}(I)$ は almost Gorenstein 次数付き環である.*

(2) I はどちらかの条件を満たす:
  (a) $I = \mathfrak{m}$.
  (b) $d = 3$ で, ある $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ が存在して, $I = (x) + \mathfrak{m}^2$ と書ける.

Example 25. $A = K[[x, y]]$ において, $Q = (x^m, y^n)$ (ただし, $2 \leq m \leq n$) とする. $I = Q \colon \mathfrak{m}$ とおくと, $I = (x^m, x^{m-1}y^{n-1}, y^n)$ である. 従って,

(1) $m \geq 3$ ならば, $\mathcal{R}(I)$ は almost Gorenstein 局所環ではない.
(2) $m = 2$ のとき, $\mathcal{R}(I)$ は almost Gorenstein 次数付き環である.

もし $Q = (x^2, y^4)$ ならば, $I = Q \colon \mathfrak{m} = (x^2, xy^3, y^4)$ で, $\overline{I} = (x^2, xy^2, y^4)$ である. ゆえに, $\mathcal{R}(I)$ は almost Gorenstein 次数付き環であるが, 整閉整域ではない. 実際, $\overline{\mathcal{R}(I)} = \mathcal{R}((x^2, xy^2, y^4))$ である.

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