TILTING COMPLEXES OVER PREPROJECTIVE ALGEBRAS OF DYNKIN TYPE

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Abstract. In this note, we explain a connection between braid groups and tilting complexes over preprojective algebras of Dynkin (A,D,E) type. More precisely, we classify all tilting complexes by giving a bijection with elements of the braid groups.

1. Introduction

Derived categories are nowadays considered as a fundamental object in many branches of mathematics including representation theory and algebraic geometry. One of the important problems is to study their equivalences. By Rickard's Morita theorem for derived categories, it is known that derived equivalences are controlled by tilting complexes [28]. Tilting theory provides several useful methods for studying tilting complexes and, in particular, mutation plays a significant role. Roughly speaking, mutation is an operation, for a certain class of objects, to obtain a new object from a given one by replacing a summand. In the case of tilting modules, their mutation was formulated by Riedtmann-Schofield and Happel-Unger [30, 16, 32]. For example, APR (Auslander-Platzeck-Reiten) tilting modules [5] and Okuyama-Rickard complexes [29, 27, 18] can be regarded as a special case of tilting mutation. One of the negative aspects of tilting mutation is that some summands of a tilting complex can not be replaced to get a new one and hence we can not repeat tilting mutation. To remove this disadvantage, Aihara-Iyama studied a wider class of mutation, called silting mutation and it is shown that silting mutation is always possible and it admits a combinatorial description [4].

We give a further development of tilting (silting) theory and we determine all tilting complexes over preprojective algebras of Dynkin type.

2. Main results

2.1. **Preprojective algebras.** Preprojective algebras was first introduced by Gelfand-Ponomarev [15], and later formulated and developed in [14, 7]. Since then, they are one of the fundamental objects in the representation theory (refer to a survey paper [31]).

Let K be an algebraically closed field and Q a finite connected acyclic quiver. We denote by $\overline{Q}$ the double quiver of Q, which is obtained by adding an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q_1. The *preprojective algebra* $\Lambda_Q = \Lambda$ associated to Q is the algebra $K\overline{Q}/I$, where I is the ideal in the path algebra $K\overline{Q}$ generated by the relations of

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the form:

$$\sum_{a\in Q_1}(aa^* - a^*a).$$

Let Q be a Dynkin quiver and e_i the primitive idempotent of Λ associated with $i \in Q_0$. Then the preprojective algebra of Q is finite dimensional and selfinjective [11, Theorem 4.8]. We denote the Nakayama permutation of Λ by $\sigma : Q_0 \to Q_0$ (i.e. $D(\Lambda e_{\sigma(i)}) \cong e_i\Lambda$, where $D := \mathrm{Hom}_K(-, K)$).

Note that Λ_Q does not depend on the orientation of Q.

2.2. **Weyl group.** We refer to [8, 19] for basic properties of the Weyl (Coxeter) group. Let Q be a quiver of type $A, B(C), D, E$ and F. The *Weyl group* W_Q associated to Q is defined by the generators s_i ($i \in Q_0$) and relations $(s_is_j)^{m(i,j)} = 1$, where

$$m(i,j) := \begin{cases} 1 & \text{if } i = j; \\ 2 & \text{if no edge between } i \text{ and } j; \\ 3 & \text{if there is an edge } i \text{ --- } j, \\ 4 & \text{if there is an edge } i \overset{4}{\text{---}} j. \end{cases}$$

Each element $w \in W_Q$ can be written in the form $w = s_{i_1} \cdots s_{i_k}$. If k is minimal among all such expressions for w, then k is called the *length* of w and we denote by $l(w) = k$. In this case, we call $s_{i_1} \cdots s_{i_k}$ a *reduced expression* of w.

Let σ be the Nakayama permutation of Λ. Then σ acts on an element of the Weyl group W_Q by $\sigma(w) := s_{\sigma(i_1)}s_{\sigma(i_2)} \cdots s_{\sigma(i_\ell)}$ for $w = s_{i_1}s_{i_2} \cdots s_{i_\ell} \in W_Q$. We define the subgroup W_Q^σ of W_Q by

$$W_Q^\sigma := \{w \in W \mid \sigma(w) = w\}.$$

Then we have the following result. (See [13, Chapter 13]).

Theorem 1. *Let Q be a Dynkin (A,D,E) quiver and W_Q the Weyl group of Q. Let $Q' = Q$ if Q is type D_{2n}, E_7 and E_8. Otherwise, let Q' be a quiver, respectively, given by the following type.*

Q	A_{2n-1}, A_{2n}	D_{2n+1}	E_6
Q'	B_n	B_{2n}	F_4

Then W_Q^σ is isomorphic to $W_{Q'}$.

