BASICALIZATION OF KLR ALGEBRAS

MASAHIDE KONISHI

Abstract. We describe an algorithm to basicalize KLR algebras arising from quivers.

1. Preliminaries

Let k be field and A be a finite dimensional or "good" infinite dimensional connected algebra over k. Throughout this paper, an algebra is associative and with an unit element 1_A. Then A is decomposed into indecomposable projective left A-modules P_i as left A-module, where P_i is Ae_i for a complete set of primitive orthogonal idempotents :

(i) $\sum_{i=1}^{n} e_i = 1_A$,
(ii) if $e_i = f + g$ where $fg = gf = 0$, $f^2 = f$, $g^2 = g$ then $f = 0$ or $g = 0$,
(iii) $(e_i)^2 = e_i$,
(iv) $e_i e_j = 0$ for $i \neq j$.

We call A basic if $P_i \ncong P_j$ for $i \neq j$. Even if A is not basic, we can basicalize A like that. Choose some primitive idempotents e_{j_k} to satisfy the following property: for every e_i there exists exactly one r such that $P_i \cong P_{j_r}$. Set e the sum of those idempotents then $A^b := eAe$ is basic algebra. Note that A and A^b are Morita equivalent therefore module categories of those two are equivalent.

Let A be a basic algebra, then we can obtain a connected quiver Q and an admissible ideal I of a path algebra kQ such that $A \cong kQ/I$. Our final destination is to describe an algorithm to obtain such Q and I for KLR algebras.

Let Γ be a finite connected quiver without loops and multiple arrows. Let $\Gamma_0 = \{1, 2, \ldots, n\}$. Let ν be n-tuple $(\nu_1, \nu_2, \ldots, \nu_n)$ of non-negative integers. In general, KLR algebras $R_\Gamma(\nu)$ is defined depend on ν however in this paper we fix $\nu_i = 1$ for every i. Let $I_n = \{\sigma(1, 2, \cdots, n) \,|\, \sigma \in S_n\}$, $s_k = (k, k+1) \in S_n$. For $\mathbf{i} \in I_n$, describe $\mathbf{i}$ as $(i_1, i_2, \ldots, i_n)$.

Definition 1. A *KLR* algebra R_Γ is defined from these generators and relations.

- generators:
  $\{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$.
- relations:
  $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}\mathbf{e}(\mathbf{i})$, $\sum_{\mathbf{i} \in I_n} \mathbf{e}(\mathbf{i}) = 1$,
  $y_k \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i}) y_k$, $\psi_k \mathbf{e}(\mathbf{i}) = \mathbf{e}(s_k \mathbf{i}) \psi_k$,
  $y_k y_l = y_l y_k$,

The detailed version of this paper will be submitted for publication elsewhere.

$\psi_k y_l = y_l \psi_k \ (l \neq k, k+1)$,
$\psi_k y_{k+1} = y_k \psi_k$, $y_{k+1}\psi_k = \psi_k y_k$,
$\psi_k \psi_l = \psi_l \psi_k \ (|k-l| > 1)$,
$\psi_k \psi_{k+1} \psi_k = \psi_{k+1} \psi_k \psi_{k+1}$,

$$\psi_k^2 \mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & (i_k \not\leftrightarrow i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases}.$$

Note that the first (resp. second) equation shows $\mathbf{e}(\mathbf{i})$s are orthogonal (resp. complete). Moreover, R_Γ is $\mathbb{Z}$-graded algebra by $deg(\mathbf{e}(\mathbf{i})) = 0$, $deg(y_k) = 2$, $deg(\psi_k) = 0$ if $i_k \not\leftrightarrow i_{k+1}$, 1 if $i_k \to i_{k+1}$ or $i_k \leftarrow i_{k+1}$, 2 if $i_k \leftrightarrow i_{k+1}$.

2. The starting point

As the first step, we define a class of quiver called gemstone quiver.

Definition 2. A gemstone quiver G_n is defined as follows.

- vertices: $\mathbf{i} \in I^n$.
- arrows:
  - $y_k^{\mathbf{i}} : \mathbf{i} \to \mathbf{i}$ for each $\mathbf{i} \in I_n$ and $1 \leq k \leq n$,
  - $\psi_l^{\mathbf{i}} : \mathbf{i} \to s_l \mathbf{i}$ for each $\mathbf{i} \in I_n$ and $1 \leq l < n$.

Then we obtain following lemma.

Lemma 3. *There exists an epimorphism $kG_n \longrightarrow R_\Gamma$ by $\mathbf{i} \mapsto \mathbf{e}(\mathbf{i})$, $y_k^{\mathbf{i}} \mapsto \mathbf{e}(\mathbf{i}) y_k \mathbf{e}(\mathbf{i})$, $\psi_l^{\mathbf{i}} \mapsto \mathbf{e}(\mathbf{i})\psi_l \mathbf{e}(s_l \mathbf{i})$. Moreover, $kG_n / I_\Gamma \cong R_\Gamma$ where I_Γ is an ideal obtained by rewriting relations of R_Γ by the above correspondence.*

Note that I_Γ is not admissible ideal since there are those relations : $\psi_k^2 \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ if $i_k \not\leftrightarrow i_{k+1}$, $(y_{k+1} - y_k)\mathbf{e}(\mathbf{i})$ if $i_k \to i_{k+1}$, $(y_k - y_{k+1})\mathbf{e}(\mathbf{i})$ if $i_k \leftarrow i_{k+1}$. Therefore we need some processes except for some cases. The following corollary is straightforward from the next section.

Corollary 4. *Let Γ be a quiver with 2-cycle for each two vertices. Then G_n and I_Γ present R_Γ.*

3. Processes

We should start from removing this type of relations: $\psi_k^2 \mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})$ if $i_k \not\leftrightarrow i_{k+1}$. In fact, that relations are useful to determine an isomorphic class of indecomposable projective modules.

Lemma 5. *All $\mathbf{e}(\mathbf{i})$ are primitive. Therefore $R_\Gamma \mathbf{e}(\mathbf{i})$ is indecomposable.*

Lemma 6. $R_\Gamma \mathbf{e}(\mathbf{i}) \cong R_\Gamma \mathbf{e}(s_k \mathbf{i})$ *if and only if* $i_k \not\leftrightarrow i_{k+1}$

Using this lemma repeatedly, we can obtain the following property.

Let $\bar{G}_n$ be a graph obtained by removing loops and replacing each 2-cycles by edge on G_n. Cut edges between $\mathbf{i}$ and $s_k \mathbf{i}$ if there exists some arrows between i_k and i_{k+1} on Γ,

denote this cut graph G_Γ. Then the followings are equivalent:
(a) $\mathbf{i}$ and $\mathbf{j}$ are on the same connected component on G_Γ,
(b) $R_\Gamma \mathbf{e}(\mathbf{i}) \cong R_\Gamma \mathbf{e}(\mathbf{j})$.
We get a new quiver by identifying the vertices of G_n for each connected components of G_Γ.

To rewrite relations, we should pick up one $\mathbf{i}$ from each connected components. Then vertices $\mathbf{i}$ means $\mathbf{e}(\mathbf{i})$ and loops $y_k^{\mathbf{i}}$ means $\mathbf{e}(\mathbf{i})y_k\mathbf{e}(\mathbf{i})$. However the meaning of two cycles for two vertices $\mathbf{i}$ and $\mathbf{j}$ are bit complicated. Since there are two cycles between them, there exists some paths from $\mathbf{i}$ to $\mathbf{j}$ in G_n constructed from three parts:
(i) a path in connected component with $\mathbf{i}$, from $\mathbf{i}$ to some $\mathbf{i}'$,
(ii) an arrow $\mathbf{i}'$ to $\mathbf{j}'$ where $\mathbf{j}'$ picked from a connected component with $\mathbf{j}$,
(iii) a path in connected component with $\mathbf{j}$ from $\mathbf{j}'$ to $\mathbf{j}$.

We pick two minimal paths for each two cycles between $\mathbf{i}$ and $\mathbf{j}$ to be inverse each other. Then the arrow $\mathbf{i}$ to $\mathbf{j}$ means $\mathbf{e}(\mathbf{i})\psi_\omega\mathbf{e}(\mathbf{j})$, where ψ_ω is a multiplication of ψs in G_n taken as above. Note that only part (ii) has positive degree in that path.

Then relations for this quiver are obtained from G_n by rewriting with the correspondence above. However there still remains a problem from these type of relations:
$\psi_k^2\mathbf{e}(\mathbf{i}) = \pm(y_{k+1} - y_k)\mathbf{e}(\mathbf{i})$ if there exists one arrow between i_k and i_{k+1}.
The problem is on right hand side, it must not be in admissible ideal since it's just a sum of two arrows. Therefore we delete arrows by rewriting relations as follows:
$y_{k+1}\mathbf{e}(\mathbf{i}) = y_k\mathbf{e}(\mathbf{i}) \pm \psi_k^2\mathbf{e}(\mathbf{i})$.
After that process all relations are obtained from a linear combination of paths of length greater than 2. Therefore it's completed.

From the construction above, we can obtain some combinatorial observations such as :

Corollary 7. *The quiver for R_Γ has at least one loop for each vertex.*

4. Cyclotomic case

We can use previous method for cyclotomic case.

Definition 8. For $\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$, a cyclotomic ideal I^Λ is generated by $\left\{y_1^{\lambda_{i_1}}\mathbf{e}(\mathbf{i}) | \mathbf{i} \in I_n\right\}$.
We call a quotient algebra $R_\Gamma^\Lambda = R_\Gamma / I^\Lambda$ a cyclotomic KLR algebra.

Only what we do is adding relations from that generators. However there is $\lambda_k \leq 1$, we need rewrite something. If there is $\lambda_k = 0$, we need to trim some vertices by using following lemma.

Lemma 9. *In R_Γ^Λ, $\mathbf{e}(\mathbf{i}) = 0$ if and only if $\lambda_{i_1} = 0$ or there exists k such that for every $s < k$ there is no arrow between i_s and i_k on Γ.*

We trim $\mathbf{i}$ with $\mathbf{e}(\mathbf{i}) = 0$ and rewrite relations including $\mathbf{i}$.

The remaining problem is about this type of relations: $y_1\mathbf{e}(\mathbf{i}) = 0$. This happens if $\lambda_{i_1} = 1$. To avoid this relation, delete arrows $y_1^{\mathbf{i}}$ and rewrite relations including $\mathbf{i}$. Then it‘s completed.

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GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
FROCHO, CHIKUSAKU, NAGOYA 464-8602 JAPAN

E-mail address: m10021t@math.nagoya-u.ac.jp