We call the quiver Q' given in Theorem 1 the *folding quiver* of Q.

Example 2. Let Q be a quiver of type A_5. Then one can check that W_Q^σ is given by $\langle s_1, (s_2s_4), (s_3s_5)\rangle$ and this group is isomorphic to $W_{Q'}$, where Q' is a quiver of type B_3.

2.3. **Support τ-tilting modules.** The notion of support τ-tilting modules was introduced in [2], as a generalization of tilting modules. We refer to [2, 21] for several nice properties of support τ-tilting modules.

Let Λ be a finite dimensional algebra and we denote by τ the AR translation [6].

Definition 3. We call a Λ-module X *τ-tilting* if X is $\mathrm{Hom}_\Lambda(X, \tau X) = 0$ and $|X| = |\Lambda|$, where $|X|$ denotes the number of non-isomorphic indecomposable direct summands of X.

Moreover, we call a Λ-module X *support τ-tilting* if there exists an idempotent e of Λ such that X is a τ-tilting $(\Lambda/\langle e\rangle)$-module.

We denote by sτ-tiltΛ the set of isomorphism classes of basic support τ-tilting Λ-modules.

Remark 4. We note that support τ-tilting modules can be described as pairs. These definition are essentially same.

Now let Q be a Dynkin quiver with $Q_0 = \{1, \dots, n\}$ and Λ the preprojective algebra of Q. We denote by $I_i := \Lambda(1 - e_i)\Lambda$ for $i \in Q_0$. We denote by $\langle I_1, \dots, I_n\rangle$ the set of ideals of Λ which can be written as

$$I_{i_1} I_{i_2} \cdots I_{i_k}$$

for some $k \geq 0$ and $i_1, \dots, i_k \in Q_0$.

Then following result plays an important role in this note.

Theorem 5. [9, 25] *Under the above notation,*

(a) *There exists a bijection* $W_Q \to \langle I_1, \dots, I_n\rangle$, *which is given by* $w \mapsto I_w = I_{i_1} I_{i_2} \cdots I_{i_k}$ *for any reduced expression* $w = s_{i_1} \cdots s_{i_k}$.

(b) *It gives a bijection between the elements of the Weyl group* W_Q *and the set* sτ-tiltΛ *of isomorphism classes of basic support* τ*-tilting* Λ*-modules.*

We remark that the above ideals I_w are tilting modules in the case of non-Dynkin type in [20, 9].

2.4. **Silting complexes.** Silting complexes are a generalization of tilting complexes, which were introduced by Keller-Vossieck [23]. They were originally invented as a tool for studying tilting complexes. Nonetheless, silting complexes have turned out to have deep connections with several important complexes such as t-structures [10, 24, 12, 22].

We recall the definition of silting complexes as follows.

Definition 6. Let Λ be a finite dimensional algebra and $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ the bounded homotopy category of the finitely generated projective Λ-modules.

(a) We call a complex P in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is *presilting* (respectively, *pretilting*) if it satisfies $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, P[i]) = 0$ for any $i > 0$ (respectively, $i \neq 0$).

(b) We call a complex P in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ *silting* (respectively, *tilting*) if it is presilting (respectively, pretilting) and the smallest thick subcategory containing P is $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

We denote by silt Λ (respectively, tilt Λ) the set of non-isomorphic basic silting (respectively, tilting) complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

For complexes P and Q of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, we write $P \geq Q$ if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, Q[i]) = 0$ for any $i > 0$. Then the relation $\geq$ gives a partial order on silt Λ [4, Theorem 2.11] (cf. [17]).

Moreover, a complex $T \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is called *2-term* provided it is concerned in the degree 0 and -1. We denote by 2-silt Λ (respectively, 2-tilt Λ) the subset of silt Λ (respectively, tilt Λ) consisting of 2-term complexes. Note that a complex T is 2-term if and only if $\Lambda \geq T \geq \Lambda[1]$.

Then we have the following nice correspondence.

Theorem 7. [2, Theorem 3.2] *Let* Λ *be a finite dimensional algebra. There exists a bijection*

$$\text{s}\tau\text{-tilt}\Lambda \leftrightarrow \text{2-silt}\,\Lambda.$$

By the above correspondence, we can give a description of 2-term silting complexes by calculating support τ-tilting modules, which is much simpler than calculations of silting complexes.

From now on, let Q be a Dynkin quiver and Λ the preprojective algebra of Q. Then, as a corollary of Theorem 5 and 7, we have the following corollary.

Corollary 8. *We have a bijection*

$$W_Q \leftrightarrow 2\text{-silt}\,\Lambda.$$

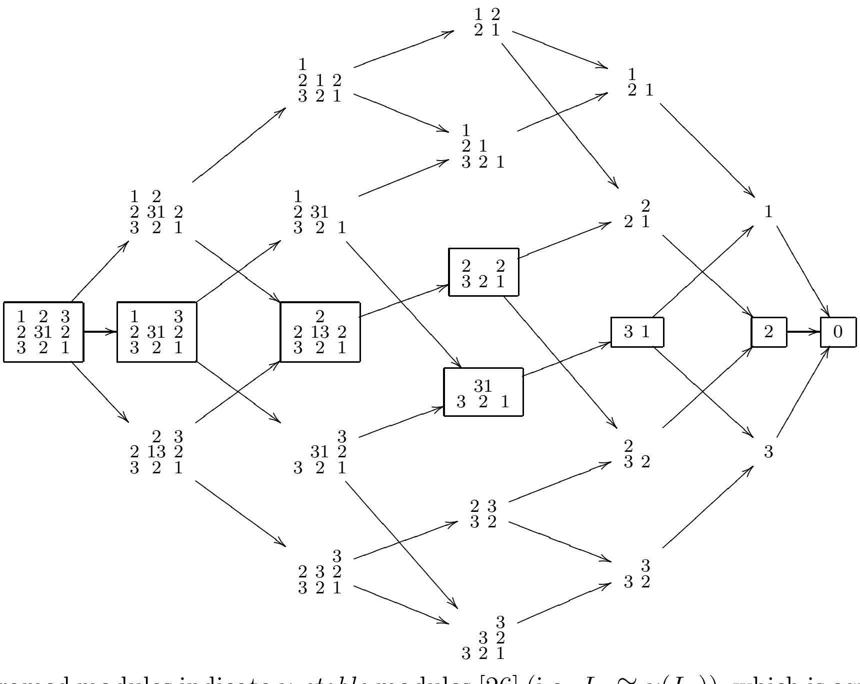
Thus we can parameterize 2-term silting complexes by the Weyl group. Moreover, we can describe 2-term tilting complexes in terms of the Weyl group by the following proposition.

Proposition 9. *Let $\nu := D\operatorname{Hom}_\Lambda(-,\Lambda)$ the Nakayama functor of Λ and $\sigma : Q_0 \to Q_0$ the Nakayama permutation of Λ. Then $\nu(I_w) \cong I_w$ if and only if $\sigma(w) = w$. In particular, We have a bijection*

$$W_Q^\sigma \leftrightarrow 2\text{-tilt}\,\Lambda.$$

Then by Theorem 1, we can understand W_Q^σ as another type of the Weyl group.

Example 10. Let Q be a quiver of type A_3 and Λ the preprojective algebra of Q. Then the *support τ-tilting quiver* of Λ ([2, Definition 2.29]) is given as follows.



The framed modules indicate *ν-stable* modules [26] (i.e. $I_w \cong \nu(I_w)$), which is equivalent to say that $\sigma(w) = w$ by Proposition 9. Hence Theorem 1 implies that these modules correspond to the subgroup $W_Q^\sigma = \langle (s_1 s_3), s_2 \rangle$ and it is isomorphic to the Weyl group of type B_2.

Next we use (silting) mutation. Let $\Lambda = X \oplus Y$. We denote by $\mu_X(\Lambda)$ the left mutation of Λ with respect to X. It is not necessarily tilting in general (cf.[1]). However, if it is tilting, then we have the following nice result.

Proposition 11. *Assume that $\mu_X(\Lambda)$ is a tilting complex, then we have an isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(\mu_X(\Lambda)) \cong \Lambda.$$

Togher with this proposition, the finiteness of 2-silt Λ implies that *tilting-discreteness* of Λ and we conclude that any tilting complex is obtained from Λ by iterated mutation (see [3]). Then we extend Proposition 9 and obtain the following consequence.

Theorem 12. *Let Q be a Dynkin quiver, Λ the preprojective algebra of Q and Q' the folding quiver of Q. We denote the braid group by $B_{Q'}$. Then we have a bijection*

$$B_{Q'} \leftrightarrow \mathrm{tilt}\,\Lambda.$$

Thus we can parametrize any tilting complex by the braid group.

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